FOR THE
IB DIPLOMA

Mathematics

ANALYSIS AND APPROACHES **HL**

Paul Fannon
Vesna Kadelburg
Ben Woolley
Stephen Ward

DYNAMIC LEARNING
HODDER EDUCATION

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Contents

Introduction v

Chapter 1 Counting principles 2
- 1A Basic techniques 4
- 1B Problem solving 11

Chapter 2 Algebra 16
- 2A Extension of the binomial theorem to fractional and negative indices 18
- 2B Partial fractions 22
- 2C Solutions of systems of linear equations 25

Chapter 3 Trigonometry 34
- 3A Further trigonometric functions 36
- 3B Compound angle identities 44

Chapter 4 Complex numbers 56
- 4A Cartesian form 58
- 4B Modulus–argument form and Euler form 65
- 4C Complex conjugate roots of quadratic and polynomial equations with real coefficients 74
- 4D Powers and roots of complex numbers 79
- 4E Trigonometric identities 87

Chapter 5 Mathematical proof 94
- 5A Proof by induction 96
- 5B Proof by contradiction 102
- 5C Disproof by counterexample 104

Chapter 6 Polynomials 108
- 6A Graphs and equations of polynomial functions 110
- 6B The factor and remainder theorems 118
- 6C Sum and product of roots of polynomial equations 122

Chapter 7 Functions 132
- 7A Rational functions of the form $f(x) = \frac{ax+b}{cx^2+dx+e}$ and $f(x) = \frac{ax^2+bx+c}{dx+e}$ 134
- 7B Solutions of $g(x) \geqslant f(x)$, both analytically and graphically 139
- 7C The graphs of the functions $y = |f(x)|$ and $y = f(|x|)$ 143
- 7D The graphs of the functions $y = \frac{1}{f(x)}$, $y = f(ax+b)$ and $y = [f(x)]^2$ 151
- 7E Properties of functions 163

Chapter 8 Vectors 176
- 8A Introduction to vectors 178
- 8B Vectors and geometry 190
- 8C Scalar product and angles 199
- 8D Equation of a line in three dimensions 207
- 8E Intersection of lines 221
- 8F Vector product and areas 226
- 8G Equation of a plane 234
- 8H Angles and intersections between lines and planes 242

Chapter 9 Probability 262
- 9A Bayes' theorem 264
- 9B Variance of a discrete random variable 270
- 9C Continuous random variables 274

Chapter 10 Further calculus 294
- 10A Fundamentals of calculus 297
- 10B L'Hôpital's rule 306
- 10C Implicit differentiation 310
- 10D Related rates of change 312
- 10E Optimization 314
- 10F Calculus applied to more functions 317
- 10G Integration by substitution 324
- 10H Integration by parts 327
- 10I Further geometric interpretation of integrals 332

Chapter 11 Series and differential equations 350
- 11A First order differential equations and Euler's method 352
- 11B Separating variables and homogeneous differential equations 357
- 11C Integrating factors 362
- 11D Maclaurin series 366
- 11E Using Maclaurin series to solve differential equations 375

Analysis and approaches HL: Practice Paper 1 386

Analysis and approaches HL: Practice Paper 2 389

Guidance for Paper 3 393

Analysis and approaches HL: Practice Paper 3 394

Answers 396

Glossary 477

Index 479

Introduction

Welcome to your coursebook for Mathematics for the IB Diploma: analysis and approaches HL. The structure and content of this coursebook follow the structure and content of the 2019 IB Mathematics: analysis and approaches guide, with headings that correspond directly with the content areas listed therein.

This is the second book required by students taking the higher level course. Students should be familiar with the content of Mathematics for the IB Diploma: analysis and approaches SL before moving on to this book.

Using this book

Special features of the chapters include:

ESSENTIAL UNDERSTANDINGS

Each chapter begins with a summary of the key ideas to be explored and a list of the knowledge and skills you will learn. These are revisited in a checklist at the end of each chapter.

CONCEPTS

The IB guide identifies 12 concepts central to the study of mathematics that will help you make connections between topics, as well as with the other subjects you are studying. These are highlighted and illustrated with examples at relevant points throughout the book. The concepts are: Approximation, Change, Equivalence, Generalization, Modelling, Patterns, Relationships, Space, Systems and Validity.

KEY POINTS

Important mathematical rules and formulae are presented as Key Points, making them easy to locate and refer back to when necessary.

WORKED EXAMPLES

There are many Worked Examples in each chapter, demonstrating how the Key Points and mathematical content described can be put into practice. Each Worked Example comprises two columns:

On the left, how to **think** about the problem and what tools or methods will be needed at each step.	On the right, what to **write**, prompted by the left column, to produce a formal solution to the question.

Exercises

Each section of each chapter concludes with a comprehensive exercise so that students can test their knowledge of the content described and practise the skills demonstrated in the Worked Examples. Each exercise contains the following types of questions:

- **Drill questions:** These are clearly linked to particular Worked Examples and gradually increase in difficulty. Each of them has two parts – **a** and **b** – desgined such that if students get **a** wrong, **b** is an opportunity to have another go at a very similar question. If students get **a** right, there is no need to do **b** as well.
- **Problem-solving questions:** These questions require students to apply the skills they have mastered in the drill questions to more complex, exam-style questions. They are colour-coded for difficulty.

1 Green questions are closely related to standard techniques and require a small number of processes. They should be approachable for all candidates.

2 Blue questions require students to make a small number of tactical decisions about how to apply the standard methods and they will often require multiple procedures. Candidates targeting the medium HL grades should find these questions challenging but achievable.

3 Red questions often require a creative problem-solving approach and extended, technical procedures. Candidates targeting the top HL grades should find these questions challenging.

4 Black questions go beyond what is expected in IB examinations, but provide an enrichment opportunity for the very best students.

The questions in the Mixed Practice section at the end of each chapter are similarly colour-coded, and contain questions taken directly from past IB Diploma Mathematics exam papers. There are also three practice examination papers at the end of the book plus guidance on how to approach Paper 3.

Answers to all exercises can be found at the back of the book.

A calculator symbol is used where we want to remind you that there is a particularly important calculator trick required in the question.

A non-calculator icon suggests a question is testing a particular skill for the non-calculator paper.

The guide places great emphasis on the importance of technology in mathematics and expects you to have a high level of fluency with the use of your calculator and other relevant forms of hardware and software. Therefore, we have included plenty of screenshots and questions aimed at raising awareness and developing confidence in these skills, within the contexts in which they are likely to occur. This icon is used to indicate topics for which technology is particularly useful or necessary.

Making connections: Mathematics is all about making links. You might be interested to see how something you have just learned will be used elsewhere in the course and in different topics, or you may need to go back and remind yourself of a previous topic.

Be the Examiner

These are activities that present you with three different worked solutions to a particular question or problem. Your task is to determine which one is correct and to work out where the other two went wrong.

Proof

Proofs are set out in a similar way to Worked Examples, helping you to gain a deeper understanding of the mathematical rules and statements you will be using and to develop the thought processes required to write your own proofs.

TOOLKIT

There are questions, investigations and activities interspersed throughout the chapters to help you develop mathematical thinking skills, building on the introductory toolkit chapter from the Mathematics for the IB Diploma: analysis and approaches SL book in relevant contexts. Although the ideas and skills presented will not be examined, these features are designed to give you a deeper insight into the topics that will be. Each toolkit box addresses one of the following three key topics: proof, modelling and problem solving.

International mindedness

These boxes explore how the exchange of information and ideas across national boundaries has been essential to the progress of mathematics and illustrate the international aspects of the subject.

You are the Researcher

This feature prompts you to carry out further research into subjects related to the syllabus content. You might like to use some of these ideas as starting points for your mathematical exploration or even an extended essay.

LEARNER PROFILE

Opportunities to think about how you are demonstrating the attributes of the IB Learner Profile are highlighted at appropriate places.

Tips

There are short hints and tips provided in the margins throughout the book.

TOK Links

Links to the interdisciplinary Theory of Knowledge element of the IB Diploma programme are made throughout the book.

Links to: Other subjects

Links to other IB Diploma subjects are made at relevant points, highlighting some of the real-life applications of the mathematical skills you will learn.

Topics that have direct real-world applications are indicated by this icon.

There is a glossary at the back of the book. Glossary terms are **purple**.

These features are designed to promote the IB's inquiry-based approach, in which mathematics is not seen as a collection of facts to be learned, but a set of skills to be developed.

About the authors

The authors are all University of Cambridge graduates and have a wide range of expertise in pure mathematics and in applications of mathematics, including economics, epidemiology, linguistics, philosophy and natural sciences.

Between them they have considerable experience of teaching IB Diploma Mathematics at Standard and Higher Level, and two of them currently teach at the University of Cambridge.

1 Counting principles

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to find the number of ways of choosing an option from list A and an option from list B
- how to find the number of ways of choosing an option from list A or an option from list B
- how to find the number of permutations of n items
- how to find the number of ways of choosing r items from a list of n items, both when the order does not matter and when the order does matter.

CONCEPTS

The following concepts will be addressed in this chapter:

- Formulas are a **generalization** made on the basis of specific examples, which can then be extended to new examples.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Without a calculator, evaluate:

  a $5!$

  b ${}^7C_3$

2 A fair dice is rolled and a fair coin is flipped. Find the probability of:

  a rolling a 6 on the dice and flipping heads on the coin

  b rolling a 6 on the dice or flipping heads on the coin or both.

Figure 1.1 How many different combinations are there?

It may seem strange to only start talking about 'counting principles' at this stage of your mathematics education! However, while simple counting is one of the very first things we learn to do, counting arrangements and selections of items in certain situations can be rather complicated. It is important to have some basic strategies and to work systematically to make sure we neither miss anything out nor count the same thing more than once.

Starter Activity

Look at the pictures in Figure 1.1. Discuss why being able to count the number of ways certain events can occur is important.

Now look at this problem:

a Write down all possible arrangements of the letters A, B, C.

b Write down all possible selections of three letters from A, B, C, D, E. Note that ABC, BCA, and so on, count as the same selection.

c Hence, without writing them all out, determine the number of possible arrangements of three letters chosen from A, B, C, D, E.

LEARNER PROFILE – Inquirers

Is mathematics just about answering other people's questions? Before you can do this, you need to get used to questioning other people's mathematics – asking questions like 'When does this work?', 'What assumptions are being made here?' or 'How does this link to what I already know?' are all second nature to mathematicians.

1A Basic techniques

The AND rule and the OR rule

If you want to choose one option from list A *and* one option from list B, then you can find the number of possible ways of doing this by multiplying the number of options in list A, $n(A)$, by the number of options in list B, $n(B)$.

KEY POINT 1.1

The AND rule:

$$n(A \text{ AND } B) = n(A) \times n(B)$$

Similarly, for the number of ways of choosing one option from list A *or* one option from list B, you add the number of options in each list. However, you need to make sure that the lists are mutually exclusive.

KEY POINT 1.2

The OR rule:
If A and B are mutually exclusive, then

$$n(A \text{ OR } B) = n(A) + n(B)$$

You saw very similar rules for probability in Chapter 7 of the Mathematics: analysis and approaches SL book.

WORKED EXAMPLE 1.1

Rohan has four jackets and seven ties in his wardrobe.

Calculate the number of different ways he can choose to dress if he wears:

a a jacket and a tie

b a jacket or a tie.

Use $n(A \text{ AND } B) = n(A) \times n(B)$

a Number of ways $= 4 \times 7$
$= 28$

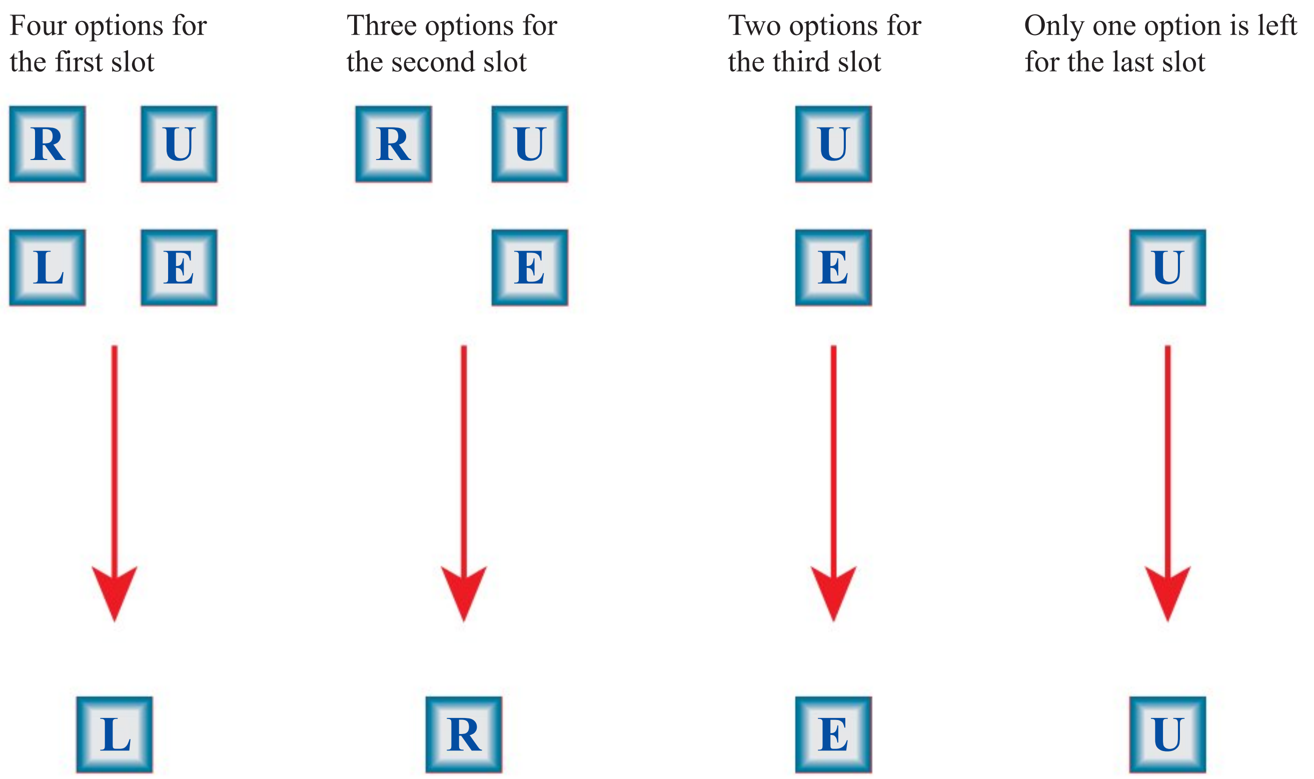
Jackets and ties are mutually exclusive so use $n(A \text{ OR } B) = n(A) + n(B)$

b Number of ways $= 4 + 7$
$= 11$

Permutations and combinations

To calculate how many ways the letters R, U, L, E can be arranged, you can consider the number of letter options available for the first position, second position and so on and use the AND rule:

- first position: 4 options
- second position: 3 options
- third position: 2 options
- fourth position: 1 option.



Total number of ways $= 4 \times 3 \times 2 \times 1 = 24$

Each of these 24 arrangements (RULE, RUEL, LUER, and so on) is called a **permutation**.

This method provides a useful result.

KEY POINT 1.3

The number of permutations of n items is $n!$

TOOLKIT: Problem Solving

How many ways are there of arranging five objects in a circle? Can you find a formula for the number of ways of arranging n objects in a circle?

This idea can be combined with the AND or the OR rule.

WORKED EXAMPLE 1.2

Jason wants to set up a new username consisting of the five letters J, A, S, O, N followed by the three digits 1, 2, 3.

Find the number of different ways he can do this.

There are 5! permutations of the letters AND 3! permutations of the numbers

$$\text{Number of ways} = 5! \times 3! = 120 \times 6 = 720$$

Instead of finding arrangements of a given number of items, you might be interested in finding the number of ways of choosing some items from a larger list, for example the number of ways of choosing four letters from the list R, U, L, E, S.

You might just want to know how many different groups of four letters can be chosen (so RULE, RUEL, LURE, and so on, would just count as one choice). A selection like this where the order does not matter is called a **combination**. There are only five combinations of four letters from this list: RULE, RULS, RUES, RLES, ULES.

The number of combinations can be calculated in general using the method you met when finding binomial coefficients.

KEY POINT 1.4

The number of ways of choosing r items from n when the order does not matter is

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Proof 1.1

Prove that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

One classic method of proof is finding the same thing in two different ways → Consider the number of ways of permuting n distinct objects. This is $n!$

However, it could also be done by

We are using 'choosing r objects out of n' as the defining feature of ${}^nC_r$ →

- choosing r objects from the n

AND

- putting these r objects into an order

AND

- ordering the remaining $n - r$ objects.

This is done in the following number of ways:

We then apply the AND rule → ${}^nC_r \times r! \times (n-r)!$

Equating the two ways of permuting n objects → Therefore, $n! = {}^nC_r \times r! \times (n-r)!$

Rearranging:

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

CONCEPT – GENERALIZATION

A very good way of understanding a new formula or proof is to try a specific example and see how it might **generalize**.

Consider the proof above and permuting the letters ABCD. Think about one pair of letters and arrange them in two ways, then arrange the remaining pair of letters in two ways. For each combination of the first pair of letters, there are therefore four ways of arranging them all. Given that there are six ways of picking two letters out of four, this makes the overall number of permutations 24.

Once you have understood the problem using a specific example, you can then generalize the idea using proof, as illustrated above.

See Chapter 13 of the Mathematics: analysis and approaches SL book for a reminder of how to work with the ${}^nC_r$ formula.

You might also want to count all the different orderings (permutations) of items picked from a larger list. With the example of choosing four letters from R, U, L, E, S, you have the five combinations (RULE, RULS, RUES, RLES, ULES) and each of these will have $4! = 24$ permutations, giving $5 \times 24 = 120$ ways of selecting four letters from RULES when the order matters.

The symbol for the number of ways of choosing r objects out of n distinct objects is ${}^nP_r$.

It can be calculated using:

$$ {}^nP_r = {}^nC_r \times r! = \frac{n!}{r!(n-r)!} \times r! = \frac{n!}{(n-r)!} $$

Tip

Remember that ${}^nP_r$ and ${}^nC_r$ are both most easily evaluated on a calculator.

KEY POINT 1.5

The number of ways of choosing r items from n when the order does matter is

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

WORKED EXAMPLE 1.3

A maths teacher needs to select a team of four students from a class of 19 to represent the school in a maths competition.

Find the number of different ways she can choose the team.

Picking Albert, Billy, Camille and Dani will result in the same team as Billy, Dani, Albert and Camille, so the order doesn't matter. Therefore, use ${}^nC_r$

Number of ways $= {}^{19}C_4$
$= 3876$

WORKED EXAMPLE 1.4

The board of directors of a company consists of 12 members. They need to appoint a Chair, a Chief Finance Officer and a Secretary.

Find the number of different ways this could be done.

Let us say that the first person chosen will be the Chair, the second will be the Chief Finance Officer and the third will be the Secretary. Therefore, the order does matter, so use ${}^nP_r$

Number of ways $= {}^{12}P_3$
$= 1320$

You are the Researcher

Although counting is one of the first topics you meet when you start studying mathematics, it is also one of the hardest topics advanced mathematicians deal with. You might be interested in researching Hilbert's Hotel and how it helps to explain counting to infinity. Ramsey theory covers an area of mathematics that deals with counting on networks and the numbers used in this are some of the largest ever found to have a useful application.

The basic tools of ${}^nC_r$ and ${}^nP_r$ can be combined with the AND and OR rules to break down harder problems.

WORKED EXAMPLE 1.5

A class has ten girls and eight boys. A committee of six must have an equal number of boys and girls. In how many ways can this be done?

Break the problem down into smaller parts which are of the form required to use ${}^nC_r$ and ${}^nP_r$ ······ We need to choose 3 girls from 10 AND 3 boys from 8

Since order does not matter, we should use ${}^nC_r$ ······ This can be done in ${}^{10}C_3 \times {}^8C_3 = 120 \times 56$
$= 6720$ ways

Exercise 1A

In questions 1 to 3, use the method demonstrated in Worked Example 1.1 to find the number of ways of choosing

1 a one item from a list of eight and one item from a different list of five

b one item from a list of seven and one item from a different list of four

2 a one item from a list of eight or one item from a different list of five

b one item from a list of seven or one item from a different list of four

3 a one item from each of a list of eight, a list of five and a list of seven

b one item from either a list of eight, a list of five or a list of seven.

In questions 4 to 6, use the method demonstrated in Worked Example 1.2 to find the number of permutations of

4 a nine items

b six items

5 a four items followed by five different items

b ten items followed by three different items

6 a either two items followed by five or three items followed by four

b either three items followed by seven or four items followed by six.

In questions 7 to 9, use the method demonstrated in Worked Example 1.3 to find the number of ways of choosing

7 a five items from nine

b three items from eight

8 a four items from seven, followed by two items from six other items

b six items from eight, followed by four items from ten other items

9 a either three items from twelve or four items from eleven

b either five items from eight or six items from ten.

In questions 10 to 12, use the method demonstrated in Worked Example 1.4 to find the number of ways of permuting

10 a five items from nine

b three items from eight

11 a four items from seven followed by two items from six other items

b six items from eight followed by four items from ten other items

12 a either three items from twelve or four items from eleven

b either five items from eight or six items from ten.

13 Amelie wants a pet cat and a pet dog. There are five breeds of cat and eleven breeds of dog at her local pet shop.

Find the number of possible ways she can choose a cat and dog.

14 There are seven men and four women who would like to play as a team of two in a bridge tournament.

Find the number of ways a pair can be chosen if one player has to be male and the other female.

15 A headteacher wants to choose a school council consisting of one student from each of Years 9, 10, 11, 12 and 13.

There are 95 students in Year 9, 92 in Year 10, 86 in Year 11, 115 in Year 12 and 121 in Year 13.

a Find the number of ways of choosing the school council.

The headteacher now decides that he only wants one student from either Year 9 or Year 10 and one each from Year 11, 12 and 13.

b Find the number of ways the school council can be chosen now.

16 A menu at a restaurant offers five starters, eight main courses and six desserts.

Find the number of different choices of meal you can make if you would like

a a starter, a main course and a desert

b a main course and either a starter or a desert

c any two different courses.

17 A mixed soccer team of six boys and five girls are having a team photo taken. The girls are arranged in the front row and the boys in the back row.

Find the number of possible arrangements.

18 a Find the number of seven-digit numbers that can be formed using the digits 1 to 7 exactly once each.

b How many of these are divisible by five?

19 David is planting a flower bed with six different types of rose and two different types of tulip.

They are all planted in a line.

a Find the number of possible arrangements.

b How many of these arrangements have the tulips at either end?

20 An exam paper consists of ten questions. Students can select any six questions to answer.

Find the number of different selections that can be made.

21 Ulrike is revising for seven subjects, but can only complete three in any one evening.

a Find the number of ways she could choose which subjects to revise in an evening.

b If she decides that one of her three subjects must be mathematics, find the number of ways she could choose the subjects to revise in an evening.

22 Find the number of four-digit numbers that can be made with the digits 1 to 9 if no digit can be repeated.

23 Find the number of ways the gold, silver and bronze medals could be awarded in a race consisting of eight athletes.

24 A teacher wants to award the Science Prize and the Humanities Prize to two different students in her class of 17.

Find the number of possible selections she could make.

25 Dr Walker has nine different shirts (three each of white, blue and green), six different pairs of trousers (two each of black, grey and blue) and four different waistcoats (one each of black, blue, beige and red).

He always wears a pair of trousers, a shirt and a waistcoat.

a Find the number of different outfits he can wear.

Dr Walker never wears blue and green together.

b Find the number of different outfits he can wear with this restriction.

26 **a** Find how many five-digit numbers can be formed using the digits 1 to 5 exactly once each.

b How many of these numbers are less than 40 000?

27 In a lottery, players select five numbers from the numbers 1 to 40 and then two further bonus numbers from the numbers 1 to 10.

Find the number of possible selections.

28 A hockey team consists of one goalkeeper, three defenders, five midfielders and two forwards.

The coach has three goalkeepers, six defenders, eight midfielders and four forwards in the squad.

Find the number of ways she can pick the team.

29 There are 15 places on a school trip to Paris, 12 on a trip to Rome and 10 on a trip to Athens.

There are 48 students in Year 12 who have applied to go on a trip, and they are all happy to go on any of the options available.

Find the number of ways the places can be allocated.

30 A committee of three boys and three girls needs to be chosen from 18 boys and 15 girls. Ahmed is the chairman, so he has to be on the committee.

a Find the number of ways the committee can be selected.

b If Baha or Connie but not both must be chosen from the girls, find the number of ways the committee can be selected now.

31 Eight athletes compete in a race. Find the number of possible ways the gold, silver and bronze medals can be awarded if Usain wins either gold or silver.

32 There are 16 girls and 14 boys in a class. Three girls and two boys are needed to play the lead roles in a play.

Find the number of ways these five roles can be cast.

33 Student ID codes must consist of either three different letters chosen from the letters A to Z followed by four different digits chosen from the numbers 1 to 9, or of four different letters followed by three different digits.

Find the number of possible ID codes available.

34 A class consists of eight girls and seven boys. A committee of five is chosen.

a How many possible committees can be chosen if there are no constraints?

b How many different committees are possible if

i it must contain Jamila

ii it must contain at least one girl.

c If the committee is chosen at random, what is the probability that it contains at least one girl?

35 Solve the equation ${}^{n}C_2 = 210$.

36 Solve the equation ${}^{n}P_2 = 132$.

37 Prove that ${}^{n}P_n = {}^{n}P_{n-1}$.

38 Find the number of ways that

a five presents can be put into two boxes

b three presents can be put into four boxes.

39 A class of 18 students are lining up in three rows of six for a class photo.

Find the number of different arrangements.

40 Ten points on a plane are drawn so that no three lie in a straight line. If the points are connected by straight lines and if vertices can only occur at the original points, find the number of different

a triangles that can be formed

b quadrilaterals that can be formed.

41 At a party, everyone shakes hands with everyone else. In total there are 465 handshakes.

Find the number of people at the party.

1B Problem solving

In addition to the techniques encountered in Section 1A, there are a few further techniques that are often useful in solving more-complex problems:

- **Count what you do not want and subtract that from the total to get what you do want.**

  For example, to find how many numbers between 1 and 100 are not divisible by five, count how many are divisible by five (20) and subtract from the total (100) to give 80 numbers that are not divisible by five.

- **Treat any items that must be together as a single item in the list.**

  For example, to find the number of permutations of ABCDEFG where A and B must be together, treat 'AB' as being on a single tile:

  C D AB E F G H

  But remember that the items that are together also need to be permuted.

  C D BA E F G H

WORKED EXAMPLE 1.6

Anushka, Beth, Caroline, Dina, Elizabeth, Freya and Georgie line up for a netball team photo.

Find the number of possible arrangements in which:

a Anushka and Beth are next to each other

b Anushka and Beth are not next to each other.

Treat A and B as one item in the list (X). There are 6! permutations of XCDEFG AND A and B can be arranged in 2! ways

a Number of arrangements with A and B together $= 6! \times 2!$
$= 1440$

This is given by the total number of permutations of ABCDEFG minus the number where A and B are together

b Number of arrangements with A and B apart
$= 7! - 1440$
$= 3600$

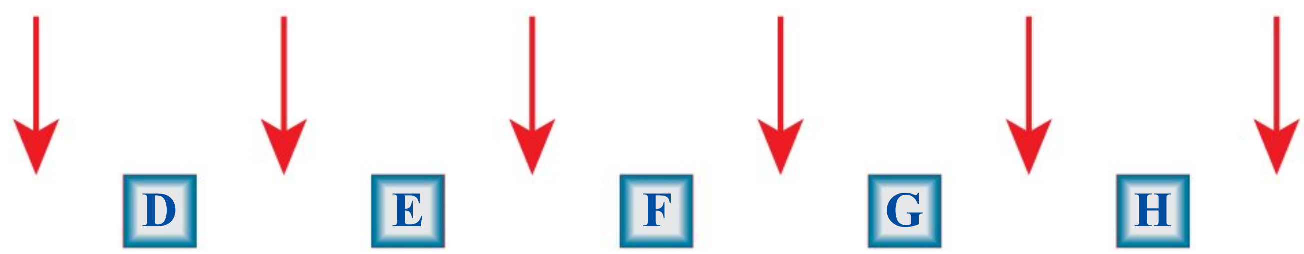
Tip

Notice that the number of permutations in which three (or more) items are kept apart can't be found by subtracting the number of permutations with the items together from the total. This would leave permutations with two of the items together still being counted.

The following technique is useful if more than two items need to be kept apart:

- **To separate three or more items, consider the gaps they can fit into.**

  For example, to separate A, B and C from the list of ABCDEFGH, consider the six gaps created by the other letters into which A, B and C could fit:



WORKED EXAMPLE 1.7

Find the number of possible arrangements of the single-row netball photograph line-up in which Anushka, Beth and Caroline are not next to each other.

We have five possible gaps for A, B and C to fill so they are not together:
_ D _ E _ F _ G _

Permute DEFG AND Permute ABC in 3 out of the 5 gaps

Number of arrangements with A, B, C apart
$= 4! \times {}^5P_3$
$= 1440$

You are the Researcher

Can you find out the number of ways to arrange n objects if some of them are identical?

TOK Links

Does having a symbol, such as ${}^5P_3$, to describe a standard calculation add to your mathematical knowledge? Does attaching a label to something help us to use an idea more effectively?

Exercise 1B

1. Find the number of ways ten different cheeses can be arranged on a shelf in a supermarket, if the brie must be next to the camembert.
2. Find the number of ways six different Standard Level IB textbooks and three different Higher Level IB textbooks can be arranged on a bookshelf if the three HL books have to be together.
3. Find the number of permutations of the letters A, B, C, D, E, F that do not start with A.
4. Find the number of seven-digit codes that do not end with ‘67’ using each of the digits 1 to 7 once.
5. Alessia has seven different soft toys and five different toy cars she likes to play with.

   Find the number of ways she can choose four of these to play with if at least one of them must be a toy car.
6. Find the number of permutations of the word ‘COMPUTE’ that do not have C, O and M in the first three letters.
7. A box contains 25 different chocolates: ten dark, eight milk and seven white.

   Find the number of ways of choosing three chocolates such that

   a. they are all a different type
   b. they are not all dark.
8. Find the number of permutations of the letters D, I, P, L, O, M, A that do not begin with D or end with A.
9. Three letters are selected from the word ‘COUNTED’ and arranged in order.

   Find the number of these arrangements that contain at least one vowel.
10. Eight students are to be chosen from 16 girls and 13 boys.

    Find the number of ways this can be done if at least two girls must be chosen.
11. Seven different milk chocolate bars, five different white chocolate bars and four different dark chocolate bars are lined up on a shelf in a sweet shop.

    Find the number of possible arrangements if all chocolate bars of the same type must be together.
12. Six students from class 12A, four students from 12B and three students from 12C all arrive to line up in the lunch queue.

    Find the number of possible ways they can arrange themselves given that the students from 12C must be separated.

13 The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged at random.

Find the probability that no odd number is next to another odd number.

14 Mr and Mrs Semba and their four children line up for a family photo.

Find the number of ways they can do this if

- **a** Mr and Mrs Semba are at opposite ends of the line
- **b** Mr and Mrs Semba are together.

The family lines up at random.

- **c** Find the probability that Mr and Mrs Semba are not next to each other.

15 Five cards are dealt from a randomly shuffled standard deck of 52 playing cards. Find the probability of getting

- **a** all spades
- **b** all red cards
- **c** at least two black cards.

16 Six members of a family go to the cinema and all want to be seated on the same row next to each other. This row contains 20 seats.

Find the number of ways they can arrange themselves.

17 A six-a-side football team is to be selected from short-listed players from the top two sides in the league. Eight players are short-listed from Team A and seven from Team B.

Find the number of ways the side can be chosen if there must be at least two players from Team A and at least one from Team B.

18 In a word game, there are 26 tiles each printed with a different letter.

Find the number of ways of choosing seven tiles if at least two of them are vowels.

Checklist

- You should be able to find the number of ways of choosing an option from list A and an option from list B. The AND rule:
  - $n(A \text{ AND } B) = n(A) \times n(B)$
- You should be able to find the number of ways of choosing an option from list A or an option from list B. The OR rule:
  - If A and B are mutually exclusive: $n(A \text{ OR } B) = n(A) + n(B)$
- You should be able to find the number of permutations of n items.
  - The number of permutations of n items is $n!$
- You should be able to find the number of ways of choosing r items from a list of n items.
  - The number of combinations of r objects out of n is written as:

    $^{n}C_{r} = \frac{n!}{r!(n-r)!}$ when the order does not matter.
  - The number of permutations of r objects out of n is written as:

    $^{n}P_{r} = \frac{n!}{(n-r)!}$ when the order does matter.

Mixed Practice

1 Eight athletes take part in the 100 m final of the Olympic games.

Find the number of ways the three medals could be won.

2 A car registration number consists of four different letters followed by three different digits chosen from 1 to 9.

Find the number of possible registration numbers.

3 A team of six students is to be chosen from a class of 15 girls and 11 boys.

Find the number of possible selections if the team must contain the same number of boys and girls.

4 Rylan has written Christmas cards for five friends and sealed each of them in a separate envelope. However, he has forgotten which card is which.

If he now wrote the names on the envelopes at random, find the probability that all the cards and envelopes would match.

5 Three teachers and eight students want to join the lunch queue. Find the number of different ways this can be done if the three teachers must go first.

6 There are six places available on a school mathematics trip and 19 students who have applied to go on the trip.

a Find the number of possible selections that could be made.
b Find the probability that Jack and Jill are both chosen.

7 Find the number of three-digit numbers that contain no zeros.

8 A football manager has a squad consisting of three goalkeepers, seven defenders, eight midfielders and four forwards.

He wants to select a team of one goalkeeper, four defenders, four midfielders and two forwards. Find the number of possible selections.

9 Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

Mathematics HL May 2014 Paper 2 TZ1 Q3

10 Find the number of permutations of the word 'CHANGED' that start and end with a consonant.

11 A committee of five students is to be selected from a class of 28. The two youngest cannot both be on the committee.

Find the number of ways the committee can be selected.

12 A group of 19 students contains nine boys and ten girls. A team is said to be mixed if it contains at least one boy and at least one girl.

Find the number of ways a mixed team of seven can be chosen from the 19 students.

13 A class consists of six girls and nine boys. A committee of six is chosen.

a How many possible committees can be chosen if there are no constraints?
b How many different committees are possible if
  i it must not contain John
  ii it must contain at least two girls.
c If the committee is chosen at random, what is the probability that it contains at least two girls?

14 12 people need to travel to a hockey match in two vehicles: a car, which can carry four people and an SUV which can carry eight people.

Given that only two of the 12 can drive, find the number of ways they can be allocated to the two vehicles.

15 Henrik draws seven letter tiles from a bag: D, M, S, T, A, E, O.

a Find the number of arrangements of the letters.
b Find the number of arrangements with the three vowels together.
c Find the number of arrangements with the three vowels all separated.

16 Amit, Brian, Connor, Dan and Ed stand in a line.

Find the number of possible permutations in which

a Amit is at one end of the line
b Amit is not at either end
c Amit is at the left end of the line or Ed is at the right end, or both.

17 Players in a lottery choose six different numbers from 1 to 50 inclusive.

a Find the probability of matching all six numbers.

There is a prize for anyone matching three numbers or more.

b Find the probability of winning a cash prize.

18 There are n different letters in a bag. If 380 possible two-letter ‘words’ can be formed from these letters, find the value of n.

19 Three boys and three girls are to sit on a bench for a photograph.

a Find the number of ways this can be done if the three girls must sit together.
b Find the number of ways this can be done if the three girls must all sit apart.

Mathematics HL May 2013 Paper 2 TZ1 Q8

20 A set of positive integers {1, 2, 3, 4, 5, 6, 7, 8, 9} is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

a Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7.
b Find the number of selections Grace could make if at least two of the four integers drawn are even.

Mathematics HL November 2014 Paper 1 Q10

21 An SUV has eight seats: two at the front, a row of three in the middle and a row of three at the back. Seven people are travelling in the car, three of whom can drive.

Find the number of ways they can be seated.

22 In a doctor’s waiting room, there are 10 seats in a row. Six people are waiting to be seen.

a Find the number of ways they can be seated.
b One of them has a bad cold and mustn’t sit next to anyone else. Find the number of ways the six people can be seated now.

23 Twelve friends arrive at a quiz, but are told that the maximum number in a team is six. Find the number of ways they can split up into

a two teams of six
b three teams of four.

2 Algebra

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to extend the binomial theorem to fractional and negative indices
- how to split rational functions with a product of two linear factors in the denominator into a sum of two algebraic fractions (partial fractions)
- how to solve systems of up to three linear equations in three unknowns.

CONCEPTS

The following concepts will be addressed in this chapter:

- The binomial theorem is a **generalization** which provides an efficient method for expanding binomial expressions.
- **Representing** partial fractions in different forms allows us to easily carry out seemingly difficult calculations.
- The solution for systems of equations can be carried out by a variety of **equivalent** algebraic and graphical methods.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Expand $(2 - 3x)^4$.
2. Factorize $5x^2 + 13x - 6$.
3. Find $\frac{1}{x+3} + \frac{2}{x-1}$.
4. Solve the simultaneous equations $\begin{cases} 2x + 3y = 4 \\ 3x - 5y = 25 \end{cases}$.

■ **Figure 2.1** How do we describe locations?

Previously, you have seen how to use the binomial theorem as a quick way of expanding brackets. Now we will look at what happens if the power is a general rational number (positive or negative), rather than a positive integer as before. It turns out that this produces infinite polynomials, which can be used to approximate the original function.

Systems of linear equations in at least two unknown quantities are used to describe systems of several interlinked variables. They also arise when fitting models to data and designing complex systems such as neural networks. Although in practice they are often solved using technology, it is important to be able to solve them analytically and to appreciate the conditions under which a solution exists and whether or not it is unique.

Starter Activity

Look at Figure 2.1. In small groups discuss how many numbers (or other pieces of information) are required to precisely describe a position in each of these situations.

Now look at this problem:

a Find at least four pairs of numbers x and y which satisfy the equation $2x + y = 10$.
Find at least four pairs of numbers x and y which satisfy the equation $x + 3y = 20$.
How many pairs can you find which satisfy both equations?

b Find at least four sets of numbers x, y, z which satisfy the equation $x + y + z = 6$.
Can you find at least two different sets of numbers which satisfy both $x + y + z = 6$ and $2x + y + z = 8$? How many sets of numbers can you find which satisfy these three equations: $x + y + z = 6$, $2x + y + z = 8$ and $3x + y + z = 10$?

LEARNER PROFILE – Balanced

What is the optimum amount of time to spend on a problem before taking a break? What are your strategies for what to do when you are stuck? Often, just trying to describe what your issue is to someone else activates new ideas. Do not expect to always be able to do a problem straight away – sometimes if you come back to it the next day the problem suddenly cracks!

2A Extension of the binomial theorem to fractional and negative indices

You know from Chapter 13 of the Mathematics: analysis and approaches SL book that if $|x| < 1$ then

$$1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$$

But this means that the function $f(x) = \frac{1}{1+x} = (1+x)^{-1}$ can be expanded as a polynomial (of infinite length) in x.

If this is possible for $f(x) = (1+x)^{-1}$ then it should be possible for other functions of the form $f(x) = (1+x)^n$ where n is a negative integer, and in fact it is possible when n is any rational number.

This is the sum to infinity of a geometric sequence with first term 1 and common ratio $-x$ from Section 13A of the Mathematics: analysis and approaches SL book.

We can see how this can be done by looking at the binomial theorem for positive integer powers:

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + x^n$$

If the power is not a positive integer, then using ${}^nC_r$ does not make sense, but we can rewrite the formula for ${}^nC_r$ so it can be used for any rational number:

$$\begin{aligned} {}^nC_r &= \frac{n!}{r!(n-r)!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2.1}{r!(n-r)(n-r-1)\ldots 2.1} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \end{aligned}$$

So,

$$\begin{aligned} {}^nC_1 &= \frac{n}{1!} \\ {}^nC_2 &= \frac{n(n-1)}{2!} \end{aligned}$$

and so on.

Tip

Remember: a rational number is any number (positive or negative) of the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0$.

Tip

The condition that $|x| < 1$ is important. Although we can always create the polynomial on the right-hand side, the series will only converge in this case, so we say the expansion is only valid if $|x| < 1$.

KEY POINT 2.1

If $|x| < 1$, then for any $n \in \mathbb{Q}$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Notice that if n is not a positive whole number, this is an infinitely long polynomial.

TOOLKIT: Problem Solving

If n is not a positive integer, why do we get an infinitely long polynomial?

WORKED EXAMPLE 2.1

Find the binomial expansion of $(1 + 3y)^{-2}$ in ascending powers of x up to the term in y^3.

Use $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$ with $n = 2$ and $x = 3y$

$$(1+3y)^{-2} = 1+(-2)(3y)+\frac{(-2)(-2-1)}{2!}(3y)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(3y)^3+\dots$$

$$= 1+(-2)(3y)+(3)(9y^2)+(-4)(27y^3)+\dots$$

$$= 1-6y+27x^2-108x^3+\dots$$

To apply the formula given in Key Point 2.1, you might have to factorize to make the number at the front of the bracket 1. This will change the range over which the series is valid.

WORKED EXAMPLE 2.2

a Find the first three terms in ascending powers of x in the binomial expansion of $(4 + x)^{\frac{1}{2}}$.

b Find the values of x for which this expansion is valid.

Take out a factor of 4 inside the bracket

a $$(4+x)^{\frac{1}{2}} = \left(4\left(1+\frac{x}{4}\right)\right)^{\frac{1}{2}}$$

Make sure the factor comes out as $4^{\frac{1}{2}}$

$$= 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Use $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ with $n = \frac{1}{2}$ and $x = \frac{x}{4}$

$$= 2\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{4}\right)^2+\dots\right]$$

$$= 2\left(1+\left(\frac{1}{2}\right)\left(\frac{x}{4}\right)+\left(-\frac{1}{8}\right)\left(\frac{x^2}{16}\right)+\dots\right)$$

$$= 2\left(1+\frac{1}{8}x-\frac{1}{128}x^2+\dots\right)$$

Finally, multiply out the bracket

$$= 2+\frac{1}{4}x-\frac{1}{64}x^2+\dots$$

Use the condition $|x| < 1$, with $x = \frac{x}{4}$

b $$\left|\frac{x}{4}\right| < 1$$

Multiplying through by 4
We could also write $-4 < x < 4$

$$|x| < 4$$

Be the Examiner 2.1

Find the first three terms in ascending powers of x in the binomial expansion of $(2-3x)^{-1}$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	$(2-3x)^{-1} = 2\left(1-\frac{3}{2}x\right)^{-1}$ $= 2\left(1+(-1)\left(-\frac{3}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{3}{2}x\right)^2+\ldots\right)$ $= 2\left(1+\frac{3}{2}x+\frac{9}{4}x^2+\ldots\right)$ $= 2+3x+\frac{9}{2}x^2+\ldots$
Solution 2	$(2-3x)^{-1} = \frac{1}{2}\left(1-\frac{3}{2}x\right)^{-1}$ $= \frac{1}{2}\left(1+(-1)\left(-\frac{3}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{3}{2}x\right)^2+\ldots\right)$ $= \frac{1}{2}\left(1+\frac{3}{2}x+\frac{9}{4}x^2+\ldots\right)$ $= \frac{1}{2}+\frac{3}{4}x+\frac{9}{8}x^2+\ldots$
Solution 3	$(2-3x)^{-1} = \frac{1}{2}\left(1-\frac{3}{2}x\right)^{-1}$ $= \frac{1}{2}\left(1+(-1)\left(-\frac{3}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{3}{2}x^2\right)+\ldots\right)$ $= \frac{1}{2}+\frac{3}{4}x-\frac{3}{4}x^2+\ldots$

Sometimes you need to use the binomial expansion as part of a larger expression.

WORKED EXAMPLE 2.3

Find the coefficient of x^2 in the binomial expansion of $\sqrt{\frac{1-x}{1+x}}$ if $|x|<1$.

Create expressions of the form $(1+x)^n$

$$\sqrt{\frac{1-x}{1+x}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

Expand both brackets

$$= \left(1+\left(\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-x)^2+\ldots\right)\left(1+\left(-\frac{1}{2}\right)(x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^2)+\ldots\right)$$

Simplify each series before continuing

$$= \left(1-\frac{1}{2}x-\frac{1}{8}x^2+\ldots\right)\left(1-\frac{1}{2}x+\frac{3}{8}x^2+\ldots\right)$$

x^2 term:

There are three ways to get an x^2 term

$$(1)\left(\frac{3}{8}x^2\right)+\left(-\frac{1}{2}x\right)\left(-\frac{1}{2}x\right)+\left(-\frac{1}{8}x^2\right)(1)$$

$$= \frac{3}{8}x^2+\frac{1}{4}x^2-\frac{1}{8}x^2$$

$$= \frac{1}{2}x^2$$

You will see in Section 2B that another technique can be used to write an expression in a form where the binomial expansion can then be applied.

Exercise 2A

For questions 1 to 4, use the method demonstrated in Worked Example 2.1 to find the first three terms of the binomial expansion of each of the following expressions, stating the values of x for which the expansion is valid.

1 **a** $(1+x)^{-2}$ **b** $(1+x)^{-3}$

2 **a** $(1+x)^{\frac{1}{3}}$ **b** $(1+x)^{\frac{1}{4}}$

3 **a** $\left(1-\frac{x}{4}\right)^{-1}$ **b** $\left(1+\frac{x}{2}\right)^{-4}$

4 **a** $(1+2x)^{-\frac{1}{2}}$ **b** $(1-3x)^{-\frac{2}{3}}$

For questions 5 to 8, use the method demonstrated in Worked Example 2.2 to find the first three terms of the binomial expansion of each of the following expressions, stating the values of x for which the expansion is valid.

5 **a** $(3+x)^{-1}$ **b** $(5+x)^{-2}$

6 **a** $(8+x)^{\frac{1}{3}}$ **b** $(9+x)^{\frac{1}{2}}$

7 **a** $(2-3x)^{-3}$ **b** $(3+4x)^{-1}$

8 **a** $\left(16+\frac{x}{2}\right)^{\frac{7}{4}}$ **b** $\left(4-\frac{x}{3}\right)^{\frac{3}{2}}$

9 Find the expansion, in ascending powers of x up to and including the term in x^3, of $\sqrt{1-2x}$.

10 Find the expansion, in ascending powers of x up to and including the term in x^3, of $\dfrac{1}{\left(1-\frac{x}{4}\right)^3}$.

11 Find the expansion, in ascending powers of x up to and including the term in x^2, of $\dfrac{4}{\sqrt[3]{8+x}}$.

12 Find the expansion, in ascending powers of x up to and including the term in x^2, of $\dfrac{1}{2-5x}$.

13 **a** Find the first three terms in ascending powers of x in the expansion of $\sqrt{9+x}$.

b Find the values of x for which the expansion is valid.

c By substituting an appropriate value of x into the expansion in part **a**, find an approximation to $\sqrt{10}$ to five decimal places.

14 **a** Find the first three terms in ascending powers of x in the expansion of $\sqrt[3]{8-3x}$.

b Find the values of x for which the expansion is valid.

c By substituting an appropriate value of x into the expansion in part **a**, find an approximation to $\sqrt[3]{5}$ to five decimal places.

15 **a** Find the first three terms in ascending powers of x in the expansion of $x(1+3x)^{-2}$.

b Find the values of x for which the expansion is valid.

16 Find the first three terms in the expansion of $\dfrac{1+x}{1-x}$.

17 Find the coefficient of x^2 in the expansion of $(1+6x)^{\frac{1}{4}}(2+x)$.

18 Find the coefficient of x^2 in the expansion of $\left(\dfrac{1+x}{1+2x}\right)^2$.

19 **a** Find the first four terms in ascending powers of x in the expansion of $\sqrt{1-4x}$.

b Find the values of x for which the expansion is valid.

c By substituting $x = 0.02$ into the expansion in part **a**, find an approximation to $\sqrt{23}$ to five decimal places.

20 The coefficient of x^3 in the expansion of $(1+ax)^{-3}$ is -640. Find the value of a.

21 **a** Find the first three non-zero terms of the binomial expansion of $\sqrt[3]{\dfrac{1+2x}{1-x}}$.

b By setting $x = 0.04$, find an approximation for $\sqrt[3]{9}$ to five decimal places.

22 $(1+ax)^n = 1 - 4x + 9x^2 + bx^3 + \ldots$

Find the value of b.

23 $(1-x)(1+ax)^n = 1 + x^2 + bx^3 + \ldots$

Find the value of b.

24 $(1+ax)^n = 1 + 21x + bx^2 + bx^3 + \ldots$

Given that $b \neq 0$, find the value of b.

2B Partial fractions

You know how to write a sum of two algebraic fractions as a single fraction, for example,

$$\frac{3}{x-1} + \frac{1}{x+2} = \frac{3(x+2)+(x-1)}{(x-1)(x+2)} = \frac{4x+5}{(x-1)(x+2)}$$

However, there are situations where you need to reverse this process. This is referred to as splitting a function into **partial fractions**.

To do this you need to know the form the partial fractions will take.

As well as using partial fractions with binomial expansion in this section, you will also see in Chapter 10 that this is an important technique in integration.

KEY POINT 2.2

Partial fractions for a rational function with two linear factors in the denominator:

$$\frac{px+q}{(ax+b)(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d}$$

Once we know the form we are aiming for, we can multiply both sides by the denominator of the original fraction. We can then either compare coefficients or substitute in convenient values to find the values of A and B.

WORKED EXAMPLE 2.4

Write $\dfrac{4x+5}{(x-1)(x+2)}$ in partial fractions.

Write the expression in the required form

$$\frac{4x+5}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$$

Multiply both sides by $(x-1)(x+2)$ to eliminate fractions

$$4x+5 \equiv A(x+2) + B(x-1)$$

When $x = 1$:

As this is an identity, we can choose any convenient value to work with

$$4+5 = A(1+2) + B(1-1)$$

$$9 = 3A$$

When $x = 1$ the coefficient of B is zero, so we can find A

$$A = 3$$

When $x = -2$ the coefficient of A is zero, so we can find B

When $x = -2$:

$$-8+5 = A(-2+2)+B(-2-1)$$
$$-3 = -3B$$
$$B = 1$$

So,

$$\frac{4x+5}{(x-1)(x+2)} \equiv \frac{3}{x-1}+\frac{1}{x+2}$$

Splitting a rational function into partial fractions allows you to then apply the binomial expansion.

WORKED EXAMPLE 2.5

a Express $\frac{1}{(x+1)(2x+1)}$ in the form $\frac{A}{x+1}+\frac{B}{2x+1}$.

b Hence, find the first three terms in the expansion of $\frac{1}{(x+1)(2x+1)}$.

c State the range of values for which this expansion converges.

Apply the standard method for finding partial fractions

a $$\frac{1}{(x+1)(2x+1)} \equiv \frac{A}{x+1}+\frac{B}{2x+1}$$
$$1 \equiv A(2x+1)+B(x+1)$$

When $x = -1$:

$$1 = A(-2+1)+B(-1+1)$$
$$1 = -A$$
$$A = -1$$

When $x = -\frac{1}{2}$:

$$1 = A(-1+1)+B\left(-\frac{1}{2}+1\right)$$
$$1 = \frac{1}{2}B$$
$$B = 2$$

So,

$$\frac{1}{(x+1)(2x+1)} \equiv \frac{-1}{x+1}+\frac{2}{2x+1}$$

Find the binomial expansion of each term separately

b $$\frac{-1}{x+1} = -(1+x)^{-1}$$
$$= -\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\ldots\right)$$
$$= -1+x-x^2+\ldots$$
$$\frac{2}{2x+1} = 2(1+2x)^{-1}$$
$$= 2\left(1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\ldots\right)$$
$$= 2-4x+8x^2+\ldots$$

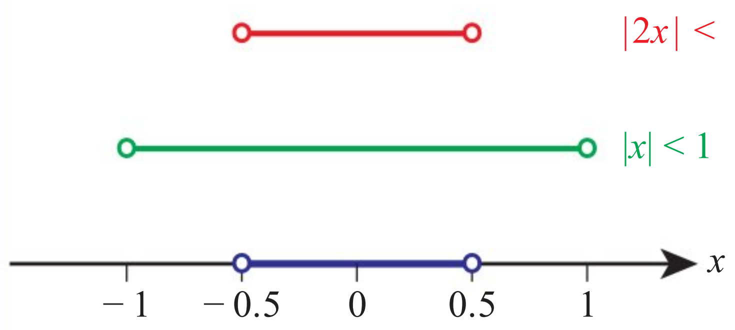
So,

Combine the two expansions $$\frac{1}{(x+1)(2x+1)} = (-1+x-x^2+\ldots)+(2-4x+8x^2+\ldots)$$
$$= 1-3x+7x^2+\ldots$$

Consider the range of convergence of each term separately **c** The first expansion converges for $|x|<1$

The second expansion converges for $|2x|<1$ so for $|x|<\frac{1}{2}$

Think about which range provides the limiting factor Therefore, both converge for $|x|<\frac{1}{2}$



Exercise 2B

For questions 1 to 4, use the method demonstrated in Worked Example 2.4 to split the following into partial fractions.

1 **a** $\frac{3x+4}{(x+1)(x+2)}$ **b** $\frac{x-7}{(x-2)(x+3)}$

2 **a** $\frac{x-4}{(x-1)(x+5)}$ **b** $\frac{2x-6}{(x-2)(x-5)}$

3 **a** $\frac{7x-6}{x(x-3)}$ **b** $\frac{x+12}{x(x+4)}$

4 **a** $\frac{4x+2}{(2x-1)(2x+3)}$ **b** $\frac{x+12}{(3x-2)(2x+5)}$

5 Express $\frac{5x+1}{(x-1)(x+2)}$ in the form $\frac{A}{x-1}+\frac{B}{x+2}$, where A and B are constants to be found.

6 Express $\frac{3-x}{(x+3)(x+5)}$ in the form $\frac{A}{x+3}+\frac{B}{x+5}$, where A and B are constants to be found.

7 Split $\frac{x+5}{(x+8)(x-4)}$ into partial fractions.

8 Express $\frac{x-10}{2x(x+2)}$ in partial fractions.

9 Split $\frac{2x-8}{(4x+5)(x+3)}$ into partial fractions.

10 Express $\frac{7x-3}{x^2+3x-18}$ in partial fractions.

11 Split $\frac{1}{9x^2-1}$ into partial fractions.

12 Express $\frac{8-x}{3x^2+12x}$ in partial fractions.

13 Express $\frac{3x-2a}{x(x-a)}$, where a is a constant, in partial fractions.

14 Split $\frac{4}{x+4\sqrt{x}+3}$ into partial fractions.

15 Express $\frac{1}{x^4+7x^2+10}$ in partial fractions.

16 **a** Express $\frac{4-5x}{(1+x)(2-x)}$ in the form $\frac{A}{1+x}+\frac{B}{2-x}$, where A and B are constants to be found.

b Hence find the first three terms in ascending powers of x in the expansion of $\frac{4-5x}{(1+x)(2-x)}$.

c Find the values of x for which the expansion is valid.

17 **a** Express $\frac{7x-2}{(1-x)(2+3x)}$ in partial fractions.

b Hence find the first three terms in ascending powers of x in the expansion of $\frac{7x-2}{(1-x)(2+3x)}$.

c Find the values of x for which the expansion is valid.

18 **a** Find the first three terms in ascending powers of x of the binomial expansion of $\frac{1}{2x^2+3x+1}$.

b Find the values of x for which the expansion is valid.

19 **a** Find the first three terms in ascending powers of x of the binomial expansion of $\frac{5x+3}{1+2x-3x^2}$.

b Find the values of x for which the expansion is valid.

20 Express $\frac{a}{x^2-3ax+2a^2}$, where a is a constant, in partial fractions.

2C Solutions of systems of linear equations

You can solve systems of two or three linear equations using your GDC.

WORKED EXAMPLE 2.6

Solve the simultaneous equations

$$\begin{cases} 2x+4z = y-9 \\ x+2y-3z-15=0 \\ 3x = 5-z. \end{cases}$$

Write the equations in standard form, with all the variables on the left and the constants on the right

$$\begin{cases} 2x-y+4z=-9 \\ x+2y-3z=15 \\ 3x+z=5 \end{cases}$$

Use the simultaneous equation solver on your GDC, making sure that you enter each term in the correct place

$x=2.8,\ y=1,\ z=-3.4$

You already know how to solve systems of two equations in two unknowns algebraically; you also need to be able to algebraically solve systems of equations in three unknowns.

To do this, start with any two equations and eliminate one of the three variables from both of them. Then solve these two equations in two unknowns as normal.

WORKED EXAMPLE 2.7

Solve the system of equations

$$\begin{cases} 2x - y + 3z = 13 \\ x + 4y - z = -5 \\ 3x + 2y + z = 9. \end{cases}$$

Labelling the equation (1), (2), (3) allows them to be referred to

$$\begin{cases} 2x - y + 3z = 13 & (1) \\ x + 4y - z = -5 & (2) \\ 3x + 2y + z = 9 & (3) \end{cases}$$

You need to eliminate a variable from a pair of equations. Any pair will do, but adding (2) and (3) is an easy way of eliminating z so is perhaps the best place to start. Here we eliminate z from (2) first

$(4) = (2) + (3)$

$$\begin{cases} 2x - y + 3z = 13 & (1) \\ 4x + 6y = 4 & (4) \\ 3x + 2y + z = 9 & (3) \end{cases}$$

Now eliminate z from (3) as well

$(5) = 3 \times (3) - (1)$

$$\begin{cases} 2x - y + 3z = 13 & (1) \\ 4x + 6y = 4 & (4) \\ 7x + 7y = 14 & (5) \end{cases}$$

(4) and (5) can both be simplified. This could have been done in one go while z was being eliminated from these two equations, but the extra step is shown here for clarity

$(6) = (4) \div 2$ and $(7) = (5) \div 7$

$$\begin{cases} 2x - y + 3z = 13 & (1) \\ 2x + 3y = 2 & (6) \\ x + y = 2 & (7) \end{cases}$$

Now solve (6) and (7) as simultaneous equations in two unknowns

$(8) = 3 \times (7) - (6)$

$$\begin{cases} 2x - y + 3z = 13 & (1) \\ 2x + 3y = 2 & (6) \\ x = 4 & (8) \end{cases}$$

Equation (8) gives the value of x

From (8):

$x = 4$

Substitute back into the other two equations to find y and then z

Substituting into (6):

$8 + 3y = 2$

$y = -2$

Substituting into (1):

$8 - (-2) + 3z = 13$

$3z = 3$

$z = 1$

State the solution

So, the solution is

$x = 4,\ y = -2,\ z = 1$

Tip

What should you do now to be sure (4, –2, 1) is the correct solution?

Not every system of equations has a unique solution. It is possible for there to be no solutions at all, or infinitely many solutions.

KEY POINT 2.3

If, after elimination, a system of equations in three unknowns results in the form:

- $x = \alpha, y = \beta, z = \gamma$, where α, β and γ are just numbers, then the system has a unique solution
- $0 = \delta$ where $\delta \neq 0$ and is just a number, then the system has no solutions
- $0 = 0$, then the system has infinitely many solutions.

For example, in Worked Example 2.7 the outcome was $x = 4$, $y = -2$, $z = 1$. This is the case shown in the first bullet point in Key Point 2.3, so the system has a unique solution.

If the system of equations has either a unique solution or infinitely many solutions, then it is said to be **consistent**. If there are no solutions, it is said to be **inconsistent**.

In the situation where there are infinitely many solutions, a **general solution** can be given. This involves expressing x, y and z in terms of a parameter, λ.

You will see in Chapter 8 that the conditions in Key Point 2.3 correspond to various positions of three planes in three-dimensional space.

WORKED EXAMPLE 2.8

$$\begin{cases} 2x + 3y + 8z = 6 \\ 2x + y + 2z = 4 \\ 6x - y - 6z = 8 \end{cases}$$

a Show that the system of equations has infinitely many solutions.

b Find the general solution.

Label each equation

a $$\begin{cases} 2x + 3y + 8z = 6 & (1) \\ 2x + y + 2z = 4 & (2) \\ 6x - y - 6z = 8 & (3) \end{cases}$$

$(4) = (2) + (3)$

Eliminate y from equation (2)

$$\begin{cases} 2x + 3y + 8z = 6 & (1) \\ 2x - z = 3 & (4) \\ 6x - y - 6z = 8 & (3) \end{cases}$$

$(5) = \frac{1}{10}(3 \times (3) + (1))$

Eliminate y from equation (3)

$$\begin{cases} 2x + 3y + 8z = 6 & (1) \\ 2x - z = 3 & (4) \\ 2x - z = 3 & (5) \end{cases}$$

$(6) = (5) - (4)$

Equations (4) and (5) are identical so there will be infinitely many solutions. We have reduced the system down to $0 = 0$, which is the case shown in the third bullet point of Key Point 2.3

$$\begin{cases} 2x + 3y + 8z = 6 & (1) \\ 2x - z = 3 & (4) \\ 0 = 0 & (6) \end{cases}$$

We have a free choice of any one of the variables. Choose, say, z and express x and y in terms of the chosen value

b Let $z = \lambda \in \mathbb{R}$

From (4):

$$2x - \lambda = 3$$
$$x = \frac{\lambda + 3}{2}$$

From (1):

$$2\left(\frac{\lambda+3}{2}\right) + 3y + 8\lambda = 6$$
$$\lambda + 3 + 3y + 8\lambda = 6$$
$$y = 1 - 3\lambda$$

So, the general solution is

$$x = \frac{\lambda+3}{2},\ y = 1 - 3\lambda,\ z = \lambda, \quad \lambda \in \mathbb{R}$$

You might wonder what an inconsistent system of equations looks like. Unfortunately, superficially it looks similar to any other system of equations.

You are the Researcher

The mathematical topic of matrices is highly linked to the study of simultaneous equations. There is a quantity called the determinant which can be used to test very quickly if a system has a unique solution or not.

WORKED EXAMPLE 2.9

How many solutions are there to the following system of equations?

$$\begin{cases} x + y + z = 10 & (1) \\ x + 2y + 3z = 4 & (2) \\ 2x + 3y + 4z = 9 & (3) \end{cases}$$

Eliminate x from equation (2)

(4) = (2) − (1)

$$\begin{cases} x + y + z = 10 & (1) \\ y + 2z = -6 & (4) \\ 2x + 3y + 4z = 9 & (3) \end{cases}$$

Eliminate x from equation (3)

(5) = (3) − 2(1)

$$\begin{cases} x + y + z = 10 & (1) \\ y + 2z = -6 & (4) \\ y + 2z = -11 & (5) \end{cases}$$

An inconsistency is probably already apparent. To show clearly that this is the case in the second bullet point from Key Point 2.3, '0 = δ', we can eliminate y from (4) and (5). This has the effect of also eliminating z

(6) = (4) − (5)

$$\begin{cases} x + y + z = 10 & (1) \\ y + 2z = -6 & (4) \\ 0 = 5 & (6) \end{cases}$$

This is inconsistent, so the system has no solutions.

Exercise 2C

For questions 1 to 4, use the method demonstrated in Worked Example 2.6 to solve the systems of equations with your GDC.

1 **a** $\begin{cases} 2x - 3y = 16 \\ 3x + 4y = 7 \end{cases}$

b $\begin{cases} 4x + 7y = -2 \\ 5x + 3y = 4 \end{cases}$

2 **a** $\begin{cases} x = 6y + 10 \\ 4y = 7x - 5 \end{cases}$

b $\begin{cases} 5y = 11 - 3x \\ y - 16 = 4x \end{cases}$

3 **a** $\begin{cases} 3x - y + z = 2 \\ 5x + 2y - 3z = 0 \\ x + y + 2z = 3 \end{cases}$

b $\begin{cases} 2x - 5y + 4z = 10 \\ x + 3y - 2z = 5 \\ 4x + 2y - z = -4 \end{cases}$

4 **a** $\begin{cases} x + 5 = 3y - z \\ 3x + 7 = y + z \\ x + 3y = 1 - 5z \end{cases}$

b $\begin{cases} 2y = 3x - z \\ x + 3y = 2z - 3 \\ 4z - 5 = 2x - y \end{cases}$

For questions 5 to 7, use the methods demonstrated in Worked Example 2.7 to solve the systems of equations or to determine that no solution exists.

5 **a** $\begin{cases} 3x - 5y + 2z = 3 \\ 2x + y - z = 6 \\ -x + 2y + z = 3 \end{cases}$

b $\begin{cases} 3x - 3y + 2z = 8 \\ 4x + 2y - z = 2 \\ 2x - y + z = 5 \end{cases}$

6 **a** $\begin{cases} 2x + 3y + z = 2 \\ 5x - 6y + z = 1 \\ x + 3y + 2z = 3 \end{cases}$

b $\begin{cases} 4x - y + 2z = -1 \\ -x + y + 3z = 5 \\ 3x + 2y - 5z = 7 \end{cases}$

7 **a** $\begin{cases} x + y - 2z = 6 \\ 2x - y + z = 5 \\ 3x + 3y - 6z = 2 \end{cases}$

b $\begin{cases} x + 5y - z = -1 \\ 2x - 2y + z = 2 \\ 5x - 3y + 2z = 4 \end{cases}$

For questions 8 to 10, use the methods demonstrated in Worked Example 2.8 to find the general solution of the system of differential equations.

8 **a** $\begin{cases} x - y + 2z = 1 \\ x + 2y - z = 4 \\ 2x + y + z = 5 \end{cases}$

b $\begin{cases} 2x - y - 3z = 3 \\ x + y - 3z = 0 \\ x + 2y - 4z = -1 \end{cases}$

9 **a** $\begin{cases} -x + y + z = -2 \\ x - y + 2z = 5 \\ 2x - 2y + 3z = 9 \end{cases}$

b $\begin{cases} 3x - 6y + z = 3 \\ x - 2y + z = 1 \\ -x + 2y + 2z = -1 \end{cases}$

10 **a** $\begin{cases} 3x - 6y + 3z = 15 \\ x - 2y + z = 5 \\ 2x - 4y + 2z = 10 \end{cases}$

b $\begin{cases} 2x - 8y - 6z = 4 \\ 5x - 20y - 15z = 10 \\ -x + 4y + 3z = -2 \end{cases}$

11 Solve the following system of equations.

$$\begin{cases} 2x - 3y + 2z = 13 \\ 3x + y - z = 2 \\ 3x - 4y - 3z = 1 \end{cases}$$

12 Solve the following system of equations.

$$\begin{cases} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{cases}$$

13 The quadratic graph $y = ax^2 + bx + c$ passes through the points (1, 4), (3, 14) and (4, 2.5).

a Form a system of three equations.

b Hence find the values a, b and c.

14 The cubic graph $y = ax^3 + bx^2 + cx$ passes through the points (−1, 7), (2, 4) and (3, 3).

a Form a system of three equations.

b Hence find the values a, b and c.

15 $\begin{cases} x+2y+kz=8 \\ 2x+5y+2z=7 \\ 5x+12y+z=2 \end{cases}$

a Find the value of the constant k for which the system is inconsistent.

b For $k=2$, solve the system of equations.

16 $\begin{cases} kx+y+2z=4 \\ 3x+ky-2z=1 \\ -x+y+z=-2 \end{cases}$

a Show that the system is consistent for all $k \in \mathbb{R}$.

b For $k=1$, solve the system of equations.

17 $\begin{cases} x-y-2z=2 \\ 2x-2y+z=0 \\ 3x-3y+4z=a \end{cases}$

a Find the value of the constant a for which the system is consistent.

b For the value of a, found in part **b**, solve the system of equations.

18 $\begin{cases} x-2y+z=2 \\ x+y-3z=k \\ 2x-y-2z=k^2 \end{cases}$

Find the values of k for which the system of equations has infinitely many solutions.

19 The mean of three numbers is twice the median. The range is five times the median. The difference between the two smallest numbers is one.

Find the largest number.

20 $\begin{cases} -x+(2k-5)y-2z=3 \\ 3x-y+(k-1)z=4 \\ x+y+2z=-1 \end{cases}$

Find the values of k for which the system of equations does not have a unique solution.

21 $\begin{cases} 3x-y+5z=2 \\ 2x+4y+z=1 \\ x+y+kz=c \end{cases}$

Find the conditions for which the system of equations has

a a unique solution

b infinitely many solutions

c no solutions.

22 The sum of the digits of a three-digit number is 16. The third digit is twice the difference between the first and second digits. When the three digits are reversed, the number decreases by 297.

Find the three-digit number.

Checklist

- You should be able to extend the binomial theorem to fractional and negative indices.
  - If $|x| < 1$, then for any $n \in \mathbb{Q}$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

- You should be able to split rational functions with a product of two linear factors in the denominator into a sum of two algebraic fractions (partial fractions):
  - $\dfrac{px+q}{(ax+b)(cx+d)} \equiv \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
- You should be able to solve systems of up to three linear equations in three unknowns.
- If, after elimination, a system of equations in three unknowns results in the form:
  - $x = \alpha, y = \beta, z = \gamma$, then the system has a unique solution
  - $0 = \delta$, then the system has no solutions
  - $0 = 0$, then the system has infinitely many solutions.
  - If there is a unique solution or infinitely many solutions, then the system is called consistent.

Mixed Practice

1 Find the expansion, in ascending powers of x up to and including the term in x^3, of $\sqrt{1-3x}$.

2 **a** Find the first three terms in ascending powers of x in the expansion of $\dfrac{1}{\sqrt{9+4x}}$.

b Find the values of x for which the expansion is valid.

3 **a** Express $\dfrac{1}{1+6x+9x^2}$ in the form $(1+ax)^n$, where a and n are integers to be found.

b Hence find the first four terms in ascending powers of x in the expansion of $\dfrac{1}{1+6x+9x^2}$.

4 Express $\dfrac{3x-1}{x(x+1)}$ in the form $\dfrac{A}{x} + \dfrac{B}{x+1}$, where A and B are constants to be found.

5 Split $\dfrac{5}{(3x-4)(x+2)}$ into partial fractions.

6 Express $\dfrac{27-x}{x^2+x-30}$ in partial fractions.

7 Solve the following system of equations.

$$\begin{cases} 2x - y + 3z = 4 \\ 3x + 2y + 4z = 11 \\ 5x - 3y + 5z = -1 \end{cases}$$

8 Solve the following system of equations.

$$\begin{cases} -2x + 3y + z = 4 \\ x - 4y + 2z = 8 \\ 7x - 18y + 4z = 16 \end{cases}$$

9 Solve the following system of equations.

$$\begin{cases} x+3y+4z=2 \\ 3x+8y+12z=5 \end{cases}$$

10 The quadratic graph $y = ax^2 + bx + c$ passes through the points (–2, 12), (–1, 1) and (1, –3).

a Form a system of three equations.

b Hence find the values a, b and c.

11 **a** Find the first three non-zero terms in ascending powers of x in the expansion of $\frac{1+2x}{(1+x)^2}$.

b Find the values of x for which the expansion is valid.

12 In the binomial expansion of $\frac{a+x}{(2-3x)^2}$, the coefficient of x^2 is $\frac{15}{2}$.

Find the value of a.

13 **a** Find the expansion in ascending powers of x, up to and including the term in x^2, of $(8+6x)^{\frac{2}{3}}$.

b Find the values of x for which the expansion is valid.

c By substituting an appropriate value into the expansion in part **a**, find an approximation for $\sqrt[3]{100}$ correct to three decimal places.

14 **a** Express $\frac{10-3x}{(1+3x)(2-5x)}$ in partial fractions.

b Hence find the first three terms in ascending powers of x in the expansion of $\frac{10-3x}{(1+3x)(2-5x)}$.

c Find the values of x for which the expansion is valid.

15

$$\begin{cases} 3x-y+z=17 \\ x+2y-z=8 \\ 2x-3y+2z=k \end{cases}$$

a Find the value of k for which the system of equations is consistent.

b For this value of k, solve the system of equations.

16 The system of equations

$$2x - y + 3z = 2$$
$$3x + y + 2z = -2$$
$$-x + 2y + az = b$$

is known to have more than one solution. Find the value of a and the value of b.

Mathematics HL May 2010 Paper 2 TZ1 Q2

17 Find the first three non-zero terms in ascending powers of x in the expansion of $\frac{1}{1+x+x^2}$.

18 Find the first two terms in ascending powers of x in the expansion of $\frac{11x-3}{2x^2-7x+3}$.

19 **a** Find the first two terms in ascending powers of x in the expansion of $\sqrt{\frac{1+5x}{1+12x}}$.

b Find the values of x for which the expansion in valid.

c By substituting $x = 0.01$ into the expansion in part **a**, find an approximation to $\sqrt{15}$ to 2 decimal places.

20 $(1+ax)^n = 1 - 3x + \frac{15}{2}x^2 + bx^3 + \ldots$

Find the value of b.

21 $(1+ax)^n = 1 - 9x + 54x^2 + bx^3 + \ldots$

Find the value of b.

22 Consider the following system of equations:

$$\begin{aligned} 2x + y + 6z &= 0 \\ 4x + 3y + 14z &= 4 \\ 2x - 2y + (\alpha - 2)z &= \beta - 12. \end{aligned}$$

a Find conditions on α and β for which
  i the system has no solutions
  ii the system has only one solution
  iii the system has an infinite number of solutions.

b In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form.

Mathematics HL May 2015 Paper 2 TZ2 Q7

3 Trigonometry

ESSENTIAL UNDERSTANDINGS

- Trigonometry allows us to quantify the physical world.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

- about three new trigonometric functions: sec, cosec and cot
- about inverse trigonometric functions
- how to expand expressions such as $\sin(A + B)$ (compound angle identities)
- the double angle identity for tan.

CONCEPTS

The following concepts will be addressed in this chapter:

- Different **representations** of the values of trigonometric relationships, such as exact or **approximate**, may not be **equivalent** to one another.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Write down the exact value of $\sin\left(\frac{\pi}{3}\right)$.

2 Given that $\sin\theta = \frac{1}{3}$, find the value of $\cos 2\theta$.

3 Solve the equation $3x^2 + 4x - 1 = 0$.

4 Solve the following equations for $0 \leqslant x \leqslant 2\pi$.

a $\sin x = -\cos x$ b $\cos x - \sin^2 x = 1$ c $\cos 3x = \frac{1}{2}$

■ **Figure 3.1** How do we model multifaceted situations using trigonometry?

Trigonometric functions are used to model real life situations where a quantity varies periodically in space or time. In some cases, several different trigonometric functions need to be combined to create a good model. In this chapter, you will learn several new identities which enable you to manipulate such expressions. You will also meet some new trigonometric functions and apply your knowledge of inverse functions to trigonometry.

Starter Activity

Look at the images in Figure 3.1. In small groups, discuss why these situations need to be modelled using a combination of different trigonometric functions. What other practical situations can be modelled in a similar way? What is the effect of combining different trigonometric functions in those situations?

Now look at this problem:

1 Use technology to draw the following graphs.

a $y = 3\sin x + 4\cos x$

b $y = 5\sin x - 2\cos x$

c $y = \cos x - 4\sin x$

Use your knowledge of transformations of graphs to write each expression as a single trigonometric function.

2 Use technology to draw these graphs.

a $y = \sin x + \sin 2x$

b $y = \sin x + \sin 5x$

c $y = \sin 6x + \sin 7x$

Describe how the frequencies of the two sin functions affect the shape of the graph.

LEARNER PROFILE – Knowledgeable

What are the links between mathematics and other subjects? Most people see mathematics used in science, but did you know that in Ancient Greece music was considered a branch of mathematics, just like geometry or statistics are now. At the highest levels, philosophy and mathematics are increasingly intertwined. See if you can find any surprising applications of mathematics in some of your other subjects.

3A Further trigonometric functions

Definition of the reciprocal trigonometric ratios sec θ, cosec θ and cot θ

You already know that $\tan x \equiv \frac{\sin x}{\cos x}$, so you could do all the relevant calculations using just sine and cosine. However, having the notation for $\tan x$ can simplify many expressions. Expressions of the form $\frac{\cos x}{\sin x}\left(\equiv \frac{1}{\tan x}\right)$, $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ also occur frequently, so it can be useful to have notation for these too.

Tip

You may also see cosecant abbreviated to csc instead of cosec.

KEY POINT 3.1

- secant : $\sec x \equiv \frac{1}{\cos x}$
- cosecant : $\operatorname{cosec} x \equiv \frac{1}{\sin x}$
- cotangent : $\cot x \equiv \frac{1}{\tan x} \equiv \frac{\cos x}{\sin x}$

In a right-angled triangle, $\sin\theta$ was defined as $\frac{\text{opposite}}{\text{hypotenuse}}$. There was nothing special about having the ratio this way round. You could have been taught all about right-angled triangles using $\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}$.

TOK Links

Does having names for reciprocals of other functions add to our body of knowledge? Why might it be useful? Why do you think the mathematical community decided to make sine the fundamental function rather than secant?

WORKED EXAMPLE 3.1

Evaluate $\sec\left(\frac{\pi}{3}\right)$.

We need to link this expression to one of the functions we are more familiar with

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$$

Remember that by default we are working in radians

$\cos\left(\frac{\pi}{3}\right)$ is one of the exact values you should know

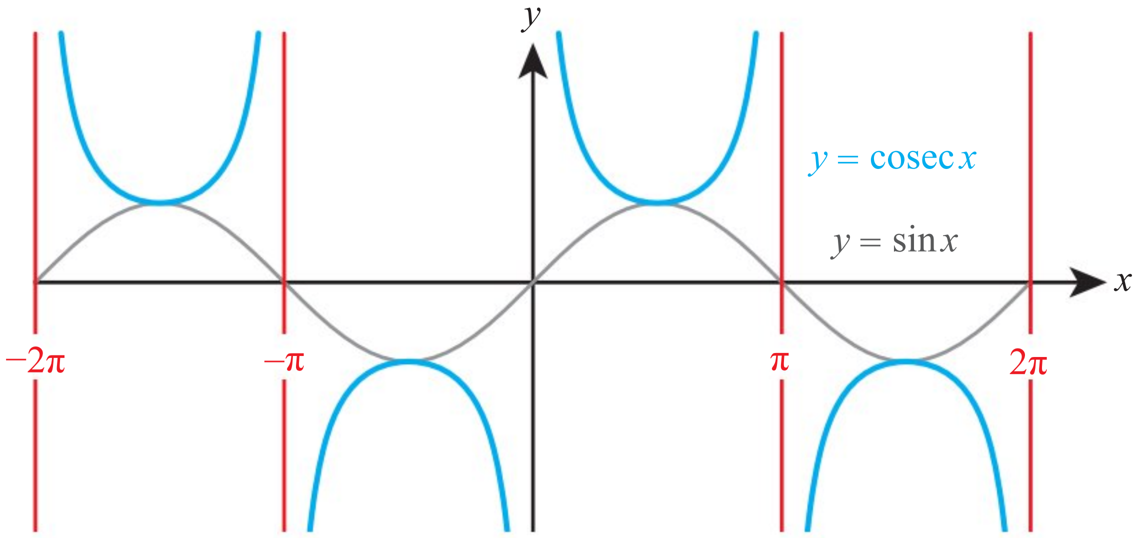
$$= \frac{1}{(1/2)}$$

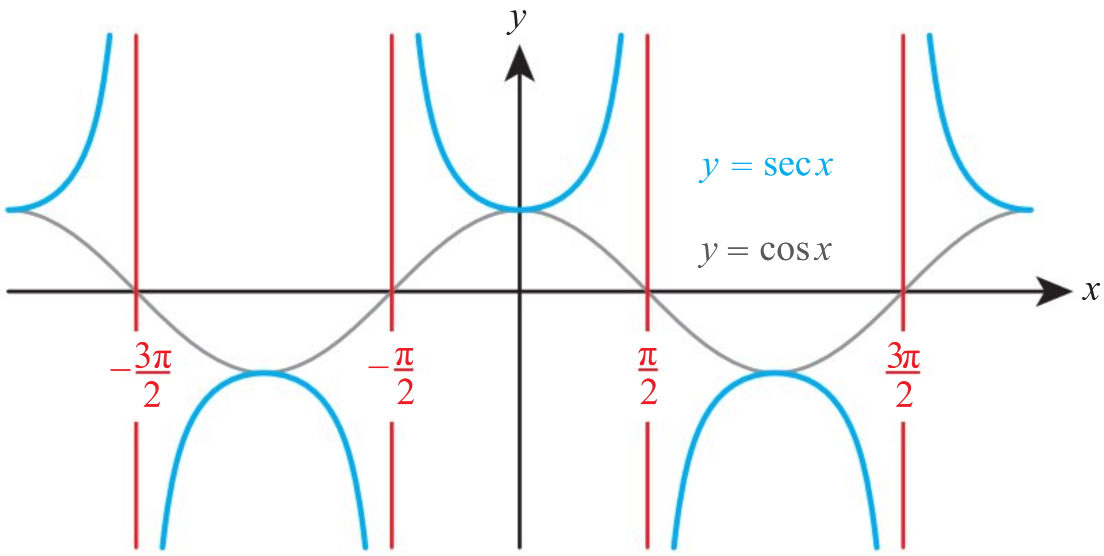
$$= 2$$

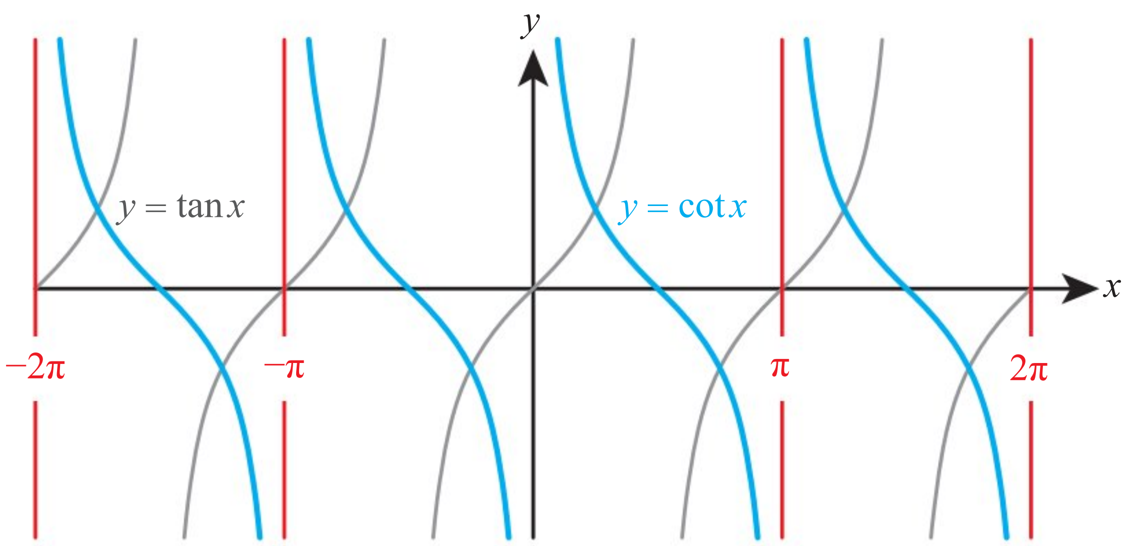
What is the origin of the word 'secant'?

You can use technology to graph these reciprocal functions.

You will see how to relate graphs of $y = \mathrm{f}(x)$ to $y = \dfrac{1}{\mathrm{f}(x)}$ in Chapter 7.







From these graphs you can deduce the range and domain of each function:

Function	Domain	Range
$\mathrm{f}(x) = \operatorname{cosec} x$	$x \neq \left(n + \frac{1}{2}\right)\pi,\ n \in \mathbb{Z}$	$\mathrm{f}(x) \geqslant 1$ or $\mathrm{f}(x) \leqslant -1$
$\mathrm{f}(x) = \sec x$	$x \neq n\pi,\ n \in \mathbb{Z}$	$\mathrm{f}(x) \geqslant 1$ or $\mathrm{f}(x) \leqslant -1$
$\mathrm{f}(x) = \cot x$	$x \neq n\pi,\ n \in \mathbb{Z}$	$\mathrm{f}(x) \in \mathbb{R}$

Tip

$\cot \frac{\pi}{2}$ is zero. Why?

WORKED EXAMPLE 3.2

Solve $\cot x = \sqrt{3}$, $0 \leqslant x \leqslant 2\pi$.

Virtually every problem with the reciprocal trigonometric functions involves, at some stage, rewriting the expression in terms of the standard trigonometric functions

$$\frac{1}{\tan x} = \sqrt{3}$$

Rearrange to get the equation with $\tan x$ as the subject

$$\tan x = \frac{1}{\sqrt{3}}$$

This is one of the standard exact values you should know. Do not forget that there can be further solutions too. When solving this type of tan equation, we can always add on π to get more answers

$$x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

Pythagorean identities

Perhaps the most common usage of these functions is in the following identities, which can be deduced from the familiar $\cos^2 x + \sin^2 x \equiv 1$ by dividing through by $\cos^2 x$ and $\sin^2 x$ respectively.

KEY POINT 3.2

- $1 + \tan^2 x \equiv \sec^2 x$
- $\cot^2 x + 1 \equiv \operatorname{cosec}^2 x$

WORKED EXAMPLE 3.3

If θ is an acute angle and $\tan\theta = \frac{1}{2}$, find the exact value of $\cos\theta$.

We need a link between tan and cos. The identity from Key Point 3.2 involving tan and sec provides this

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\frac{1}{4} + 1 = \sec^2\theta$$

We can take the reciprocal of both sides to get $\cos^2 x$

$$\frac{5}{4} = \sec^2\theta$$

$$\frac{4}{5} = \cos^2\theta$$

Square root both sides

$$\cos\theta = \pm\frac{2}{\sqrt{5}}$$

For all acute angles, cosine is positive

Since θ is acute, $\cos\theta = \frac{2}{\sqrt{5}}$

Tip

The reciprocal trigonometric functions follow the same conventions as the normal trigonometric functions, so $\sec^2 x = (\sec x)^2$.

WORKED EXAMPLE 3.4

Solve the equation $\tan^2 x - 4\sec x + 5 = 0$ for $0 < x < 2\pi$.

We would like to have only one trigonometric function, so use the identity $\tan^2 x + 1 \equiv \sec^2 x$ from Key Point 3.2

$(\sec^2 x - 1) - 4\sec x + 5 = 0$

This is a disguised quadratic in $\sec x$. First write in standard quadratic form

$\sec^2 x - 4\sec x + 4 = 0$

We could use a substitution $u = \sec x$ to help make it look more familiar

If $u = \sec x$,

$u^2 - 4u + 4 = 0$

Factorize (or you could use the quadratic formula)

$(u - 2)^2 = 0$

We do not know how to find inverse sec, so express $\sec x$ in terms of $\cos x$

$u = 2$ so $\sec x = 2$ so $\frac{1}{\cos x} = 2$

Therefore, $\cos x = \frac{1}{2}$

Finally, we solve this trigonometric equation, remembering to give all the possible solutions

$x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Inverse trigonometric functions

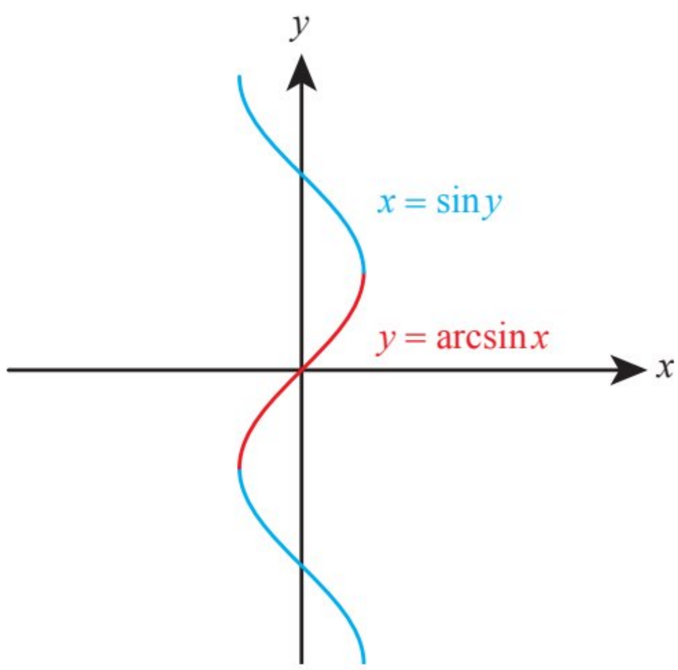
You have already met the inverse sin, cos and tan operations when you used them to solve trigonometric equations. However, in this section we shall now look at them as functions.

The best way to study the inverse sine function is to consider the relationship between functions and their inverses graphically – they are reflections in the line $y = x$.

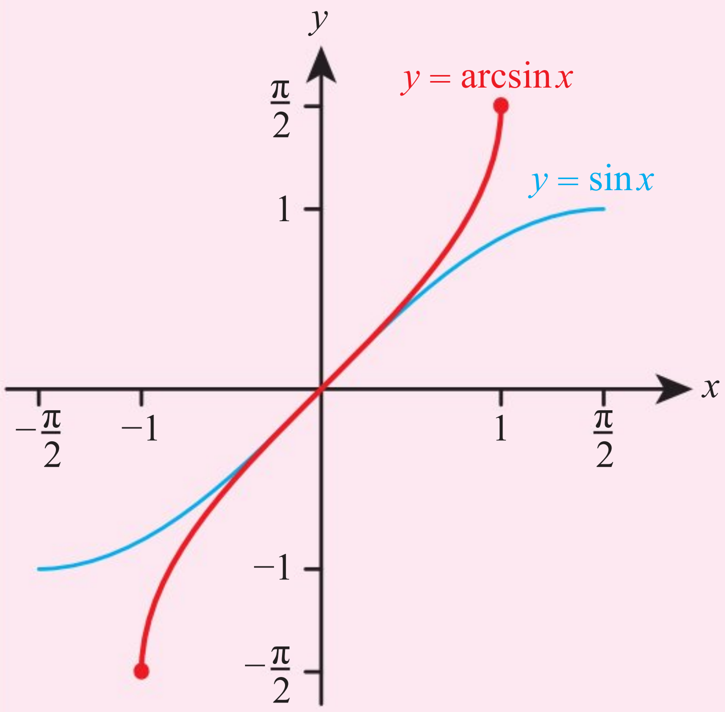
You met this relationship in Chapter 14 of the Mathematics: analysis and approaches SL book.

If the graph of $y = \sin x$ is reflected in the line $y = x$, the result is not a function because each x value corresponds to more than one y value.

However, if the original graph of $y = \sin x$ is restricted to $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ then when it is reflected in $y = x$, the result (shown in red on the graph) is a function, called $\arcsin x$.



KEY POINT 3.3



The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$ and the range is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

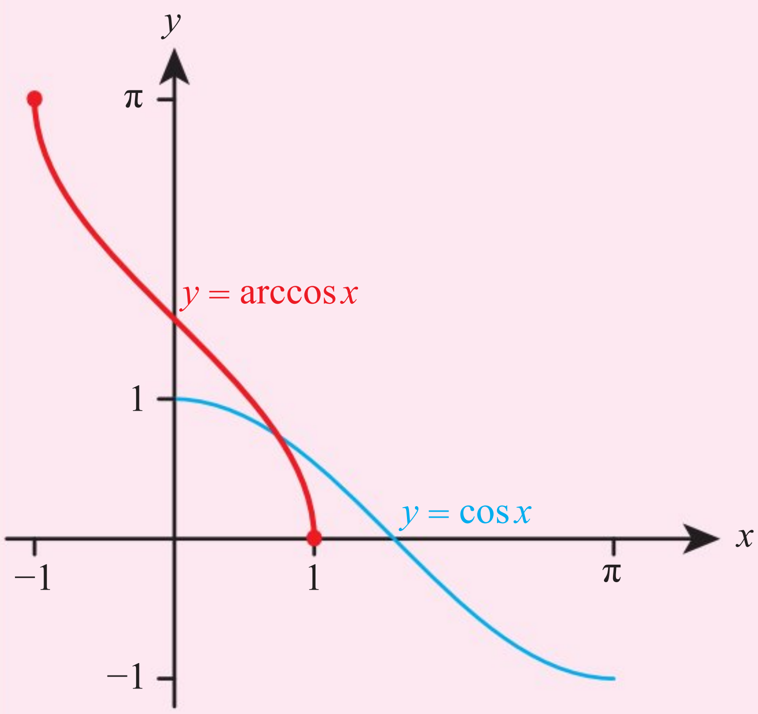
Tip

$\arcsin x$ is also known as $\sin^{-1}x$, especially on calculators. This is not the same as $\frac{1}{\sin x}$.

Remember that the domain is the set of all numbers allowed into a function and the range is the corresponding set of outputs. This was covered in Chapter 3 of the Mathematics: analysis and approaches SL book.

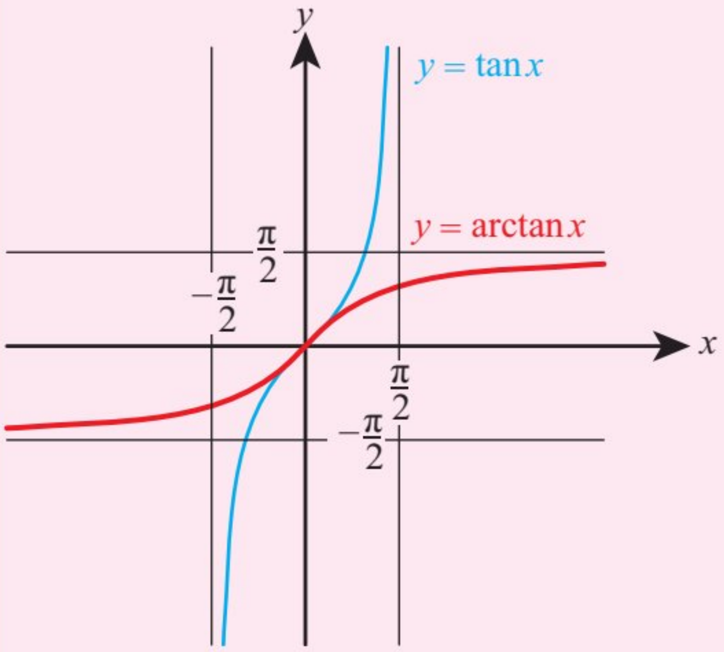
A similar argument leads to inverse functions of the cosine and tangent functions over restricted domains.

KEY POINT 3.4



The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$ and the range is $0 \leqslant y \leqslant \pi$.

KEY POINT 3.5



The domain of $y = \arctan x$ is $x \in \mathbb{R}$ and the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

WORKED EXAMPLE 3.5

Evaluate $\arccos 0.5$.

$\arccos 0.5$ is equivalent to asking for the solution to $\cos x = 0.5$ with $0 \leqslant x \leqslant \pi$. This is one of the exact values you should know

$$\arccos 0.5 = \frac{\pi}{3}$$

Exact values you should know are covered in Key Point 18.9 of the Mathematics: analysis and approaches SL book.

Once you are familiar with the inverse trigonometric functions, you can apply them in algebraic expressions. People often find this quite tricky – the example below is towards the top end of examination difficulty.

WORKED EXAMPLE 3.6

Simplify $\sin(\arccos x)$ if $-1 \leqslant x \leqslant 1$.

We cannot directly apply sine to an arccosine, so we need to use an identity to apply cosine rather than a sine to $\arccos x$. The appropriate one is $\sin^2\theta + \cos^2\theta \equiv 1$

$$\sin^2(\arccos x) + \cos^2(\arccos x) = 1$$

Since cosine and arccosine are inverse functions, we can use $\cos(\arccos x) \equiv x$

$$\sin^2(\arccos x) + x^2 = 1$$

$$\sin^2(\arccos x) = 1 - x^2$$

So,

$$\sin(\arccos x) = \pm\sqrt{1 - x^2}$$

We then need to decide if there is a reason why we should choose the positive or the negative root. To do this, we need to consider the range of $\arccos x$

The range of $\arccos(x)$ is between 0 and π inclusive. Sine of these values is never negative, so we can exclude the negative root. Therefore,

$$\sin(\arccos x) = \sqrt{1-x^2}$$

TOOLKIT: Problem Solving

Find the values of x for which it is true that:

a $\sin(\arcsin x) = x$

b $\arcsin(\sin x) = x$

Be the Examiner 3.1

Evaluate $\cos^{-1} 0$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\cos^{-1} 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$	$\cos^{-1} 0 = \frac{\pi}{2}$	$\cos^{-1} 0 = \pi$

Exercise 3A

1 Use the method demonstrated in Worked Example 3.1 to evaluate the following expressions on your calculator. Give your answers to three significant figures.

a **i** $\sec 2.4$ **ii** $\operatorname{cosec} 3$

b **i** $\cot(-1)$ **ii** $\sec(-2)$

c **i** $\operatorname{cosec}\left(\frac{3\pi}{5}\right)$ **ii** $\cot\left(\frac{7\pi}{3}\right)$

2 Use the method demonstrated in Worked Example 3.1 to find the exact value of the following expressions.

a **i** $\operatorname{cosec}\left(\frac{\pi}{2}\right)$ **ii** $\operatorname{cosec}\left(\frac{\pi}{4}\right)$

b **i** $\sec(0)$ **ii** $\sec\left(\frac{\pi}{6}\right)$

c **i** $\cot\left(\frac{\pi}{4}\right)$ **ii** $\cot\left(\frac{\pi}{3}\right)$

d **i** $\operatorname{cosec}\left(\frac{3\pi}{2}\right)$ **ii** $\cot\left(\frac{\pi}{2}\right)$

3 Use the method demonstrated in Worked Example 3.2 to solve the following equations, giving all answers in the region $0 \leqslant x \leqslant 2\pi$. Give your answer to three significant figures.

a **i** $\sec x = 2.5$ **ii** $\sec x = 4$

b **i** $\operatorname{cosec} x = -3$ **ii** $\operatorname{cosec} x = 3$

c **i** $\cot x = 2$ **ii** $\cot x = 0.4$

d **i** $\sec 2x = 1.5$ **ii** $\operatorname{cosec} 2x = 5$

4 Use the method demonstrated in Worked Example 3.2 to solve the following equations, giving all answers in the region $0 \leqslant x \leqslant 2\pi$. Give your answer in an exact form.

a **i** $\operatorname{cosec}\theta = 2$ **ii** $\operatorname{cosec}\theta = 1$

b **i** $\cot\theta = \frac{1}{\sqrt{3}}$ **ii** $\cot\theta = 1$

c **i** $\sec\theta = 1$ **ii** $\sec\theta = \frac{2}{\sqrt{3}}$

d **i** $\cot\theta = 0$ **ii** $\cot\theta = -1$

5 Use the method demonstrated in Worked Example 3.3 to find the exact value of the required trigonometric ratio.

a **i** Given that $\tan\theta = \frac{3}{4}$ and $0 < \theta < \frac{\pi}{2}$, find $\sec\theta$.

ii Given that $\tan\theta = \frac{1}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\sec\theta$.

b **i** Given that $\operatorname{cosec}\theta = 4$ and $\frac{\pi}{2} < \theta < \pi$, find $\cot\theta$.

ii Given that $\operatorname{cosec}\theta = 7$ and $\frac{\pi}{2} < \theta < \pi$, find $\cot\theta$.

c **i** Given that $\cot\theta = 2$ and $0 < \theta < \pi$, find the exact value of $\sin\theta$.

ii Given that $\cot\theta = 3$ and $\pi < \theta < 2\pi$, find the exact value of $\sin\theta$.

d **i** Given that $\tan\theta = \sqrt{2}$, find the possible values of $\cos\theta$.

ii Given that $\cot\theta = -4$, find the possible values of $\sin\theta$.

6 Use technology to evaluate each of the following, giving your answers in radians to three significant figures.

a **i** $\arcsin 0.8$ **b** **i** $\cos^{-1}(-0.75)$ **c** **i** $\arctan(\pi)$

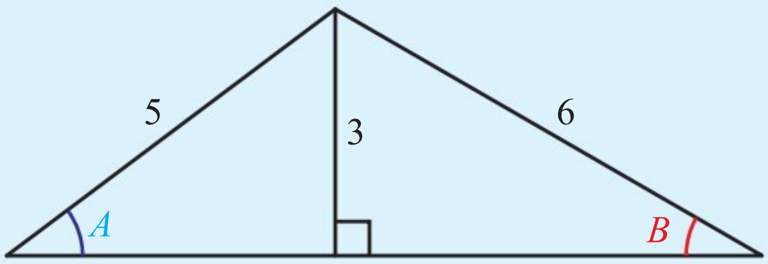
ii $\arcsin(-0.6)$ **ii** $\cos^{-1}(0.01)$ **ii** $\arctan(10)$

7 Use the method demonstrated in Worked Example 3.5 to find the exact value of the given expression. Give your answer in radians.

a **i** $\arccos\frac{\sqrt{3}}{2}$ **b** **i** $\arcsin\frac{1}{2}$ **c** **i** $\arctan\sqrt{3}$ **d** **i** $\arccos(-0.5)$

ii $\arccos 1$ **ii** $\arcsin 1$ **ii** $\arctan 1$ **ii** $\arcsin(-0.5)$

8 Find the values of $\operatorname{cosec} A$ and $\sec B$.



9 Prove that $\sin^2\theta + \cot^2\theta\sin^2\theta = 1$.

10 Solve the equation $\tan x + \sec x = 2$ for $0 \leqslant x \leqslant 2\pi$.

11 A function is defined by $f(x) = \tan x + \operatorname{cosec} x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

a Find the coordinates of the minimum and maximum points on the graph of $y = f(x)$.

b Hence write down the range of f.

12 Show that $\sin A \cot A = \cos A$.

13 Show that $\tan B \operatorname{cosec} B = \sec B$.

14 Evaluate $\arcsin(\sin\pi)$.

15 Sketch $y = \sec 2x$ for $0 \leqslant x \leqslant 2\pi$, labelling all maximum and minimum points.

16 Sketch $y = 3\cot 2x$ for $0 \leqslant x \leqslant \pi$, labelling all asymptotes.

17 Sketch $y = \operatorname{cosec}(x - \pi)$ for $0 \leqslant x \leqslant 2\pi$.

18 Show that $\tan x + \cot x = \sec x \operatorname{cosec} x$.

19 Prove that $\sec x - \cos x \equiv \sin x \tan x$.

20 Prove that $\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta}{1+\cos\theta} = 2\cot\theta$.

21 Solve the equation $2\tan^2 x + \frac{3}{\cos x} = 0$ for $-\pi < x < \pi$.

22 How many solutions are there to the equation $\arccos x = 2x$? Justify your answer.

23 Write $\arccos(-x)$ in terms of $\arccos x$.

24 **a** Given that $\sec^2 x - 3\tan x + 1 = 0$, show that $\tan^2 x - 3\tan x + 2 = 0$.

b Find the possible values of $\tan x$.

c Hence solve the equation $\sec^2 x - 3\tan x + 1 = 0$ for $x \in [0, 2\pi]$.

25 Prove that $\operatorname{cosec}(2x) = \dfrac{\sec x \operatorname{cosec} x}{2}$.

26 Show that $\sec(2\theta) = \dfrac{\sec^2\theta}{2 - \sec^2\theta}$.

27 Find the inverse function of $\sec x$ in terms of the arccosine function.

28 A straight line has equation $(4\cos\theta)x + (5\sin\theta)y = 20$, where θ is a constant. The line intersects the x-axis at P and the y-axis at Q. The midpoint of PQ is M.

a Show that the coordinates of M are $\left(\frac{5}{2}\sec\theta, 2\operatorname{cosec}\theta\right)$.

b Hence show that M lies on the curve with equation $\dfrac{25}{x^2} + \dfrac{16}{y^2} = 4$.

29 a State the largest possible domain and range of the function $f(x) = \arccos(\cos x)$.

b Simplify $\arccos(\cos x)$ for:

i $2\pi \leqslant x \leqslant 3\pi$ ii $\pi \leqslant x \leqslant 2\pi$ iii $-\pi \leqslant x \leqslant 0$.

30 The function $\operatorname{isin} x$ is defined as the inverse function of $f(x) = \sin x$ for $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.
Write $\operatorname{invsin} x$ in terms of $\arcsin x$.

3B Compound angle identities

Compound angle identities are used to expand expressions such as $\sin(A + B)$ or $\tan(A - B)$. They are derived in a similar way to the double angle identities.

You met the identities for $\sin 2\theta$ and $\cos 2\theta$ in Chapter 18 of the Mathematics: analysis and approaches SL book.

Tip

Notice the signs in the cosine identities: in the identity for the sum you use the minus sign, and in the identity for the difference the plus sign.

Compound angle identities for sin and cos

KEY POINT 3.6

- $\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$

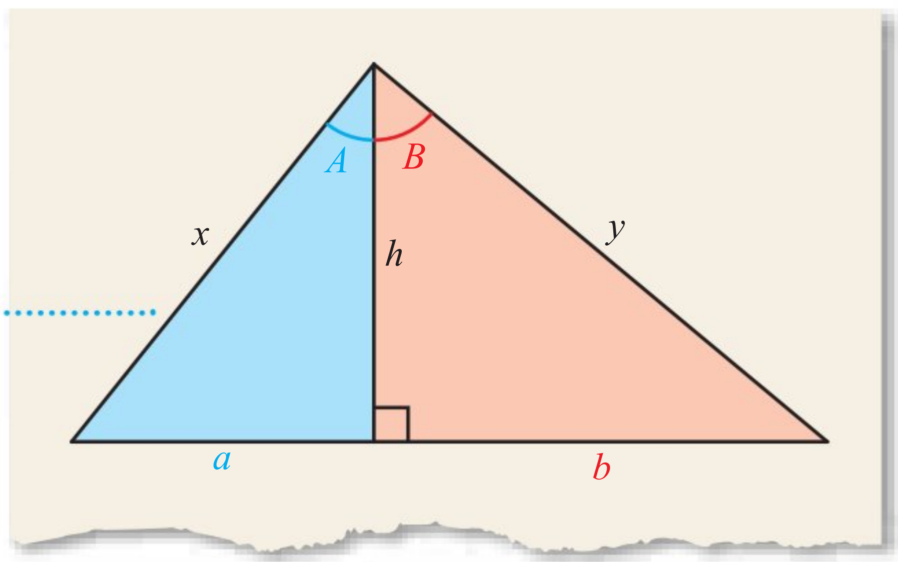
Proof 3.1

Prove the identity $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$ for $0 < A,\ B < \dfrac{\pi}{2}$.

To work with trigonometric ratios, it is useful to try to form a diagram containing some right-angled triangles including the given angles

Consider the triangle in the diagram. The angle on top is $A + B$ and the height divides it into two angles of size A and B

Label the sides of the triangle x and y



sin($A+B$) appears in the formula for the area of the triangle

For the whole triangle:

$$\text{Area} = \frac{1}{2}xy\sin(A+B) \quad (1)$$

But you can also find the area of the two right-angled triangles and add them together

You first need to express the base and the height of each triangle in terms of x, y, A and B

From the diagram:

$$a = x\sin A,\quad b = y\sin B$$

$$h = x\cos A = y\cos B$$

You have two different expressions for the height. In the identity you want to prove, both A and B appear in each term. So, use the expression containing B for the first triangle and the expression containing A for the second

For the first triangle:

$$\text{Area} = \frac{1}{2}ah$$

$$= \frac{1}{2}(x\sin A)(y\cos B) \quad (2)$$

For the second triangle:

$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2}(y\sin B)(x\cos A) \quad (3)$$

$(2) + (3) = (1)$:

Add the two small triangles to get the large triangle

$$\frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B = \frac{1}{2}xy\sin(A+B)$$

Divide both sides by $\frac{1}{2}xy$

$$\therefore\ \sin A\cos B + \cos A\sin B = \sin(A+B)$$

The proof above works whenever A and B are between 0 and $\frac{\pi}{2}$, so they can be angles in the two right-angled triangles. It can be shown, by using symmetries of trigonometric graphs, that the identity in fact holds for all values of A and B.

You can use compound angle identities to find certain exact values of trigonometric functions.

The other compound angle formulae from Key Point 3.6 can be proved by starting from this formula and applying the symmetries of the trigonometric functions, such as $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$.

WORKED EXAMPLE 3.7

Find the exact value of: **a** $\sin 75°$ **b** $\cos\frac{\pi}{12}$.

Notice that $30° + 45° = 75°$ – you know the exact values of sin and cos for 30° and 45°

a $\sin 75° = \sin(30° + 45°)$

$$= \sin 30°\cos 45° + \cos 30°\sin 45°$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

This time use the fact that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

b $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

Make sure to use the correct sign in the cos identity

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Tip

If you use more than one ± in a single expression or equation, the interpretation is that they pair up exactly (all the upper options produce one equation and all the lower options make a second option; there is no suggestion that every combination should be used). Here, the + on the left side is paired specifically with a + in the numerator and a – in the denominator.

Compound angle identities for tan

You can use the compound angle identities for sin and cos to derive the identities for tan.

KEY POINT 3.7

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Proof 3.2

Write $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

Express tan in terms of sin and cos

$$\tan(A+B) \equiv \frac{\sin(A+B)}{\cos(A+B)}$$

Use the identities for $\sin(A+B)$ and $\cos(A+B)$

$$\equiv \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

You want to write this in terms of tan. Looking at the top of the fraction, if you divide by $\cos A$ you will get $\tan A$ in the first term, and if you divide by $\cos B$ you will get $\tan B$ in the second term. So, divide top and bottom by $\cos A \cos B$

$$\equiv \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$\equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

The identity for tan $(A - B)$ is proved similarly.

WORKED EXAMPLE 3.8

Show that $\tan 105° = -2 - \sqrt{3}$.

Find two special angles whose sum or difference is 105. For example, you could use 60 + 45, or 135 – 30

$$\tan 105° = \tan(60° + 45°)$$

Use the compound angle formula for $\tan(A+B)$

$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

To get the expression into the required form, rationalise the denominator

$$= \frac{1+3+2\sqrt{3}}{1-3}$$

$$= -2-\sqrt{3}$$

Be the Examiner 3.2

Find the exact value of $\tan 15°$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\tan(15°) = \tan(45° - 30°)$ $= \tan(45°) - \tan(30°)$ $= 1 - \frac{1}{\sqrt{3}}$ $= \frac{3-\sqrt{3}}{3}$	$\tan(15°) = \tan(45° - 30°)$ $= \frac{\tan(45°) - \tan(30°)}{1 + \tan(45°)\tan(30°)}$ $= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$ $= \frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\tan(15°) = \tan(45° - 30°)$ $= \frac{\tan(45°) + \tan(30°)}{1 - \tan(45°)\tan(30°)}$ $= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$ $= 2 + \sqrt{3}$

Link with double angle identities

The compound angle identities can be used to derive the double angle identities which you already know. For example, setting $A = B = \theta$ in the identity for $\cos(A + B)$ gives $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

You can similarly prove a new double angle identity for tan.

KEY POINT 3.8

$$\tan 2\theta \equiv \frac{2\tan\theta}{1 - \tan^2\theta}$$

You can use these new identities to solve more complicated trigonometric equations.

WORKED EXAMPLE 3.9

Solve the equation $\tan 2x = 3\tan x$ for $0 \leqslant x \leqslant 2\pi$.

Use the double angle identity to express the left-hand side in terms of $\tan x$

$$\frac{2\tan x}{1 - \tan^2 x} = 3\tan x$$

Multiply through by the denominator

$$2\tan x = 3\tan x - 3\tan^3 x$$

Get everything on one side and factorize

$$3\tan^3 x - \tan x = 0$$

$$\tan x(3\tan^2 x - 1) = 0$$

Don't forget ± when taking the square root

$$\tan x = 0 \quad \text{or} \quad \tan x = \pm\frac{1}{\sqrt{3}}$$

Solve each equation separately

$$\tan x = 0 \Rightarrow x = 0, \pi, 2\pi$$

Remember to add π to get to the next solution

$$\tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$$

$$\tan x = -\frac{1}{\sqrt{3}} \Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}$$

You can also derive further identities involving multiple angles.

WORKED EXAMPLE 3.10

Starting from the identity for $\cos(A+B)$, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Write 3θ as $2\theta + \theta$ and use the compound angle identity

$$\cos(2\theta+\theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

Use the double angle identities for sin and cos. Since the required answer only contains cos, use the version of the $\cos 2\theta$ identity which only contains cos

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

Write $\sin^2\theta$ in terms of $\cos^2\theta$

$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

In Chapter 4 you will learn how to use complex numbers to derive multiple angle identities.

Symmetries of trigonometric graphs

In Section 18B of the Mathematics: analysis and approaches SL book, you learnt about various symmetries of trigonometric functions, such as $\sin(\pi + x) = -\sin x$, which you can illustrate using either a graph or the unit circle. You can now also derive them using compound angle identities.

WORKED EXAMPLE 3.11

Use a compound angle identity to simplify $\sin(\pi + x)$.

Use $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

$$\sin(\pi + x) \equiv \sin\pi\cos x + \cos\pi\sin x$$

$\sin\pi \equiv 0$ and $\cos\pi \equiv -1$

$$\equiv 0\cos x - 1\sin x$$
$$\equiv -\sin x$$

Exercise 3B

For questions 1 to 4, use the technique demonstrated in Worked Example 3.7 to find the exact values of the trigonometric functions.

1 **a** $\sin 105°$ **b** $\sin 165°$

2 **a** $\cos 15°$ **b** $\cos 75°$

3 **a** $\cos\frac{5\pi}{12}$ **b** $\cos\frac{7\pi}{12}$

4 **a** $\sin\frac{\pi}{12}$ **b** $\sin\frac{5\pi}{12}$

For questions 5 to 7, use the technique demonstrated in Worked Example 3.8 to find the exact values of the trigonometric functions.

5 **a** $\tan 15°$ **b** $\tan 165°$

6 **a** $\tan 330°$ **b** $\tan 225°$

7 **a** $\tan\frac{7\pi}{12}$ **b** $\tan\frac{5\pi}{12}$

For questions 8 to 10, use the technique demonstrated in Worked Example 3.11 to simplify the given expressions.

8 **a** $\sin(x-\pi)$ **b** $\sin\left(x+\frac{\pi}{2}\right)$

9 **a** $\cos(\pi+x)$ **b** $\cos\left(\frac{\pi}{2}-x\right)$

10 **a** $\tan(\pi-x)$ **b** $\tan(x+\pi)$

11 **a** Use a compound angle formula to show that $\sin(\pi+x)=-\sin x$ for all values of x.

b Sketch the graph of $y=\sin x$ and illustrate the result from part **a** in the case when $x=\frac{\pi}{6}$.

12 Show that $\sin\left(\theta+\frac{\pi}{3}\right)+\sin\left(\theta-\frac{\pi}{3}\right)\equiv\sin\theta$.

13 Prove that, for all values of x, $\cos x+\cos\left(x+\frac{2\pi}{3}\right)+\cos\left(x+\frac{4\pi}{3}\right)=0$.

14 **a** Simplify $\sin(x+45°)+\cos(x+45°)$.

b Hence solve the equation $\sin(x+45°)+\cos(x+45°)=\frac{\sqrt{2}}{2}$ for $0°<x<360°$.

15 Prove the identity $\cot x-\tan x\equiv 2\cot 2x$.

16 Show that $\tan\left(\theta+\frac{\pi}{4}\right)-\tan\left(\theta-\frac{\pi}{4}\right)\equiv\frac{4\tan\theta}{1-\tan^2\theta}$.

17 Given that $\tan x=\frac{1}{3}$, find the exact value of:

a $\tan 2x$

b $\tan 3x$.

18 Given that $\tan(\theta-45°)=\frac{1}{2}$, find the exact value of $\tan\theta$.

19 Given that $\sin A=\frac{1}{3}$ and $\sin B=\frac{4}{5}$, and that $0<A,\ B<\frac{\pi}{2}$, find the exact value of

a $\cos A$

b $\cos(A+B)$.

20 Given that x and y are acute angles such that $\sin x=\frac{3}{5}$ and $\sin y=\frac{5}{13}$, find the exact value of $\sin(x-y)$.

21 Given that $\sin(\theta + 60°) = 2\cos\theta$, find the exact value of $\tan\theta$.

22 Given that $\tan(x - y) = 2$, express $\tan y$ in terms of $\tan x$.

23 Given that $\tan 2A = -\frac{3}{4}$, find the possible values of $\tan A$.

24 Find the exact value of $\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{24}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{24}\right)$.

25 Find the exact value of $\cos 15° + \sqrt{3}\sin 15°$.

26 Show that $\cot(A + B) \equiv \frac{\cot A \cot B - 1}{\cot A + \cot B}$.

27 **a** Use the identity for $\sin(A + B)$ to express $\sin 2x$ in terms of $\sin x$ and $\cos x$.

b Hence solve the equation $\sin 2x = \tan x$ for $0 \leqslant x \leqslant 2\pi$.

28 Use a double angle identity to find the exact value of $\tan 22.5°$.

29 **a** Show that $\tan\left(x - \frac{\pi}{4}\right) \equiv \frac{\tan x - 1}{\tan x + 1}$.

b Given that $\tan\left(\theta - \frac{\pi}{4}\right) = 6\tan\theta$, find the two possible values of $\tan\theta$.

30 **a** Use the identity for $\sin(A + B)$ to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

b Hence solve the equation $\sin 3x = 2\sin x$ for $0 \leqslant x \leqslant \pi$.

31 **a** Use a compound angle formula to show that $\sin(\pi - x) = \sin x$.

b An isosceles triangle has angles A, A and C. Show that $\frac{\sin C}{\sin A} = 2\cos A$.

32 **a** Find the value of R such that $3\sin x - 7\cos x = R\sin(x - \theta)$ for some $\theta \in \left(0, \frac{\pi}{2}\right)$.

b Hence find the maximum value of the expression $\frac{3}{10 + 3\sin x - 7\cos x}$.

33 Let $f(x) = \frac{1}{3\sin 2x + \sqrt{3}\cos 2x + 6}$.

a Express $3\sin\theta + \sqrt{3}\cos\theta$ in the form $R\sin(\theta + \alpha)$.

b Hence find the minimum value of $f(x)$ and the smallest positive value of x for which it occurs.

34 Find the exact value of $\tan\left(\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{5}\right)\right)$.

35 **a** Use the identity for $\sin(A + B)$ to show that $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

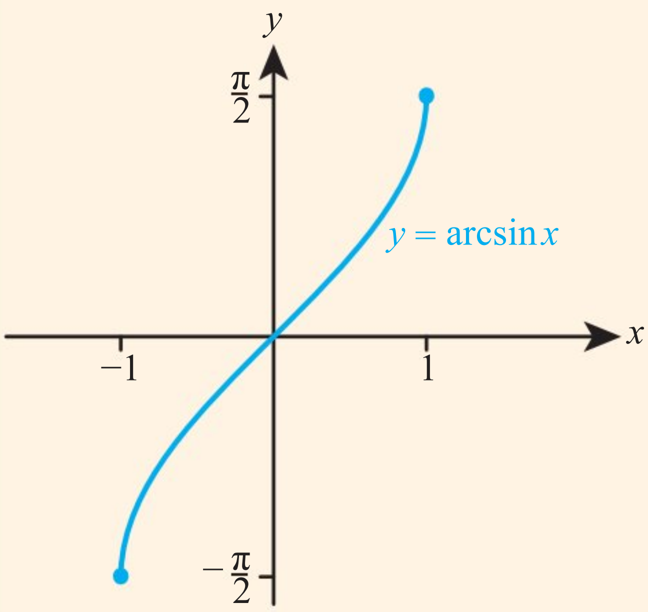
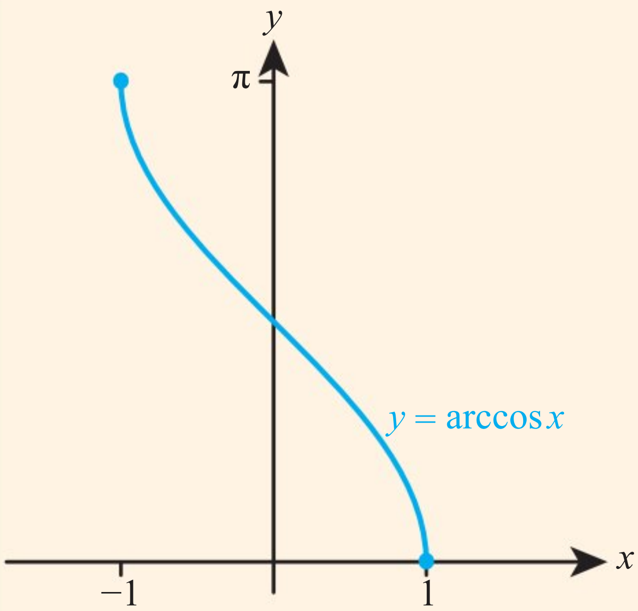
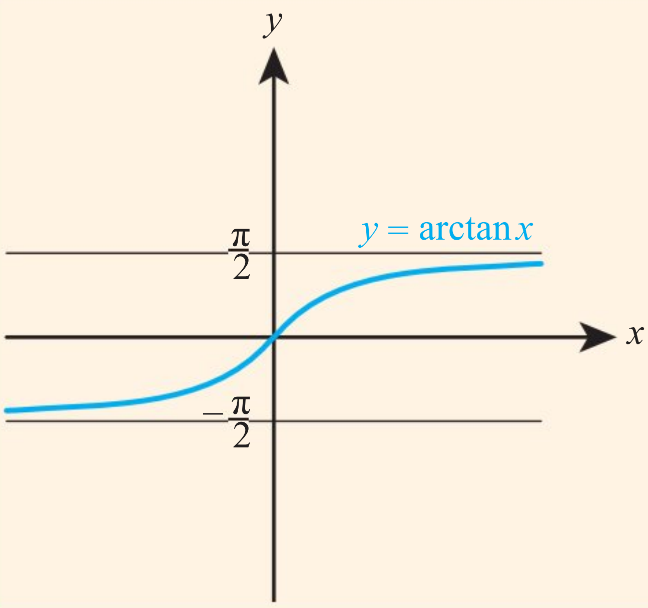
b Starting from the identity for $\sin(A + B)$, prove the identity $\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$.

36 **a** Use the formula for $\tan(A + B)$ to show that $\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

b Hence show that $\tan 10°$ is a root of the equation $t^3 - \sqrt{3}t^2 - 3t + \frac{\sqrt{3}}{3} = 0$.

Checklist

- You should know the reciprocal trigonometric functions:
  - secant: $\sec x = \dfrac{1}{\cos x}$
  - cosecant: $\operatorname{cosec} x = \dfrac{1}{\sin x}$
  - cotangent: $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
- You should be able to work with the Pythagorean identities:
  - $\tan^2 x + 1 \equiv \sec^2 x$
  - $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$
- You should know the inverse trigonometric functions:

Function	Graph	Domain	Range
$\arcsin x$		$-1 \leqslant x \leqslant 1$	$-\dfrac{\pi}{2} < y \leqslant \dfrac{\pi}{2}$
$\arccos x$		$-1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$\arctan x$		$x \in \mathbb{R}$	$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

- You should be able to work with the compound angle identities for sin and cos:
  - $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$
  - $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$
- You should be able to work with the compound and double angle identities for tan:
  - $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
  - $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Mixed Practice

1 Given that $\sec\theta = 3$, find the exact value of $\cos 2\theta$.

2 **a** Show that $\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right) \equiv \sqrt{2}\cos x$.

b Hence solve the equation $\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right) = \sqrt{2}\sin x$ for $0 \leqslant x < \pi$.

3 **a** Show that

$$\sin\left(x+\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{6}\right) \equiv \sqrt{3}\cos x.$$

b Hence solve the equation

$$\sin\left(x+\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{6}\right) = \sin x$$

for $0 < x < 2\pi$.

4 An acute angle θ has $\tan\theta = 3$. Find the exact value of

a $\tan 2\theta$

b $\sec\theta$.

5 Given that $\cos A = \frac{1}{2}$ and $\cos B = \frac{1}{3}$, find the exact value of $\cos(A+B)$.

6 Solve the equation $(\arcsin x)^2 = \frac{\pi^2}{9}$.

7 Find the exact value of $\tan 105°$.

8 Solve the equation

$$\sin\left(x+\frac{\pi}{3}\right) = \sin\left(x-\frac{\pi}{3}\right)$$

for $0 < x < 2\pi$.

9 Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leqslant x < 360°$.

Mathematics HL May 2015 Paper 1 TZ2 Q3

10 **a** Given that $\arctan\frac{1}{2} - \arctan\frac{1}{3} = \arctan a$, $a \in \mathbb{Q}^+$, find the value of a.

b Hence, or otherwise, solve the equation $\arcsin x = \arctan a$.

Mathematics HL November 2011 Paper 2 Q4

11 **a** Use the formulae for $\sin 2\theta$ and $\cos 2\theta$ to derive a formula for $\tan 2\theta$ in terms of $\tan\theta$.

b Hence find the exact value of $\tan 112.5°$.

12 Solve the equation $\sin\left(x+\frac{\pi}{6}\right)+\sin\left(x-\frac{\pi}{6}\right) = 3\cos x$ for $0 \leqslant x \leqslant \pi$.

13 Given that $\cos y = \sin(x+y)$, show that $\tan y = \sec x - \tan x$.

14 Prove that $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$.

15 **a** Prove that $\operatorname{cosec} 2x - \cot 2x \equiv \tan x$.

b Hence find the exact value of $\tan\left(\frac{3}{8}\pi\right)$.

16 **a** Prove the following identities:

i $\cos^4\theta - \sin^4\theta = \cos 2\theta$

ii $\sin^2 2\theta(\cot^2\theta - \tan^2\theta) \equiv 4(\cos^4\theta - \sin^4\theta)$.

b Hence solve the equation $\sin^2 2\theta(\cot^2\theta - \tan^2\theta) = 2$ for $0 \leqslant \theta \leqslant 2\pi$.

17 **a** Write $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\theta)$ where $R > 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$.

b Hence solve the equation $3\sin x + \sqrt{3}\cos x = 3$ for $-\pi < x < \pi$.

18 If x satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$, show that $11 \tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

Mathematics HL May 2010 Paper 1 TZ2 Q6

19 Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

a For what values of x does $f(x)$ not exist?

b Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

Mathematics HL May 2012 Paper 1 TZ1 Q5

20 **a** Show that $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$.

b Hence solve the equation $\cos 3x + \cos x = 3\cos 2x$ for $0 \leqslant \theta < 2\pi$.

21 Find the exact value of:

a $\tan(\arctan 3 - \arctan 2)$

b $\tan\left(2\arctan\left(\frac{1}{2}\right)\right)$.

22 Let θ be the acute angle between the lines with equations $y = x$ and $y = 2x$.
Find the exact value of $\tan\theta$.

23 **a** Use the identity for $\tan(A+B)$ to express $\tan 3x$ in terms of $\tan x$.

b Hence solve the equation $\tan x + \tan 3x = 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

24 **a** Show that $\sin\left(2x + \frac{\pi}{2}\right) \equiv \cos 2x$.

b Hence solve the equation $\sin 3x = \cos 2x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

c Show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$ and hence express $\cos 2x - \sin 3x$ in terms of $\sin x$.

Let $f(s) = 4s^3 - 2s^2 - 3s + 1$.

d Show that $(s-1)$ is a factor of $f(s)$ and factorize $f(s)$ completely.

e Hence find the exact value of $\sin\left(\frac{\pi}{10}\right)$.

25 Compactness is a measure of how compact an enclosed region is. The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$ where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

a If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$.

If $n > 1$ and odd, it can be shown that $C = \dfrac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

b Find the regular polygon with the least number of sides for which the compactness is more than 0.99.

c Comment briefly on whether C is a good measure of compactness.

Mathematics HL November 2014 Paper 2 Q9

26 **a** Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p.

b Hence find the value of $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$.

Mathematics HL May 2013 Paper 1 TZ2 Q10

4 Complex numbers

ESSENTIAL UNDERSTANDINGS

- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to work with the imaginary number i
- how to find sums, products and quotients of complex numbers in Cartesian form
- how to represent complex numbers geometrically on the complex plane (Argand diagram)
- how to find the modulus and argument of a complex number
- how to write a complex number in modulus–argument (polar) form
- how to find sums, products and quotients of complex numbers in modulus–argument form
- how to write a complex number in Euler form
- how to find sums, products and quotients of complex numbers in Euler form
- how to use the fact that roots of any polynomial with real coefficients are either real or occur in complex conjugate pairs
- how to use De Moivre's theorem to find powers of complex numbers
- how to use De Moivre's theorem to find roots of complex numbers
- how to use De Moivre's theorem to find trigonometric identities.

CONCEPTS

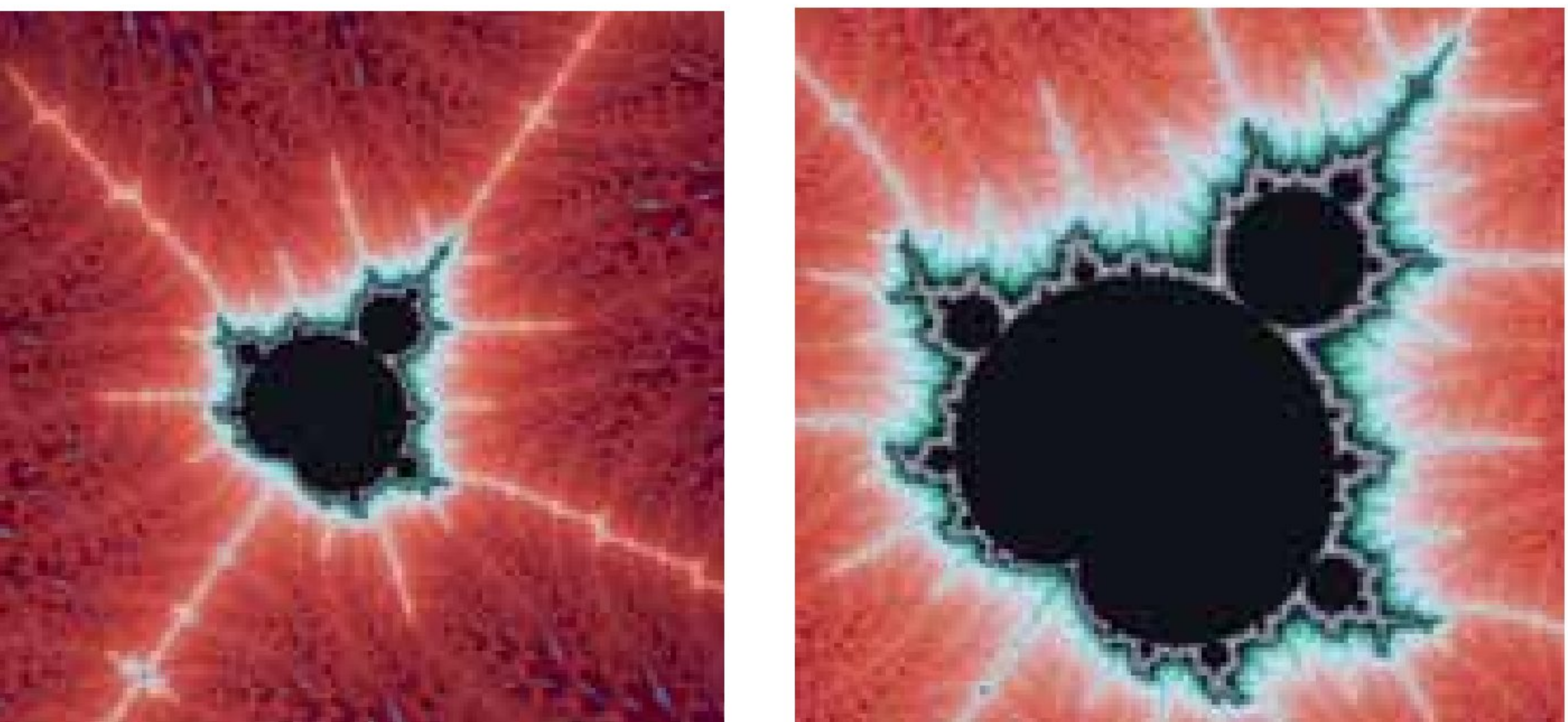
The following concepts will be addressed in this chapter:

- **Patterns** in numbers inform the development of algebraic tools that can be applied to find unknowns.
- **Representing** complex numbers in different forms allows us to easily carry out seemingly difficult calculations.

LEARNER PROFILE – Thinkers

Is mathematics an art or a science? How much creativity do you use when tackling a mathematics problem? What would mathematics and art assignments look like if their teaching approaches were reversed?

Figure 4.1 Can you imagine a shape which has a finite area but infinite perimeter?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Solve the equation $3x^2 - 6x + 2 = 0$, giving your answer in its simplest form.
2. Express $\dfrac{5-\sqrt{5}}{3+\sqrt{5}}$ in the form $a + \sqrt{5}b$, where $a, b \in \mathbb{Z}$.
3. State the value of:
   - a $\sin\dfrac{\pi}{4}$
   - b $\cos\left(-\dfrac{2\pi}{3}\right)$.
4. Express:
   - a $\sin\dfrac{\pi}{5}\cos\dfrac{\pi}{10} + \cos\dfrac{\pi}{5}\sin\dfrac{\pi}{10}$ in the form $\sin\theta$, stating the value of θ
   - b $\cos\dfrac{\pi}{12}\cos\dfrac{\pi}{8} - \sin\dfrac{\pi}{12}\sin\dfrac{\pi}{8}$ in the form $\cos\theta$, stating the value of θ.
5. Express 4^x in the form e^{kx}, stating the value of the constant k.
6. Given that 3 and −5 are zeroes of the quadratic polynomial p(x), write p(x) in factorized form.
7. Find an expression for the infinite sum $1 + x + x^2 + \ldots$

You are the Researcher

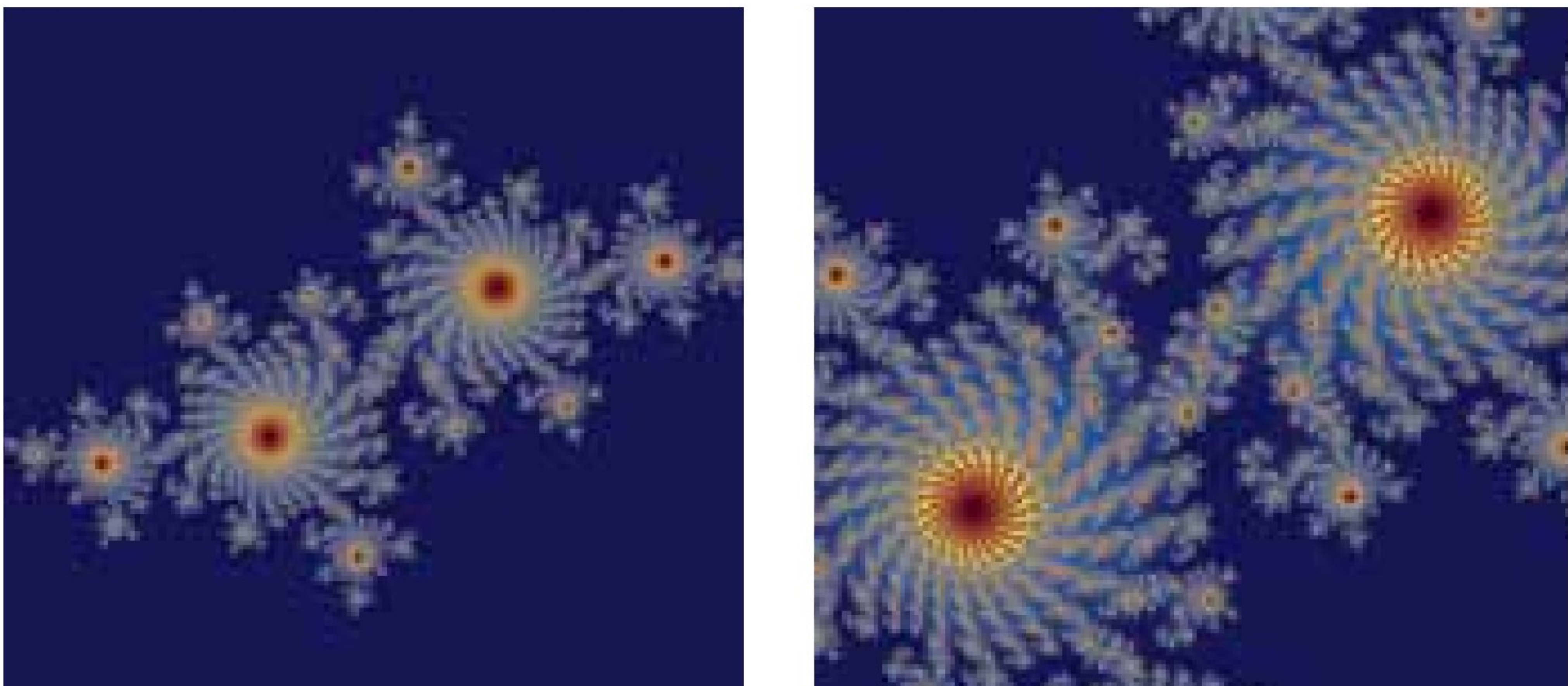
The pictures in Figure 4.1 are called fractals. This comes from the fact that they are a fractional dimension shaped. You are used to 2D and 3D shapes, but what does it mean for the shapes to have fractional dimension?

Starter Activity

Can you have half of a drop of water? Can you show someone minus two pens? Can you imagine a shape which has a finite area but infinite perimeter? Can you draw a line with an irrational length? Do you know a number which squares to give minus one?

All of these are problems which at some time in history were considered impossible, but their study has opened up new areas of mathematics with sometimes surprising applications.

In this chapter we shall extend the number line to another dimension! The new number, i, allows us to solve equations that we had previously said 'have no real solution'. However, if that were the only purpose of complex numbers, they probably would have been discarded as a mathematical curiosity. We shall also explore how their geometric representations give us a new way of doing trigonometry.



4A Cartesian form

The number i

There are no real numbers that solve the equation $x^2 = -1$. But there is an imaginary number that solves this equation: i.

KEY POINT 4.1

$$i = \sqrt{-1}$$

This number behaves just like a real constant.

WORKED EXAMPLE 4.1

Simplify the following:

a i^3

b i^4.

Consider 3 as 2 + 1, then use a rule of exponents to isolate i^2

$i^2 = (\sqrt{-1})^2 = -1$

a $i^3 = i^2 \times i$
$= (-1)i$
$= -i$

Consider 4 as 2 × 2, then use a rule of exponents to isolate i^2

Again $i^2 = -1$

b $i^4 = (i^2)^2$
$= (-1)^2$
$= 1$

We can now find the square root of any negative number, and therefore solve any quadratic equation with a negative discriminant.

WORKED EXAMPLE 4.2

a Find $\sqrt{-16}$.

b Hence, solve the equation $x^2 - 6x + 13 = 0$.

Split the square root as usual using $\sqrt{ab} = \sqrt{a}\sqrt{b}$

a $\sqrt{-16} = \sqrt{16}\sqrt{-1}$
$= 4i$

Use the quadratic formula

$\sqrt{-16} = 4i$ from part **a**

Cancel a factor of 2 as usual

b $x = \dfrac{-(6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2}$
$= \dfrac{6 \pm \sqrt{-16}}{2}$
$= \dfrac{6 \pm 4i}{2}$
$= 3 \pm 2i$

The solutions in Worked Example 4.2, $3 + 2i$ and $3 - 2i$, are examples of **complex numbers** written in **Cartesian form**.

The variable z (rather than x) is often used for complex numbers and the set of all complex numbers is given the symbol $\mathbb{C}$.

KEY POINT 4.2

A complex number z can be written in Cartesian form as

$$z = x + iy$$

where $x, y \in \mathbb{R}$.

The **real part** of the complex number $z = x + iy$ is x: this is denoted by $\text{Re}(z)$. The **imaginary part** of z is y: this is denoted by $\text{Im}(z)$.

So, for example, if $z = 3 - 2i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -2$.

The solutions to the quadratic equation in Worked Example 4.2 differ only by the sign of the imaginary part. One is said to be the **complex conjugate** of the other.

KEY POINT 4.3

If $z = x + iy$, then its complex conjugate, z^*, is

$$z^* = x - iy$$

Sums, products and quotients in Cartesian form

Adding, subtracting and multiplying complex numbers works in ways you might expect, with real and imaginary parts being grouped together.

Many calculators can do arithmetic with complex numbers, so make sure you know how to use this feature.

WORKED EXAMPLE 4.3

$z = 2 + i$ and $w = 5 - 3i$

Find

a $z + w$ **b** $z - w$ **c** zw.

Group real and imaginary parts

a $z + w = 2 + i + 5 - 3i$
$= 7 - 2i$

Group real and imaginary parts

b $z - w = 2 + i - (5 - 3i)$
$= 2 + i - 5 + 3i$
$= -3 + 4i$

Expand the brackets as usual

c $zw = (2 + i)(5 - 3i)$
$= 10 - 6i + 5i - 3i^2$

$i^2 = -1$

$= 10 - 6i + 5i + 3$

Group real and imaginary parts

$= 13 - i$

Division is a little more involved and relies on the following result.

KEY POINT 4.4

If $z = x + iy$, then

$$zz^* = x^2 + y^2$$

This means that the product of a complex number and its complex conjugate is always real.

Proof 4.1

Prove that zz^* is always real.

Let $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$

Then $z^* = x - \mathrm{i}y$.

So, $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y)$

$= x^2 - \mathrm{i}xy + \mathrm{i}yx - \mathrm{i}^2y^2$ ($\mathrm{i}^2 = -1$)

$= x^2 - (-y^2)$

$= x^2 + y^2$

which is real. (Remember that x and y are real numbers)

WORKED EXAMPLE 4.4

$z = 7 + 11\mathrm{i}$ and $w = 4 - \mathrm{i}$

Find $\frac{z}{w}$.

$$\frac{z}{w} = \frac{7+11\mathrm{i}}{4-\mathrm{i}}$$

The denominator can be made real by multiplying by its complex conjugate. The numerator must be multiplied by this too

$$= \frac{7+11\mathrm{i}}{4-\mathrm{i}} \times \frac{4+\mathrm{i}}{4+\mathrm{i}}$$

Expand the numerator: $(7+11\mathrm{i})(4+\mathrm{i})$

The denominator is given by $zz^* = x^2 + y^2$

$$= \frac{28 + 7\mathrm{i} + 44\mathrm{i} + 11\mathrm{i}^2}{4^2 + 1^2}$$

$\mathrm{i}^2 = -1$

$$= \frac{28 + 7\mathrm{i} + 44\mathrm{i} - 11}{17}$$

$$= \frac{17 + 51\mathrm{i}}{17}$$

$$= 1 + 3\mathrm{i}$$

Tip

This procedure is like rationalising the denominator with surds.

The idea of separating real and imaginary parts is very useful in solving equations.

WORKED EXAMPLE 4.5

Find the complex number z such that $5z + 3z^* = 8 - 4\mathrm{i}$.

Let $z = x + \mathrm{i}y$.

Then

Expand and group together real and imaginary parts on the LHS

$$5z + 3z^* = 8 - 4\mathrm{i}$$
$$5(x + \mathrm{i}y) + 3(x - \mathrm{i}y) = 8 - 4\mathrm{i}$$
$$5x + 5y\mathrm{i} + 3x - 3y\mathrm{i} = 8 - 4\mathrm{i}$$
$$8x + 2y\mathrm{i} = 8 - 4\mathrm{i}$$

Equate real and imaginary parts on either side

Re : $8x = 8$

$x = 1$

Im : $2y = -4$

$y = -2$

So, $z = 1 - 2\mathrm{i}$

The complex plane

While real numbers can be represented on a one-dimensional number line, complex numbers need two-dimensional coordinates. The x-axis represents the real part of the number and the y-axis the imaginary part.

This is referred to as the **complex plane** or an **Argand diagram**.

WORKED EXAMPLE 4.6

$z = 3 + 4\text{i}$

Represent on the complex plane:

a z **b** $-z$ **c** z^*

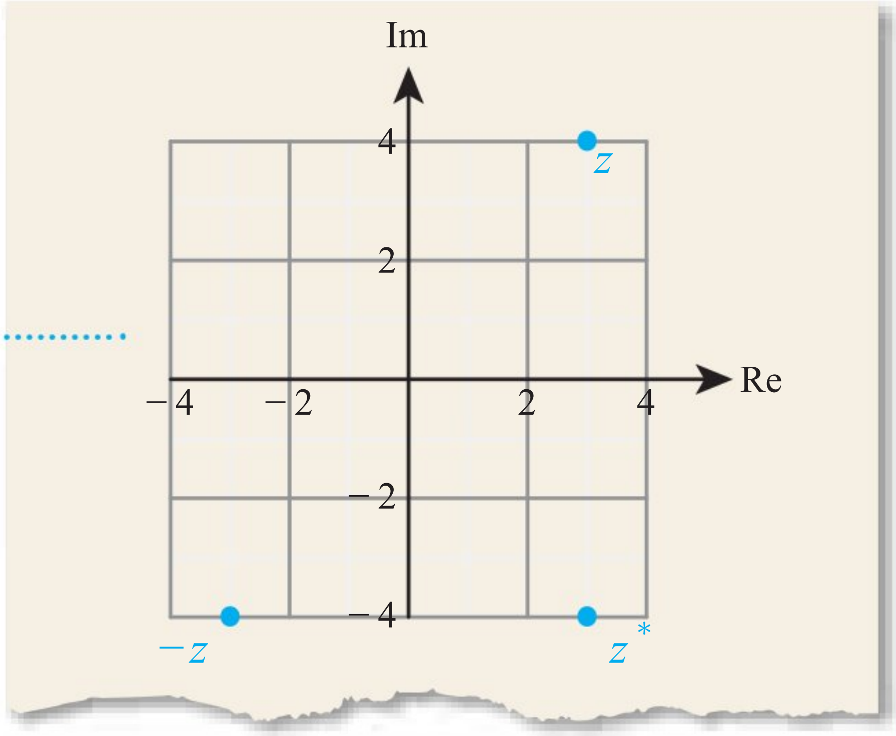
z has coordinates (3, 4)

$-z = -(3 + 4\text{i}) = -3 - 4\text{i}$

So this has coordinates (−3, −4)

$z^* = 3 - 4\text{i}$

So this has coordinates (3, −4)



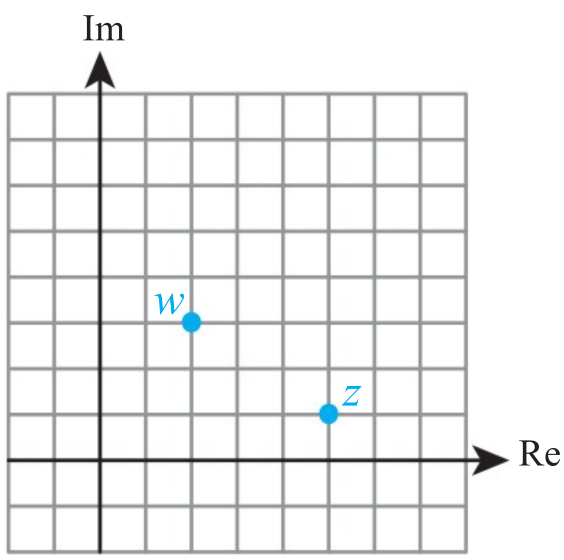
Who was Argand? What contribution did he make to our understanding of complex numbers?

Addition and subtraction can be represented neatly on the complex plane.

Complex numbers can also be represented as vectors in the complex plane. The methods in Worked Examples 4.7 and 4.8 are essentially vector addition and subtraction, which you will meet in Section 8A.

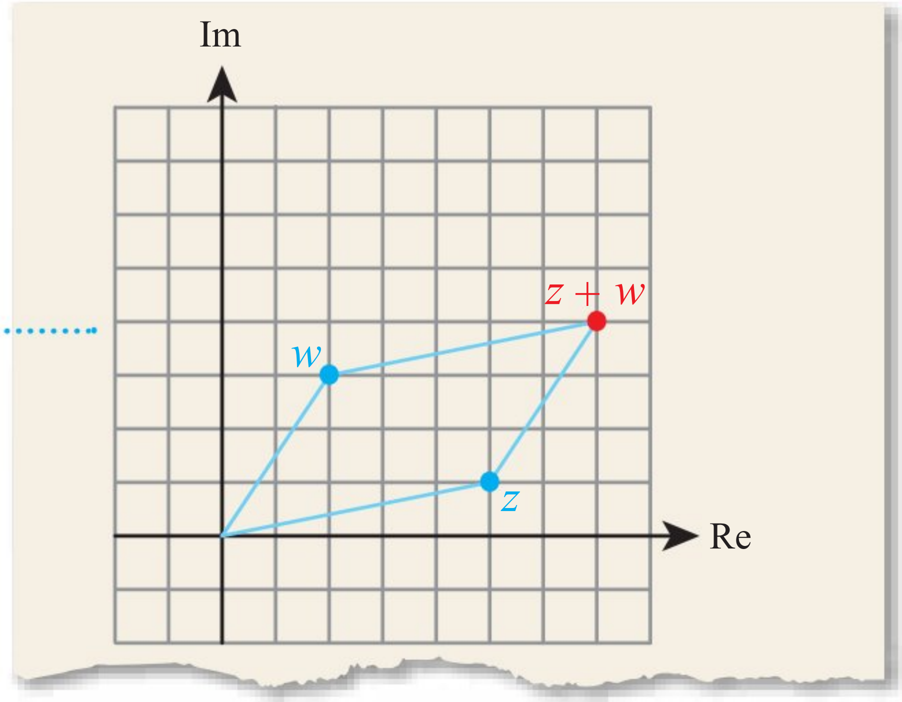
WORKED EXAMPLE 4.7

The complex numbers z and w are shown on the Argand diagram below.



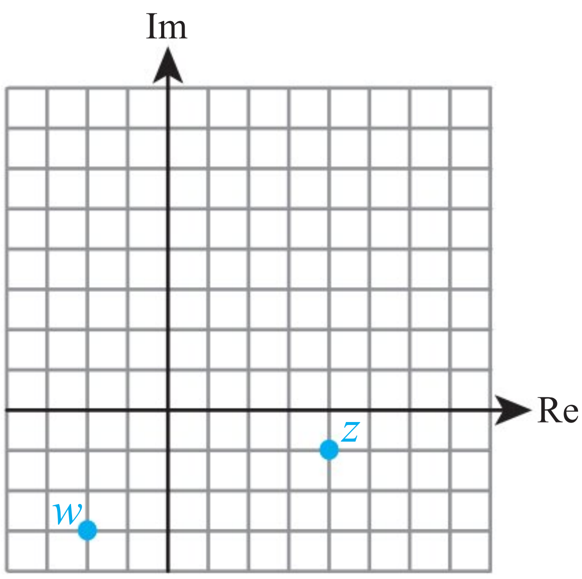
Mark on the complex number $z + w$.

The point representing $z + w$ can be found by creating a parallelogram with the relevant points as vertices



WORKED EXAMPLE 4.8

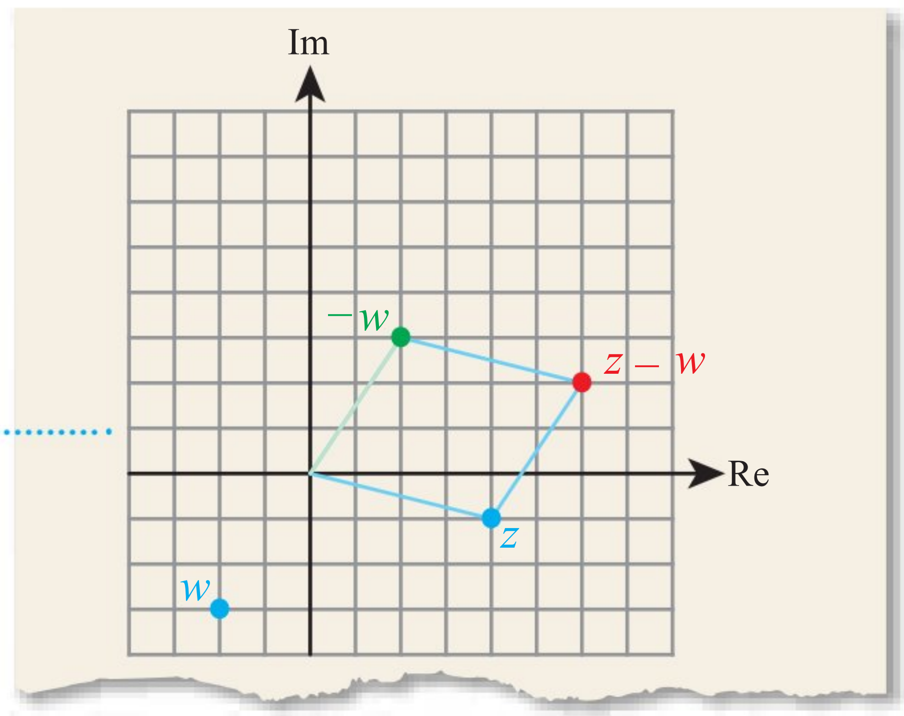
The complex numbers z and w are shown on the Argand diagram below.



Mark on the complex number $z - w$.

The point representing $z - w$ can be found as for addition by thinking of the process as $z + (-w)$

The complex number $-w$ is w rotated by 180° around the origin



CONCEPTS – PATTERNS AND REPRESENTATIONS

When you first saw a number line, it probably started at the origin and extended to the right. You might have been surprised when somebody suggested that it could also go to the left to represent negative numbers, but after a while you probably got used to that. We are now extending this **pattern** to going up and down. As before, you might start off unsure about that **representation**, but you will eventually get used to it. You might ask whether this pattern continues and if there will be another dimension added in your future work. Mathematicians have shown that we do not need another 'new' type of number to solve all the current types of equations studied, but that does not mean that a use would not be found for them in future research.

Exercise 4A

For questions 1 to 4, use the method demonstrated in Worked Example 4.1 to simplify the expression.

1 **a** i^5 **b** i^6

2 **a** $-4i^3$ **b** $5i^4$

3 **a** $(3i)^2$ **b** $(2i)^3$

4 **a** i^{4n} for n, a positive integer **b** i^{4n+2} for n, a positive integer

For questions 5 to 8, use the method demonstrated in Worked Example 4.2 to solve the equation.

5 **a** $x^2 = -9$ **b** $x^2 = -36$

6 **a** $x^2 = -8$ **b** $x^2 = -75$

7 **a** $x^2 - 2x + 5 = 0$ **b** $x^2 - 4x + 13 = 0$

8 **a** $2x^2 + 4x + 3 = 0$ **b** $3x^2 - 2x + 2 = 0$

For questions 9 to 12, use the method demonstrated in Worked Example 4.3 to simplify the expression.

9 **a** $(2 - i) + (9 + 5i)$ **b** $(-3 - 7i) + (-1 + 9i)$

10 **a** $(2 + i) - (1 + 3i)$ **b** $(-4 + 7i) - (2 - 3i)$

11 **a** $(2 + 3i)(1 - 2i)$ **b** $(3 + i)(5 - i)$

12 **a** $(3 + i)^2$ **b** $(4 - 3i)^2$

For questions 13 to 15, use the method demonstrated in Worked Example 4.4 to write each expression in the form $x + iy$.

13 **a** $\frac{10}{2+i}$ **b** $\frac{6i}{1-i}$

14 **a** $\frac{10-5i}{1-2i}$ **b** $\frac{7+i}{3+4i}$

15 **a** $\frac{3+2i}{5-i}$ **b** $\frac{5-4i}{2+3i}$

For questions 16 to 19, use the method of equating real and imaginary parts demonstrated in Worked Example 4.5 to find the solutions.

16 **a** $a, b \in \mathbb{R}$ such that $(a + 3i)(1 - 2i) = 11 - bi$
b $a, b \in \mathbb{R}$ such that $(4 - ai)(3 + i) = b + 13i$

17 **a** $a, b \in \mathbb{R}$ such that $(7 - ai)(2 - i) = b - 4i$
b $a, b \in \mathbb{R}$ such that $(2a - i)(3 - 5i) = -2 + bi$

18 **a** $z \in \mathbb{C}$ such that $z + 3i = 2z^* + 4$
b $z \in \mathbb{C}$ such that $3z + 2z^* = 5 + 2i$

19 **a** $z \in \mathbb{C}$ such that $z + 2z^* = 2 - 7i$
b $z \in \mathbb{C}$ such that $2z + i = -3 - iz^*$

For questions 20 to 23, use the method demonstrated in Worked Example 4.6 to represent each complex number on an Argand diagram.

20 **a** **i** $z = 5 + 2i$ **ii** $-z$ **iii** z^*
b **i** $z = -2 + 3i$ **ii** $-z$ **iii** z^*

21 **a** **i** $z = 3 - 4i$ **ii** $2z$ **iii** iz
b **i** $z = -4 - 5i$ **ii** $2z$ **iii** iz

22 **a** **i** $z = -3 + 2i$ **ii** $w = 5 + 3i$ **iii** $z + w$
b **i** $z = 2 + 2i$ **ii** $w = 1 + 3i$ **iii** $z + w$

23 **a** **i** $z = -4 - i$ **ii** $w = -2 - 3i$ **iii** $z - w$
b **i** $z = -2 + 5i$ **ii** $w = 6 - 2i$ **iii** $z - w$

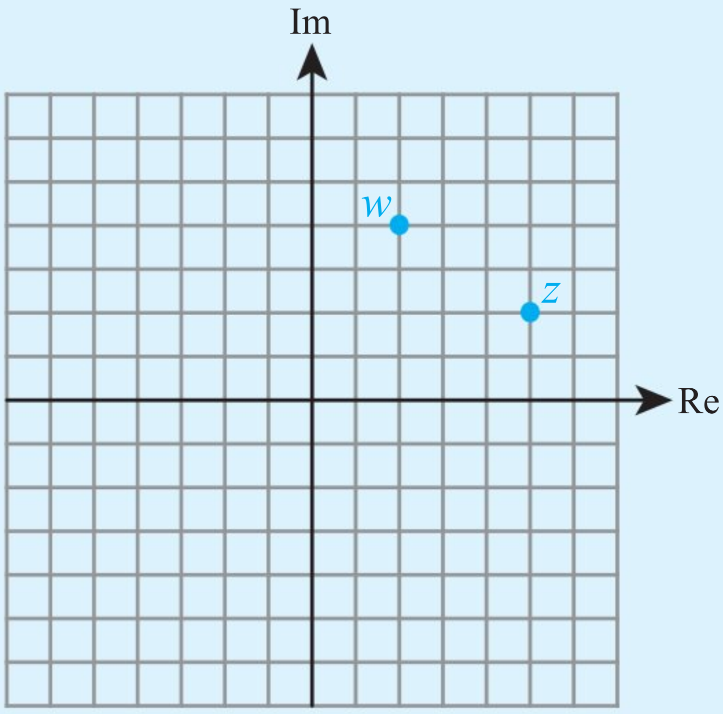
For questions 24 and 25, use the methods demonstrated in Worked Examples 4.7 and 4.8 to add the points corresponding to the stated complex numbers to a copy of each complex plane.

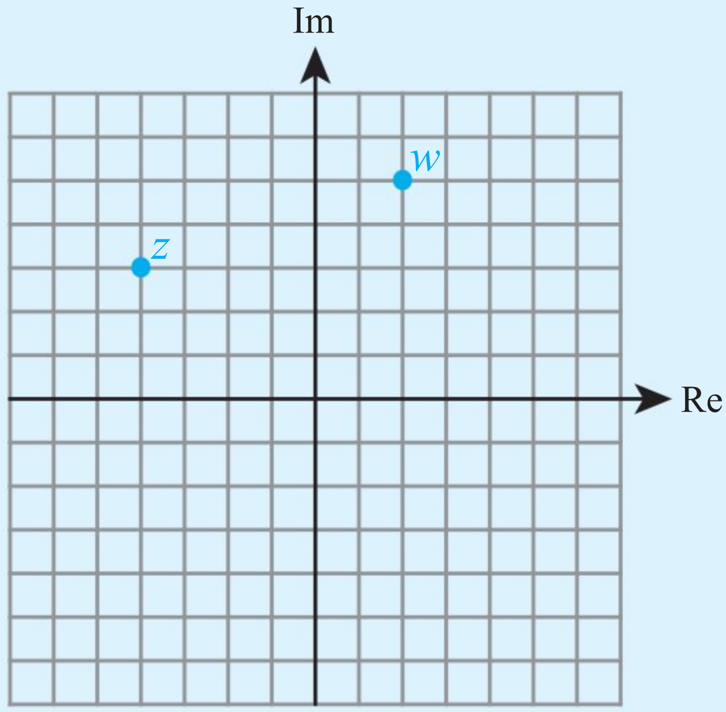
24 **a** **i** z^* **ii** $z + w$

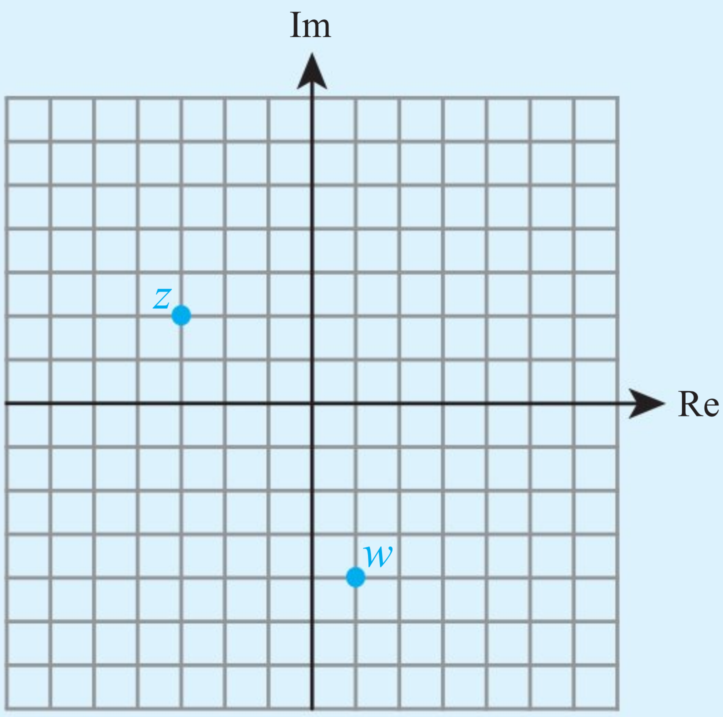
b **i** z^* **ii** $z + w$

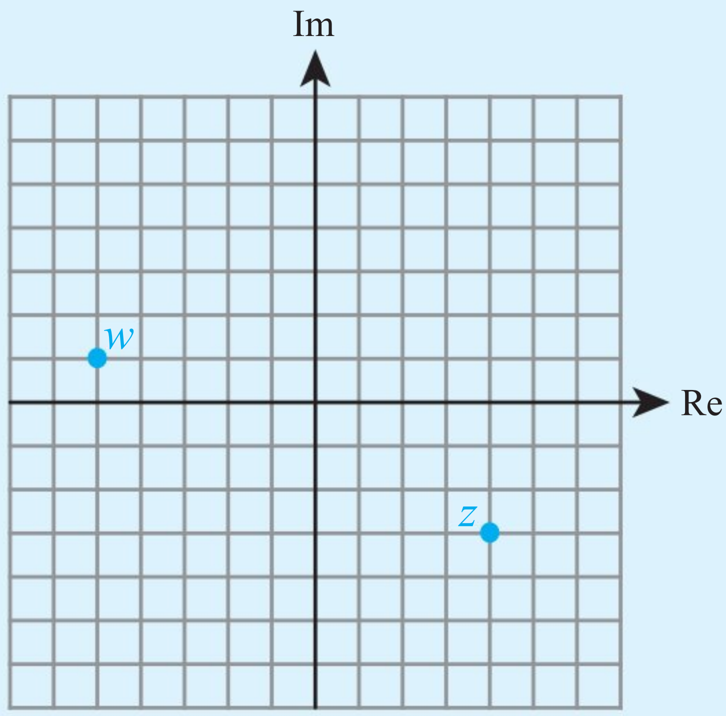
25 **a** **i** $-w$ **ii** $z - w$

b **i** $-w$ **ii** $z - w$









26 Solve the equation $5x^2 + 6x + 5 = 0$.

27 Given that $z = 7 + 3\text{i} - \dfrac{10\text{i}}{2+\text{i}}$, find z^*.

28 Find the complex number z such that $4z - 23 = 5\text{i}z + 2\text{i}$.

29 Find the complex number z such that $3\text{i}z - 2z^* = \text{i} - 4$.

30 Let $z = \dfrac{a + 3\text{i}}{a - 3\text{i}}$, $a \in \mathbb{R}$.

Find the values of a for which z is purely imaginary.

31 Prove that $(z^*)^2 = (z^2)^*$.

32 Solve the simultaneous equations

$$\begin{cases} 3z + \mathrm{i}w = 5 - 11\mathrm{i} \\ 2\mathrm{i}z - 3w = -2 + \mathrm{i} \end{cases}$$

where $z, w \in \mathbb{C}$.

33 Solve the simultaneous equations

$$\begin{cases} 2z - 3\mathrm{i}w = 9 + \mathrm{i} \\ (1+\mathrm{i})z + 4w = 1 + 10\mathrm{i} \end{cases}$$

where $z, w \in \mathbb{C}$.

34 Find the possible values of $a, b \in \mathbb{R}$ such that $(1 + a\mathrm{i})(1 + b\mathrm{i}) = b - a + 9\mathrm{i}$.

35 Let $z = \frac{7+\mathrm{i}}{2-\mathrm{i}} - \frac{3+\mathrm{i}}{a+2\mathrm{i}}$. Find the values of $a \in \mathbb{R}$ such that $\mathrm{Re}(z) = \mathrm{Im}(z)$.

36 By letting $z = x + \mathrm{i}y$, solve the equation $z^2 = -3 - 4\mathrm{i}$.

37 By letting $z = x + \mathrm{i}y$, solve the equation $z^2 = 8 - 6\mathrm{i}$.

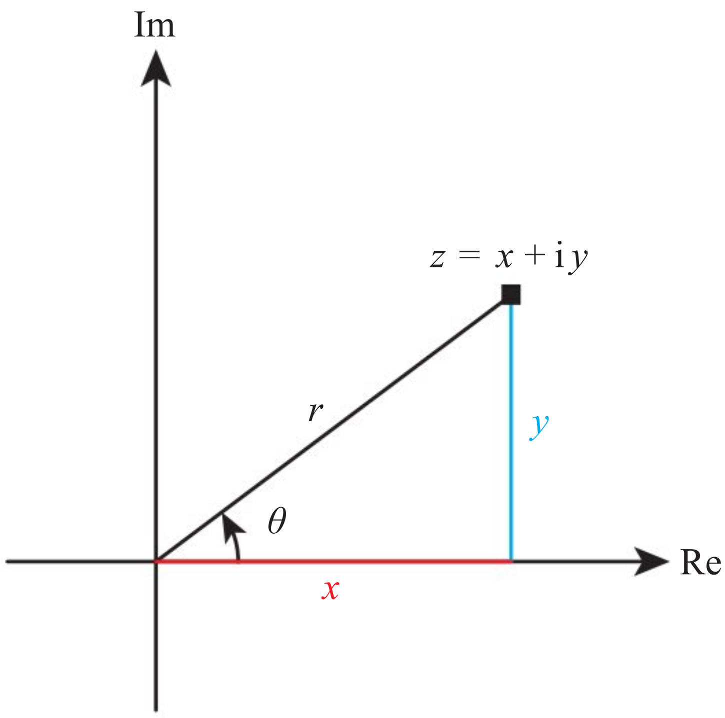
38 Solve $z^2 = z^*$.

4B Modulus–argument form and Euler form

Modulus–argument form

When a complex number, z, is represented on an Argand diagram, its distance from the origin is called the **modulus**, denoted by $|z|$ or r.

The angle made with the positive x-axis (measured anticlockwise and in radians) is called the **argument**, denoted by $\arg z$ or θ.



You can find the modulus and argument of a complex number from the diagram above.

KEY POINT 4.5

If $z = x + \mathrm{i}y$, then the modulus, r, and argument, θ, are given by

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \frac{y}{x}$

Tip

Always draw the complex number in an Argand diagram before finding the argument. It is important to know which angle you need to find.

The modulus must be positive. The argument can either be measured between 0 and 2π or between $-\pi$ and π. It will be made clear in the question which is required.

Using trigonometry we can write complex numbers in terms of their modulus and argument:

$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r\cos\theta + \mathrm{i}r\sin\theta \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

This form is so common that a shorthand is often used: $\operatorname{cis}\theta = \cos\theta + \mathrm{i}\sin\theta$.

KEY POINT 4.6

A complex number z can be written in **modulus–argument (polar) form** as

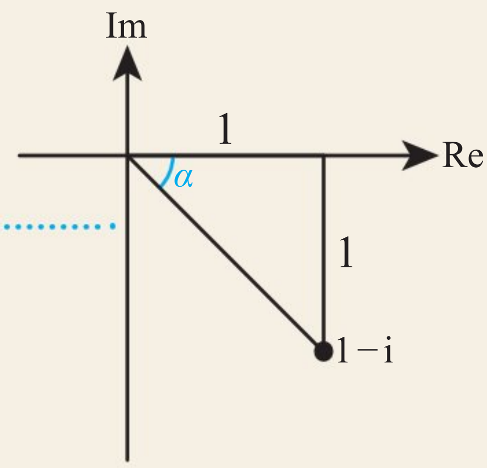
$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\operatorname{cis}\theta$$

WORKED EXAMPLE 4.9

Write $1 - \mathrm{i}$ in modulus–argument form, with the argument between 0 and 2π.

Use $r = \sqrt{x^2 + y^2}$ ⋯ $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

To find the argument, draw a diagram to see where the complex number actually is ⋯



Find the angle α ⋯ $\tan\alpha = \frac{1}{1}$

$\alpha = \frac{\pi}{4}$

From the diagram, $\theta = 2\pi - \alpha$ ⋯ $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

Modulus–argument form is $z = r(\cos\theta + \mathrm{i}\sin\theta)$ ⋯ So,

$$z = \sqrt{2}\left(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4}\right)$$

WORKED EXAMPLE 4.10

A complex number z has modulus 4 and argument $\frac{\pi}{6}$.

Write the number in Cartesian form.

Use $x = r\cos\theta$ and $y = r\sin\theta$ ⋯ $x = 4\cos\frac{\pi}{6} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

$y = 4\sin\frac{\pi}{6} = \frac{4}{2} = 2$

Cartesian form is $z = x + \mathrm{i}y$ ⋯ So,

$z = 2\sqrt{3} + 2\mathrm{i}$

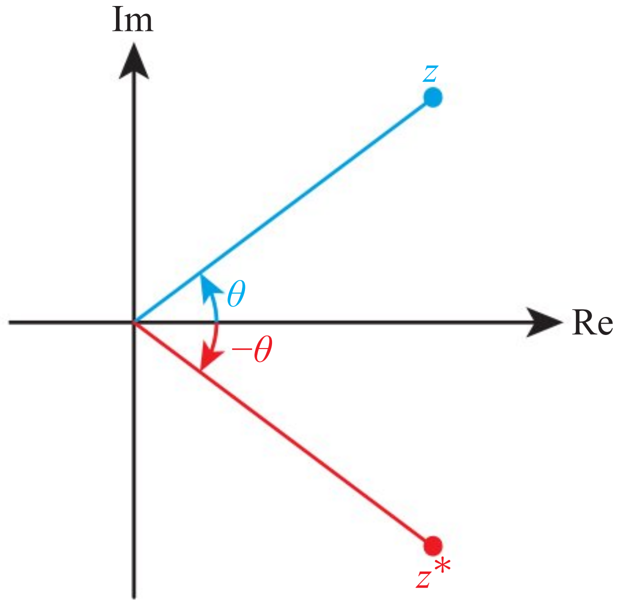
Be the Examiner 4.1

Find the argument of $z = -5 - 2\text{i}$, where $-\pi < \arg z \leqslant \pi$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\arctan\left(\frac{2}{5}\right) = 0.381$ $\pi - 0.381 = 2.76$ So, $\arg z = -2.76$	$\arg z = -\arctan\left(\frac{2}{5}\right)$ $= -2.76$	$\arctan\left(\frac{-2}{-5}\right) = 0.381$ $\arg z = 2\pi - 0.381$ $= 5.90$

You have already seen that the complex conjugate is represented in an Argand diagram by reflection in the x-axis. Modulus–argument form, with $-\pi < \theta \leqslant \pi$, therefore gives a nice form for the conjugate.



KEY POINT 4.7

If $z = r\,\text{cis}\,\theta$, then

$$z^* = r\,\text{cis}(-\theta)$$

When writing a number in modulus–argument form, it is important to notice that there must be a plus sign between the two terms.

WORKED EXAMPLE 4.11

Write $z = \cos\left(\frac{\pi}{5}\right) - \text{i}\sin\left(\frac{\pi}{5}\right)$ in the form cis θ, with the argument between $-\pi$ and π.

Use $-\sin\theta = \sin(-\theta)$ to remove the negative sign in between terms

$$z = \cos\left(\frac{\pi}{5}\right) - \text{i}\sin\left(\frac{\pi}{5}\right)$$

$$= \cos\left(\frac{\pi}{5}\right) + \text{i}\sin\left(-\frac{\pi}{5}\right)$$

Use $\cos\theta = \cos(-\theta)$ so that the arguments of sin and cos are the same

$$= \cos\left(-\frac{\pi}{5}\right) + \text{i}\sin\left(-\frac{\pi}{5}\right)$$

$$= \text{cis}\left(-\frac{\pi}{5}\right)$$

Notice from Worked Example 4.11 and Key Point 4.7 that $\cos\theta - \text{i}\sin\theta = \text{cis}(-\theta)$ is the complex conjugate of cis θ.

Sums, products and quotients in modulus–argument form

Addition and subtraction are straightforward in Cartesian form, but multiplication and division are more difficult. However, these operations are much easier in modulus–argument form.

KEY POINT 4.8

- $|zw| = |z||w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $\arg(zw) = \arg z + \arg w$
- $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

CONCEPTS – REPRESENTATION

Point 4.8 highlights one of the main reasons complex numbers are important. If you think about what $\arg(zw) = \arg z + \arg w$. means, it can be interpreted as saying that multiplying by a complex number has the effect of rotating the number around the origin on the Argand diagram. Complex numbers add like vectors (i.e. consecutive journeys), but multiply like rotations. This means that they very naturally **represent** physical situations involving rotations or oscillations, including things like alternating currents or vibrating bridges. You will see as this chapter progresses that they are therefore closely linked to trigonometric ratios.

Proof 4.2

Prove that $|zw| = |z||w|$ and $\arg(zw) = \arg z + \arg w$.

Define z and w in modulus–argument form

Let

$$z = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \text{ and}$$
$$w = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

Multiply together and group real and imaginary parts

$$zw = r_1 r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathrm{i}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))$$

Using the compound angle formulae:
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\sin(A+B) = \sin A\cos B + \sin B\cos A$

$$= r_1 r_2(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2))$$
$$= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$$

This is in the form $r\operatorname{cis}\theta$ so, by comparison, you can state the modulus and argument of zw

So,

$$|zw| = r_1 r_2$$
$$\arg(zw) = \theta_1 + \theta_2$$
i.e. $|zw| = |z||w|$ and $\arg(zw) = \arg z + \arg w$

A similar proof gives the result for dividing complex numbers.

WORKED EXAMPLE 4.12

$z = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ and $w = 5\operatorname{cis}\left(\frac{\pi}{4}\right)$

Write zw in the form $r\operatorname{cis}\theta$.

Multiply the moduli and add the arguments: $|zw| = 2 \times 5 = 10$

$|z| = 2$ and $\arg z = \frac{\pi}{3}$; $|w| = 5$ and $\arg w = \frac{\pi}{4}$:

$\arg(zw) = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

So,

$zw = 10\operatorname{cis}\left(\frac{7\pi}{12}\right)$

WORKED EXAMPLE 4.13

$z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $w = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Write $\frac{z}{w}$ in Cartesian form.

Divide the moduli, and subtract the arguments: $\left|\frac{z}{w}\right| = \frac{8}{4} = 2$

$|z| = 8$ and $\arg z = \frac{2\pi}{3}$; $|w| = 4$ and $\arg w = \frac{\pi}{6}$:

$\arg\frac{z}{w} = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$

So, $\frac{z}{w} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Then convert to Cartesian form:

$= 2(0 + i)$

$= 2i$

Euler form

The rules for finding the argument when multiplying or dividing complex numbers are just the same as the rules of indices, that is, add the arguments when multiplying and subtract the arguments when dividing.

This suggests that complex numbers can be written in an exponential form with the argument as the exponent.

You might wonder why we choose e to be the base in Euler form. When you study Maclaurin series in Chapter 11 you will find some justification for this decision.

KEY POINT 4.9

A complex number z can be written in **Euler form** as

$$z = re^{i\theta}$$

where r is the modulus of z and θ is the argument of z, so that $e^{i\theta} = \cos\theta + i\sin\theta$.

Conventionally the arguments in Euler form satisfy $0 \leqslant \theta < 2\pi$, however numbers are not uniquely represented in Euler form and you should be able to deal with arguments outside of that range. For example, 1 can be written as e^{0i} or $e^{2\pi i}$ or $e^{-4\pi i}$.

TOK Links

How is new knowledge created in mathematics? Is analogy a valid way of creating knowledge? If I define e^{ix} to be $\cos x + i\sin x$ because it seems to have similar properties, does that make it true?

This allows us to use the rules of indices on complex numbers.

WORKED EXAMPLE 4.14

$z = 12e^{i\frac{2\pi}{5}}$ and $w = 4e^{i\frac{\pi}{3}}$

Find, in Euler form,

a zw **b** $\frac{z}{w}$.

Follow the rules of exponents by adding the powers

a $zw = 12e^{i\frac{2\pi}{5}} \times 4e^{i\frac{\pi}{3}}$

$= 48e^{i\left(\frac{2\pi}{5}+\frac{\pi}{3}\right)}$

$= 48e^{i\left(\frac{6\pi}{15}+\frac{5\pi}{15}\right)}$

$= 48e^{i\frac{11\pi}{15}}$

Follow the rules of exponents by subtracting the powers

b $\frac{z}{w} = \frac{12e^{i\frac{2\pi}{5}}}{4e^{i\frac{\pi}{3}}}$

$= 3e^{i\left(\frac{2\pi}{5}-\frac{\pi}{3}\right)}$

$= 3e^{i\frac{\pi}{15}}$

Euler form is particularly useful for finding powers of complex numbers.

WORKED EXAMPLE 4.15

$z = 2e^{i\frac{\pi}{12}}$

Find z^3 in Cartesian form.

Follow the laws of exponents by applying the power 3 to both terms of the product…

$z^3 = \left(2e^{i\frac{\pi}{12}}\right)^3$

$= 2^3\left(e^{i\frac{\pi}{12}}\right)^3$

…and then multiplying $i\frac{\pi}{12}$ by 3

$= 8e^{i\frac{3\pi}{12}}$

$= 8e^{i\frac{\pi}{4}}$

Convert to Cartesian form by first changing to modulus–argument form using $e^{i\theta} = \cos\theta + i\sin\theta$ and then evaluating

$= 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 8\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= 4\sqrt{2} + 4\sqrt{2}i$

The use of the laws of indices for Euler form will be further justified by De Moivre's theorem in Section 4D.

We can also use Euler form to find complex powers of numbers.

WORKED EXAMPLE 4.16

Express 2^i in Cartesian form correct to three significant figures.

We need to change the base from 2 to e, so express 2 as $e^{\ln 2}$

$$2^i = \left(e^{\ln 2}\right)^i$$

Then proceed as before, simplifying the indices and using $e^{i\theta} = \cos\theta + i\sin\theta$

$$= e^{i\ln 2}$$
$$= \cos(\ln 2) + i\sin(\ln 2)$$
$$\approx 0.769 + 0.639i$$

TOK Links

From Euler's form we find the famous mathematical equation $e^{i\pi} + 1 = 0$. This is often described as mathematical poetry – it has the fundamental constants of arithmetic (1), calculus (e), geometry (π) and imaginary numbers (i). It uses all the fundamental operations: addition, multiplication and raising to a power. When these are combined together in just the right way, the answer is nothing!

Is there a role for aesthetics in mathematics? Could someone with no understanding of calculus and complex numbers appreciate this result in the same way you now can? Does our previous experience change what we find to be beautiful?

Exercise 4B

For questions 1 to 4, use the method demonstrated in Worked Example 4.9 to write the following in the form $r\,\text{cis}\,\theta$, where $-\pi < \theta \leqslant \pi$.

1 a -4 b 5

2 a $3i$ b $-2i$

3 a $\sqrt{3} + i$ b $2\sqrt{3} - 2i$

4 a $-1 - i$ b $-3 + 3i$

For questions 5 to 7, use the method demonstrated in Worked Example 4.9 to write the following in the form $r\,\text{cis}\,\theta$, where $0 \leqslant \theta < 2\pi$.

5 a $-5i$ b $-7i$

6 a $3 - 3\sqrt{3}i$ b $4 - 4i$

7 a $-2 - 2i$ b $-1 - \sqrt{3}i$

For questions 8 to 11, use the method demonstrated in Worked Example 4.10 to write z in Cartesian form in the following cases.

8 a $|z| = 10, \arg z = \left(-\frac{\pi}{2}\right)$ b $|z| = 8, \arg z = \frac{\pi}{2}$

9 a $|z| = 4, \arg z = \frac{\pi}{3}$ b $|z| = \sqrt{2}, \arg z = \frac{\pi}{4}$

10 a $|z| = 4\sqrt{3}, \arg z = \frac{3\pi}{4}$ b $|z| = 2, \arg z = \frac{2\pi}{3}$

11 a $|z| = 8, \arg z = \frac{11\pi}{6}$ b $|z| = 2, \arg z = \left(-\frac{\pi}{4}\right)$

For questions 12 to 14, use the method demonstrated in Worked Example 4.11 to write the following in the form $r\,\text{cis}\,\theta$, where $-\pi < \theta \leqslant \pi$.

12 a $7\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$ b $5\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)$

13 a $8\left(\cos\left(-\frac{2\pi}{7}\right) - i\sin\left(-\frac{2\pi}{7}\right)\right)$ b $2\left(\cos\left(-\frac{3\pi}{8}\right) - i\sin\left(-\frac{3\pi}{8}\right)\right)$

14 a $-3\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$ b $-4\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

For questions 15 and 16, use the methods demonstrated in Worked Examples 4.12 and 4.13 to write the following in the form $r \operatorname{cis} \theta$, where $0 < \theta \leqslant 2\pi$.

15 **a** $\left(2 \operatorname{cis} \frac{\pi}{3}\right)\left(6 \operatorname{cis} \frac{\pi}{5}\right)$

b $\left(\frac{1}{2} \operatorname{cis} \frac{3\pi}{4}\right)\left(10 \operatorname{cis} \frac{11\pi}{8}\right)$

16 **a** $\frac{15 \operatorname{cis} \pi/9}{5 \operatorname{cis} \pi/6}$

b $\frac{\operatorname{cis} 5\pi/7}{3 \operatorname{cis} \pi/7}$

For questions 17 and 18, use the methods demonstrated in Worked Examples 4.12 and 4.13 to write the following in Cartesian form.

17 **a** $\left(9 \operatorname{cis} \frac{2\pi}{5}\right)\left(\frac{2}{3} \operatorname{cis} \frac{\pi}{10}\right)$

b $\left(4 \operatorname{cis} \frac{17\pi}{9}\right)\left(\operatorname{cis} \frac{4\pi}{9}\right)$

18 **a** $\frac{\operatorname{cis} 5\pi/18}{\operatorname{cis}(-7\pi/18)}$

b $\frac{8 \operatorname{cis} 5\pi/12}{4 \operatorname{cis} \pi/6}$

For questions 19 and 21, use Key Point 4.9 to write the following numbers in Euler form.

19 **a** -3 **b** 2

20 **a** i **b** $-\sqrt{2}\mathrm{i}$

21 **a** $1 + \mathrm{i}$ **b** $\sqrt{3} + \mathrm{i}$

For questions 22 and 23, use Key Point 4.9 to write the following numbers in Euler form.

22 **a** $\operatorname{cis} 0.4$ **b** $\operatorname{cis} 1.8$

23 **a** $4 \operatorname{cis} \frac{\pi}{5}$ **b** $7 \operatorname{cis} \frac{\pi}{10}$

For questions 24 and 25, use Key Point 4.9 to write the following numbers in Cartesian form.

24 **a** $\mathrm{e}^{\mathrm{i}\pi}$ **b** $\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$

25 **a** $\sqrt{2}\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}}$ **b** $2\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}$

For questions 26 and 27, use the method demonstrated in Worked Example 4.14 to evaluate in Euler form.

26 **a** $3\mathrm{e}^{0.1\mathrm{i}} \times 5\mathrm{e}^{-0.2\mathrm{i}}$ **b** $\sqrt{2}\mathrm{e}^{0.5\mathrm{i}} \times \sqrt{2}\mathrm{e}^{1.5\mathrm{i}}$

27 **a** $4\mathrm{e}^{\pi\mathrm{i}} \div 2\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$ **b** $12\mathrm{e}^{\frac{\pi\mathrm{i}}{6}} \div 3\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$

For questions 28 and 29, use the method demonstrated in Worked Example 4.15 to evaluate in Cartesian form. Check your answers using your calculator.

28 **a** $\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}\right)^2$ **b** $\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{8}}\right)^2$

29 **a** $\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}\right)^3$ **b** $\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}\right)^4$

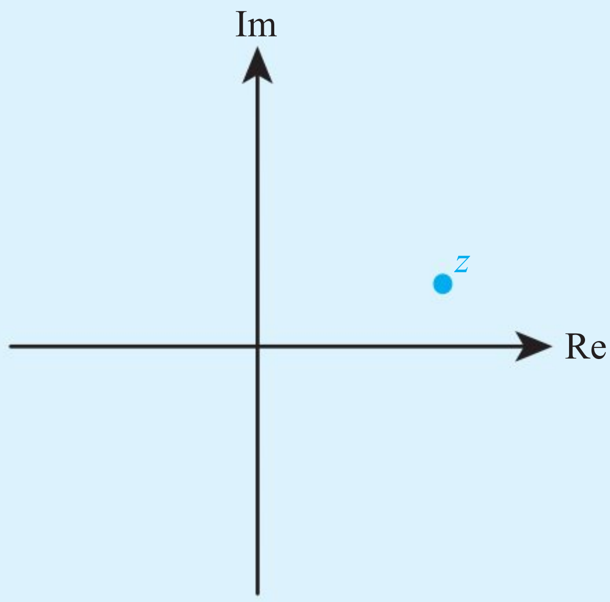
30 The complex number z is plotted on the Argand diagram.

The modulus of z is 2.

On a copy of this diagram sketch and label

a z^2

b $\mathrm{i}z$.



31 You are given $z = -2 + 2\text{i}$.

a Find $|z|$.

b Find $\arg z$.

c Hence write down the modulus and argument of z^2.

d Hence write z^2 in Cartesian form.

32 Simplify $\text{cis}\,0.6 \times \text{cis}\,0.4$.

33 In this question, $z = 1 + \text{i}$ and $w = 1 + \sqrt{3}\text{i}$.

a Find $\arg(w)$. **b** Find $\arg(zw)$.

34 Write in Cartesian form

a $\text{cis}\,\frac{\pi}{3} \times \text{cis}\,\frac{\pi}{6}$ **b** $\text{cis}\,\frac{\pi}{3} + \text{cis}\,\frac{\pi}{6}$.

35 Simplify $\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$.

36 Simplify $\dfrac{\cos 3\pi/5 + \text{i}\sin 3\pi/5}{\cos \pi/4 + \text{i}\sin \pi/4}$.

37 If z is a non-real complex number and arguments are defined to take values $0 \leqslant \theta < 2\pi$, evaluate with justification $\arg z + \arg z^*$.

38 Find an expression in terms of trigonometric functions for $\arg(a + \text{i}b)$ if $a < 0$ and $b < 0$. You may assume that $0 \leqslant \arg(a + \text{i}b) < 2\pi$.

39 Write $\text{i}\,\text{cis}\,\theta$ in modulus–argument form.

40 Write $1 + \text{i}\tan\theta$ in modulus–argument form.

41 Use a counterexample to prove that it is not always the case that $\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2)$.

42 The complex numbers z and w have arguments between 0 and π.

Given that $zw = -4\sqrt{2} + 4\sqrt{2}\text{i}$ and $\frac{z}{w} = 1 + \sqrt{3}\text{i}$, find the modulus and argument of z and the modulus and argument of w.

43 If $z = 6 + 8\text{i}$ and $|w| = 5$, find w if $|z + w| = |z| + |w|$.

44 **a** Prove that $\dfrac{1}{(\cos x + \text{i}\sin x)} = \cos x - \text{i}\sin x$.

b If $|z| = 1$, simplify $z + \frac{1}{z}$.

45 Sketch the curves in the Argand diagram described by

a $|z| = 2$ **b** $\arg z = \frac{\pi}{6}$ **c** $\text{Re}(z)^2 = \text{Im}(z)$.

46 If $|z - 5\text{i}| = 3$, find the smallest possible value of $|z|$.

47 Prove that $\dfrac{1 + \text{e}^{2\text{i}x}}{1 - \text{e}^{2\text{i}x}} = \text{i}\cot x$.

48 The complex numbers z and w are defined by $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\text{i}$ and $w = 1 + \sqrt{3}\text{i}$.

a Find

i $|z|$ **ii** $\arg z$ **iii** $|w|$ **iv** $\arg w$.

b Find $\frac{w}{z}$ in modulus–argument form.

c Find $\frac{w}{z}$ in Cartesian form.

d Hence find an exact surd expression for $\cos\frac{\pi}{12}$.

49 Find $z \in \mathbb{C}$ such that $|z| + z = 8 + 4\text{i}$.

50 **a** Given that $|z| = |z - 1|$, find $\text{Re}(z)$.

b Sketch all the points with $|z| = |z - 1|$ on an Argand diagram.

51 An equilateral triangle is drawn on an Argand diagram with the centre at the origin. One vertex is at the complex number w.

a Find the complex numbers where the other two vertices are located.

b Find an expression for the length of the side of the equilateral triangle.

52 Find an exact expression for 3^{i} in the form $r\operatorname{cis}\theta$.

53 **a** Show that i^{i} is a real number.

b Use the approximation $\mathrm{e} \approx \pi \approx 3$ to estimate the value of i^{i} to one decimal place.

54 **a** Write -2 in Euler form.

b Hence suggest a value for $\ln(-2)$.

55 **a** Write i in Euler form.

b Hence suggest a value for $\ln(\mathrm{i})$.

c Explain why there is more than one plausible value for $\ln(\mathrm{i})$.

56 **a** Find $\operatorname{Re}\left(\mathrm{e}^{(1+\mathrm{i})x}\right)$.

b Find the integral $\int \mathrm{e}^{x}\sin x \, \mathrm{d}x$.

57 **a** If

$$C = \sum_{k=0}^{k=n} \cos k\theta$$

and

$$S = \sum_{k=0}^{k=n} \sin k\theta$$

show that $C + \mathrm{i}S$ forms a geometric series and state the common ratio.

b Hence show that $C = \dfrac{1-\cos\theta + \cos n\theta - \cos(n+1)\theta}{2-2\cos\theta}$.

58 Let $w = \dfrac{4}{z-\mathrm{i}}$.

a Express z in terms of w.

b If $|z| = 1$, show that $\operatorname{Im}(w) = 2$.

59 Sketch $|z| = \arg z$ on an Argand diagram if $0 < \arg z < 2\pi$

In Chapter 10 you will learn another method, called integration by parts, for dealing with integrals like the one in question 56 b.

4C Complex conjugate roots of quadratic and polynomial equations with real coefficients

Factorizing polynomials

When you solved quadratic equations by factorizing in Chapter 15 of Mathematics: analysis and approaches SL, you saw how real roots were related to factors. For example, $x^2 - 6x + 8 = 0$ has solutions $x = 2$ and $x = 4$ and factorizes as $(x-2)(x-4) = 0$. The same relationship exists when the roots are complex. This means that you can now factorize some expressions that were impossible to factorize using just real numbers.

WORKED EXAMPLE 4.17

a Solve the equation $x^2 - 6x + 34 = 0$.

b Hence, factorize $x^2 - 6x + 34$.

Solve using the quadratic formula

a $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(34)}}{2(1)}$

$= \frac{6 \pm \sqrt{36 - 136}}{2}$

$= \frac{6 \pm \sqrt{-100}}{2}$

$= \frac{6 \pm 10i}{2}$

$= 3 \pm 5i$

Relate the roots to the factors

b $x^2 - 6x + 34 = (x - (3 + 5i))(x - (3 - 5i))$

$= (x - 3 - 5i)(x - 3 + 5i)$

You can use the same method to factorize polynomials of higher degree.

WORKED EXAMPLE 4.18

Let $p(x) = x^3 - 5x^2 + 11x - 15$.

Given that $x = 3$ is a root of $p(x) = 0$, express $p(x)$ as the product of a linear and a quadratic factor.

Since $x = 3$ is a root, you know that $(x - 3)$ is a factor The remaining quadratic factor will be of the form $ax^2 + bx + c$

$x^3 - 5x^2 + 11x - 15 = (x - 3)(ax^2 + bx + c)$

Expand and tidy up coefficients

$= ax^3 + bx^2 - 3ax^2 + cx - 3bx - 3c$

$= ax^3 + (b - 3a)x^2 + (c - 3b) - 3c$

Now compare coefficients to find the values of a, b and c

Comparing coefficients:

$x^3: 1 = a$

$x^2: -5 = b - 3a$

$-5 = b - 3$

$b = -2$

You could consider the coefficients of x but the constant term gives the value of c straight away

$x^0: -15 = -3c$

$c = 5$

So,

$x^3 - 5x^2 + 11x - 15 = (x - 3)(x^2 - 2x + 5)$

Tip

With a bit of practice you can find the quadratic factor by inspection without having to go through the formal process of equating coefficients. It was already clear in this case that $a = 1$ and $c = 5$ just by looking at the coefficient of x^3 and the constant term in $p(x)$.

Solving polynomial equations involving complex roots

You saw in Section 4A, and again above, that when a quadratic equation has two complex roots they will be a conjugate pair. For example, the equation $x^2 - 8x + 25 = 0$ has solutions $x = 4 + 3i$ and $x = 4 - 3i$. This happens because of the ± in the quadratic formula.

Cubics, quartics and any higher degree polynomials may have a mixture of real and complex roots, but it remains true that if the coefficients are real then any complex root is always accompanied by its complex conjugate.

Tip

This result is not true if the polynomial has complex coefficients. For example, the equation $z^2 - 3iz - 2 = 0$ has solutions i and 2i.

KEY POINT 4.10

The roots of any polynomial with real coefficients are either real or occur in complex conjugate pairs.

TOK Links

Key Point 4.10 is an example of a type of theorem which becomes increasingly common in advanced mathematics, where we can describe properties of a solution to an equation without ever actually finding the solution. How useful are general properties compared to specific details? Can you find analogies to this type of knowledge in other areas?
For example, is stereotyping in literature comparable?

Proof 4.3

Prove that, for any polynomial, p, with real coefficients, if $p(z) = 0$, then $p(z^*) = 0$.

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$

Then,

Use $(z^*)^k = (z^k)^*$

$$p(z^*) = a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \ldots + a_1 z^* + a_0$$

$$= a_n (z^n)^* + a_{n-1} (z^{n-1})^* + \ldots + a_1 z^* + a_0$$

The coefficients are real so $a_k (z^k)^* = (a_k z^k)^*$

$$= (a_n z^n)^* + (a_{n-1} z^{n-1})^* + \ldots + (a_1 z)^* + (a_0)^*$$

Use $z^* + w^* = (z + w)^*$

$$= (a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0)^*$$

$$= (p(z))^*$$

$$= 0^*$$

$$= 0$$

The result in Proof 4.3 is useful for factorizing and solving polynomial equations when you know one complex root.

To factozise and solve when you know one complex root, you will always need to expand a product of the form $(x - z)(x - z^*)$ so it is useful to be able to do this quickly.

KEY POINT 4.11

$$(x - z)(x - z^*) = x^2 - 2\text{Re}(z)x + |z|^2$$

WORKED EXAMPLE 4.19

Given that one of the roots of the polynomial $p(x) = x^3 - 4x^2 + x + 26$ is $3 - 2i$, find all the roots.

Complex roots come in conjugate pairs

$3 - 2i$ is a root so $3 + 2i$ is also a root.

Create the corresponding factors

So $(x - (3 - 2i))$ and $(x - (3 + 2i))$ are factors of $p(x)$.

Multiply out the brackets using Key Point 4.11 If $z = 3 + 2i$, then: $\text{Re}(z) = 3$ and $|z|^2 = 3^2 + 2^2 = 13$

Therefore,

$$(x-(3-2i))(x-(3+2i)) = x^2 - 2\times 3x + 13$$
$$= x^2 - 6x + 13$$

is a factor

Compare coefficients to find the remaining linear factor. You can find the required values using just the cubic and constant terms. It is then a good idea to check them by looking at the quadratic and liner terms

$$x^3 - 4x^2 + x + 26 = (x^2 - 6x + 13)(ax + b)$$

By inspection: $a = 1$, $b = 2$

Having factorized, solve the equation

So $x^3 - 4x^2 + x + 26 = 0$

$$(x^2 - 6x + 13)(x + 2) = 0$$
$$x = 3 \pm 2i \text{ or } -2$$

Be the Examiner 4.2

Find a cubic polynomial with real coefficients, given that two of its roots are 2 and i – 3.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
The other complex root is i + 3, so the polynomial is $(x-2)(x-(i-3))(x-(i+3))$ $= (x-2)(x^2-2ix-10)$ $= x^3-(2+2i)x^2+(4i-10)x+20$	The other complex root is – i – 3, so the polynomial is $(x+2)(x-(i-3))(x-(-i-3))$ $= (x+2)(x^2+6x+10)$ $= x^3+8x^2+22x+20$	The other complex root is – 3 – i, so the polynomial is $(x-2)(x-(-3+i))(x-(-3-i))$ $= (x-2)(x^2+6x+10)$ $= x^3+4x^2-2x-20$

You are the Researcher

The number of roots of a polynomial is the subject of a very powerful mathematical law called the fundamental theorem of algebra. You might like to find out more about this theorem, its proof and its applications.

Exercise 4C

For questions 1 to 5, use the technique demonstrated in Worked Example 4.17 to factorize the quadratic.

1 **a** $x^2 + 4$ **b** $x^2 + 25$

2 **a** $x^2 + 12$ **b** $x^2 + 18$

3 **a** $4x^2 + 49$ **b** $9x^2 + 64$

4 **a** $x^2 - 2x + 2$ **b** $x^2 + 6x + 25$

5 **a** $2x^2 - 6x + 7$ **b** $3x^2 - 2x + 1$

For questions 6 to 8, use the method demonstrated in Worked Example 4.18 to express the cubic as a product of a linear and a quadratic factor, given one real root.

6 **a** $x^3 + 2x^2 - x - 14$, root $x = 2$ **b** $x^3 + 3x^2 + 7x + 5$, root $x = -1$

7 **a** $2x^3 - 5x + 6$, root $x = -2$ **b** $3x^3 - x^2 - 2$, root $x = 1$

8 **a** $4x^3 - 8x^2 + 11x - 4$, root $x = \frac{1}{2}$ **b** $6x^3 + 5x^2 + 10x + 3$, root $x = -\frac{1}{3}$

For questions 9 to 12, use the method demonstrated in Worked Example 4.19 to find all the roots of the polynomial given one complex root.

9 a $x^3 - 11x^2 + 43x - 65$, root $x = 3 + 2\text{i}$
 b $x^3 - x^2 - 7x + 15$, root $x = 2 + \text{i}$

10 a $x^3 - 3x^2 + 7x - 5$, root $x = 1 - 2\text{i}$
 b $x^3 - 2x^2 - 14x + 40$, root $x = 3 - \text{i}$

11 a $x^4 + 3x^3 - 2x^2 + 6x - 8$, root $x = 1 + \text{i}$
 b $x^4 - 9x^3 + 23x^2 - x - 34$, root $x = 4 - \text{i}$

12 a $x^4 - 2x^3 + 14x^2 - 8x + 40$, root $x = 1 + 3\text{i}$
 b $x^4 - 6x^3 + 11x^2 - 6x + 10$, root $x = 3 - \text{i}$

13 Let $p(x) = x^3 - 8x^2 + 22x - 20$.
 a Given that $x = 2$ is a root of $p(x) = 0$, express $p(x)$ as the product of a linear and a quadratic factor.
 b Hence solve the equation $p(x) = 0$.

14 Let $p(x) = x^3 - 8x^2 + 9x + 58$.
 a Given that $x = -2$ is a root of $p(x) = 0$, find the other two roots.
 b Hence express $p(x)$ as the product of three linear factors.

15 Let $p(x) = 2x^3 + 7x^2 + 8x - 6$.
 a Given that $(2x - 1)$ is a factor of $p(x)$, write $p(x)$ as the product of a linear and a quadratic factor.
 b Hence solve the equation $p(x) = 0$.

16 One root of the equation $x^3 + x^2 + 11x + 51 = 0$ is $1 - 4\text{i}$.
 a Write down another complex root.
 b Find the third root.

17 Let $p(x) = x^3 + 4x^2 + 9x + 36$.
 a Show that $p(3\text{i}) = 0$.
 b Hence solve the equations $p(x) = 0$.

18 Let $p(x) = x^4 + 3x^3 - x^2 - 13x - 10$.
 Given that $(x + 1)$ and $(x - 2)$ are factors of $p(x)$, express $p(x)$ as a product of two linear factors and a quadratic factor.
 Hence find all solutions of the equation $f(x) = 0$.

19 Let $p(x) = x^4 - 3x^3 + 8x - 24$.
 a Given that $x = -2$ and $x = 3$ are roots of $p(x) = 0$, find the other two roots.
 b Hence express $p(x)$ as the product of four linear factors.

20 One root of the equation $x^4 - 4x^3 + 30x^2 - 4x + 29 = 0$ is $2 + 5\text{i}$.
 a Write down another complex root.
 b Find the remaining two roots.

21 Two roots of the equation $x^4 - 8x^3 + 21x^2 - 32x + 68 = 0$ are 2i and $4 - \text{i}$.
 a Write down the other two roots.
 b Hence express $x^4 - 8x^3 + 21x^2 - 32x + 68$ as a product of two quadratic factors.

22 Find a cubic polynomial with roots 3 and $4 + \text{i}$.

23 The polynomial $f(x) = x^3 + bx^2 + cx + d$ has roots -1 and $3 - 3\text{i}$.
 Find the values of the real numbers b, c and d.

24 Find a quartic polynomial with real coefficients and zeros 4i and $2 - 3\text{i}$.

25 The polynomial $f(x) = x^4 + bx^3 + cx^2 + dx + e$ has roots 3i and $2 - \text{i}$.
 Find the values of the real numbers b, c, d and e.

26 Solve the equation $x^4 + 13x^2 + 40 = 0$.

27 Use a counterexample to prove that if an equation has complex conjugate roots it does not necessarily have real coefficients.

4D Powers and roots of complex numbers

De Moivre's theorem

In Section 4B, you saw that

$|zw| = |z||w|$ and $\arg(zw) = \arg z + \arg w$.

It follows that if $|z| = r$ and $\arg z = \theta$, then multiplying $z \times z$ gives that z^2 has modulus r^2 and argument 2θ. Repeating this process gives

$|z^n| = r^n$ and $\arg(zn) = n\theta$.

This result holds not only for positive integer powers, but also for negative integer powers.

KEY POINT 4.12

De Moivre's theorem:

$$(r(\cos\theta + \mathrm{i}\sin\theta))^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) \text{ for } n \in \mathbb{Z}$$

You will see how to prove De Moivre's theorem for positive integer powers in Chapter 5.

TOOLKIT: Proof

You might think that we can use the Euler form to prove De Moivre's theorem easily. In Euler form, it just says

$$(r\mathrm{e}^{\mathrm{i}\theta})^n = r^n\mathrm{e}^{\mathrm{i}n\theta}$$

which follows from the laws of exponents. The problem is that we have defined cis θ to be $\mathrm{e}^{\mathrm{i}\theta}$ from the analogy of how the cis θ form was behaving under multiplication. If we then used this analogy to prove how cis θ behaves, we would be guilty of circular reasoning. This is one of the most subtle issues that can sometimes arise in mathematical proofs, and one you should watch out for!

You can use De Moivre's theorem to evaluate powers of complex numbers.

WORKED EXAMPLE 4.20

Given $z = 16\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$, find z^{-4}, giving your answer in Cartesian form

Apply De Moivre's theorem

$$z^{-4} = \left[16\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)\right]^{-4}$$

$$= 16^{-4}\left(\cos\left(\frac{-4\pi}{12}\right) + \mathrm{i}\sin\left(\frac{-4\pi}{12}\right)\right)$$

Simplify and express in Cartesian form

$$= 2\left(\cos\left(\frac{-\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\pi}{3}\right)\right)$$

$$= 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= 1 - \sqrt{3}\mathrm{i}$$

To use De Moivre's theorem, you might have to convert to modulus–argument form first.

WORKED EXAMPLE 4.21

Find $(1+i)^6$ in Cartesian form.

Convert to modulus–argument form: $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$\arg(1+i) = \arctan\frac{1}{1} = \frac{\pi}{4}$

$\therefore 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Apply De Moivre's theorem: $\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^6 = (\sqrt{2})^6\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right)$

Simplify and express in Cartesian form: $= 2^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$= 8(0 + i(-1))$

$= -8i$

Roots of unity

De Moivre's theorem can also be used to find roots of complex numbers. However, you have to be slightly careful if you want to find all roots. You are already familiar with the idea that there are two numbers which square to give 1. In a similar fashion, there are multiple numbers which when raised to the nth power give one.

Solutions of the equation $z^n = 1$ are called roots of unity. There will always be n such solutions. To find all the solutions you need to remember that, in modulus–argument form, 1 can be written in many different ways. It can have an argument of 0, 2π, 4π…

WORKED EXAMPLE 4.22

a Solve the equation $z^5 = 1$, giving your answers in modulus–argument form.

b Show the solutions to part **a** on an Argand diagram.

Write z in modulus–argument form: **a** Let $z = r\operatorname{cis}\theta$

Then,

$(r\operatorname{cis}\theta)^5 = 1$

Apply De Moivre's theorem: $r^5\operatorname{cis}5\theta = 1$

Write 1 in modulus–argument form as well: $r^5\operatorname{cis}5\theta = 1\operatorname{cis}0$

Equate moduli: So,

$r^5 = 1$

$r = 1$

$0 \leqslant \theta < 2\pi$, so $0 \leqslant 5\theta < 10\pi$: And

$\operatorname{cis}5\theta = \operatorname{cis}0$

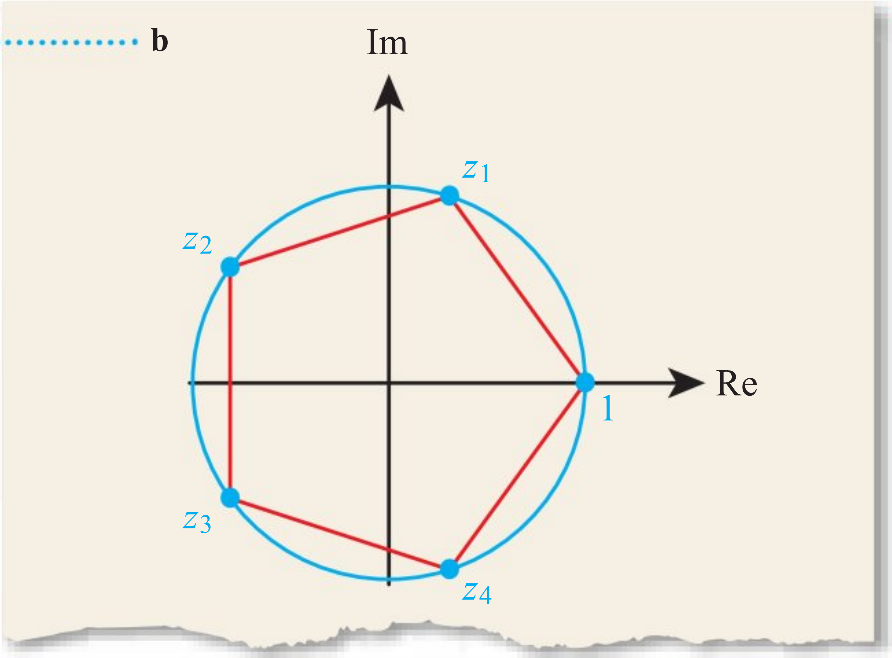
Since cos and sin have a period of 2π, so does cis: $5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

This gives five solutions: $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$z = 1, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{6\pi}{5}, \operatorname{cis}\frac{8\pi}{5}$

The five roots form a regular pentagon on the Argand diagram

b



The method of Worked Example 4.22 can be generalized for any positive integer n.

KEY POINT 4.13

The nth roots of unity are

$$1, \text{cis}\frac{2\pi}{n}, \text{cis}\frac{4\pi}{n}, \ldots, \text{cis}\frac{2(n-1)\pi}{n} = 0$$

They form a regular n-gon on the Argand diagram.

Notice that all the arguments are multiples of $\frac{2\pi}{n}$. But multiplying an argument by a number k corresponds to raising the complex number to the power of k. Hence, all the nth roots of unity are powers of $\text{cis}\frac{2\pi}{n}$.

That is to say, denoting $\text{cis}\frac{2\pi}{n} = \omega$, the nth roots of unity can be written as $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

This leads to an interesting and useful result.

KEY POINT 4.14

$$1 + \text{cis}\frac{2\pi}{n} + \text{cis}\frac{4\pi}{n} + \ldots + \text{cis}\frac{2(n-1)\pi}{n} = 0$$

Proof 4.4

Prove that $1 + \text{cis}\frac{2\pi}{n} + \text{cis}\frac{4\pi}{n} + \ldots + \text{cis}\frac{2(n-1)\pi}{n} = 0$.

Express the n^{th} roots of unity as powers of $\text{cis}\frac{2\pi}{n}$ ······ Let $\omega = \text{cis}\frac{2\pi}{n}$

Then,

$$1 + \text{cis}\frac{2\pi}{n} + \text{cis}\frac{4\pi}{n} + \ldots + \text{cis}\frac{2(n-1)\pi}{n} = 1 + \omega + \omega^2 + \ldots + \omega^{n-1}$$

This is a geometric series and since $\omega \neq 1$ you can use $S_n = \frac{a - r^n}{1 - r}$ ······ $= \frac{1 - \omega^n}{1 - \omega}$

ω is an n^{th} root of unity, which means that $\omega^n = 1$ ······ $= 0$ (since $\omega^n = 1$)

Roots of general complex numbers

You can use the method shown earlier to find roots of any complex number. This is effectively the equivalent of using De Moivre's theorem for rational powers.

WORKED EXAMPLE 4.23

a Solve the equation $z^3 = 4\sqrt{3} + 4i$.

b Show the solutions to part **a** on an Argand diagram.

Write z in modulus–argument form

a Let $z = r\,\text{cis}\,\theta$

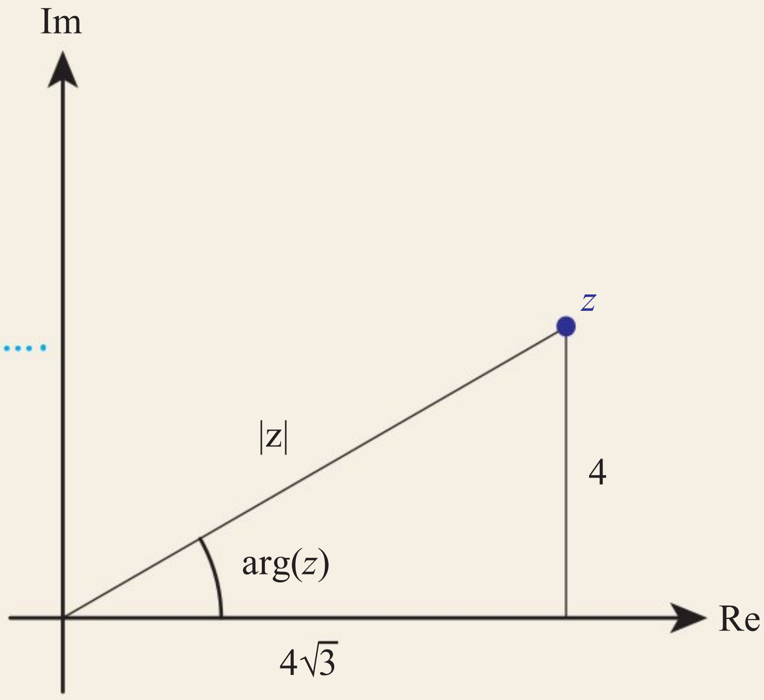
Then,

Apply De Moivre's theorem

$$(r\,\text{cis}\,\theta)^3 = 4\sqrt{3} + 4i$$

$$r^3\,\text{cis}\,3\theta = 4\sqrt{3} + 4i$$

Write $4\sqrt{3} + 4i$ in modulus–argument form as well



$$|4\sqrt{3} + 4i| = \sqrt{(4\sqrt{3})^2 + 4^2} = 8$$

$$\arg(4\sqrt{3} + 4i) = \arctan\left(\frac{4}{4\sqrt{3}}\right) = \frac{\pi}{6}$$

$$r^3\,\text{cis}\,3\theta = 8\,\text{cis}\,\frac{\pi}{6}$$

Equate moduli

So,

$$r^3 = 8$$

$$r = 2$$

$0 \leqslant \theta < 2\pi$, so $0 \leqslant 3\theta < 6\pi$

And

$$\text{cis}\,3\theta = \text{cis}\,\frac{\pi}{6}$$

Again, since cis is 2π periodic the solutions occur at intervals of 2π

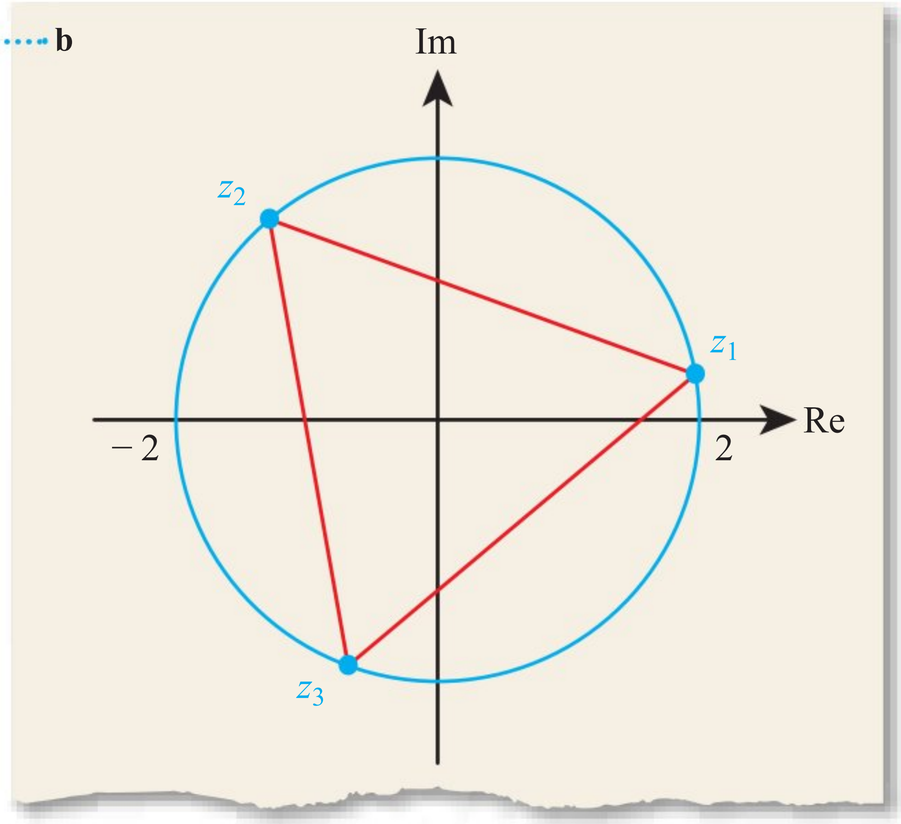
$$3\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$$

This gives three solutions

$$\theta = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}$$

$$z = 2\,\text{cis}\,\frac{\pi}{18}, 2\,\text{cis}\,\frac{13\pi}{18}, 2\,\text{cis}\,\frac{25\pi}{18}$$

The roots form an equilateral triangle whose vertices lie on a circle of radius 2 **b**



KEY POINT 4.15

The solutions of $z^n = w$ form a regular n-gon with vertices on a circle of radius $|z|$ centred at the origin.

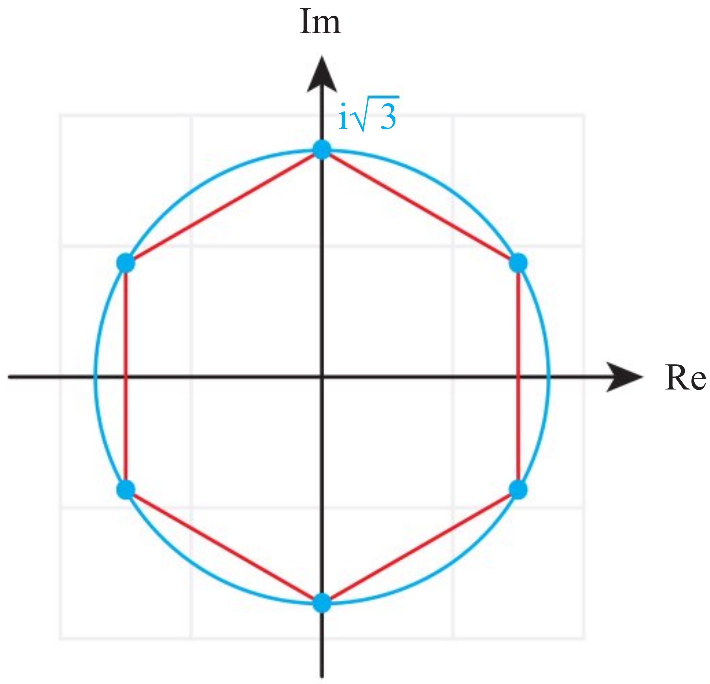
CONCEPTS – REPRESENTATION

When adding two complex numbers together, what **representation** is easiest to work with – Cartesian form or modulus–argument form? What about when raising a complex number to a large power?

WORKED EXAMPLE 4.24

The diagram shows a regular hexagon inscribed in a circle of radius of $\sqrt{3}$. The vertices correspond to solutions of the equation $z^n = w$.

Find the value of n and the number w.



The polygon has 6 vertices so $n = 6$ .... There are six solutions so $n = 6$

You can see that one of the solutions is $z = i\sqrt{3}$ so substitute this into $z^6 = w$ .... $z = i\sqrt{3}$ is a solution.

So,

$$w = \left(i\sqrt{3}\right)^6$$

$i^6 = i^2 = -1$ .... $= i^6\left(\sqrt{3}\right)^6$

$$= -27$$

The development of negative numbers and complex numbers shows an interesting interplay between mathematics and historical events. The notion of 'debts' as negative numbers was formalized and understood by Arabian mathematicians such as Al-Samawal (1130–1180) in Bagdhad. The Indian Mathematician Brahmagupta (598–670) and Chinese Mathematician Liu Hui (225–295) might have claims to similar ideas predating Al-Samawal. Until that point, equations such as $x+1=0$ were said to have no solution, just like you might have said $x^2+1=0$ has no solution until you met complex numbers.

■ **Figure 4.2** Girolamo Cardano

As the influence of the Islamic world spread – initially through conquest, then trade – these ideas also spread. It is no surprise that the jump from negative numbers to imaginary numbers happened in a country with major trade links to the Islamic world – Italy. Girolamo Cardano (1501–1576) found that he was having to find square roots of negative numbers when he was solving cubic equations. He called these numbers 'fictitious' and although they appeared in the middle of his working, they disappeared by the end and he got the correct answer so he accepted them. Probably the relatively recent introduction of equally puzzling negative numbers helped him overcome his scepticism of this other new type of number.

In countries less influenced by the Islamic world, such as Northern Europe, it took several centuries more for negative numbers to achieve widespread acceptance. Even by the eighteenth century in England, there were respected mathematicians, such as Maseres (1731–1824) who decided that negative numbers could only be used as long as they did not appear in the final answer.

Exercise 4D

For questions 1 to 4, use the method demonstrated in Worked Example 4.20 to express each complex number in Cartesian form.

1 a $\left(4\operatorname{cis}\frac{\pi}{9}\right)^3$ b $\left(3\operatorname{cis}\frac{\pi}{10}\right)^5$

2 a $\left(2\cos\frac{3\pi}{8}+2\mathrm{i}\sin\frac{3\pi}{8}\right)^6$ b $\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)^4$

3 a $\left(\frac{1}{6}\operatorname{cis}\frac{\pi}{6}\right)^{-2}$ b $\left(\frac{1}{2}\operatorname{cis}\frac{5\pi}{12}\right)^{-3}$

4 a $\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)^{-8}$ b $\left(\frac{2}{3}\cos\frac{3\pi}{4}+\frac{2}{3}\mathrm{i}\sin\frac{3\pi}{4}\right)^{-1}$

For questions 5 to 8, use the method demonstrated in Worked Examples 4.22 and 4.23 to solve the equation. Give your answers in the form $z = r\operatorname{cis}\theta$, where $0 \leqslant \theta < 2\pi$.

5 a $z^3 = 1$ b $z^4 = 1$

6 a $z^6 = -27$ b $z^5 = 243$

7 a $z^5 = 32\mathrm{i}$ b $z^3 = -64\mathrm{i}$

8 a $z^4 = 2\sqrt{2}+2\sqrt{2}\mathrm{i}$ b $z^6 = 4\sqrt{2}-4\sqrt{2}\mathrm{i}$

9 a Express $z = 2-2\mathrm{i}$ in polar form.

b Hence find, in Cartesian form, z^5.

10 a Express $z = \sqrt{3}+\mathrm{i}$ in polar form.

b Hence find, in Cartesian form, z^{-3}.

11 a Express $w = -\sqrt{2}-\sqrt{2}\mathrm{i}$ in modulus–argument form.

b Given that $z = \cos\frac{\pi}{7}+\mathrm{i}\sin\frac{\pi}{7}$, find, in Cartesian form, w^6z^7.

12 a Solve the equation $z^6 = 1$.

Give your answers in the form $z = x+\mathrm{i}y$, where $x, y \in \mathbb{R}$.

b Show these solutions on an Argand diagram.

13 a Solve the equation $z^4 = -16$.

Give your answers in Cartesian form.

b Show these solutions on an Argand diagram.

14 a Solve the equation $z^3 = 8\mathrm{i}$.

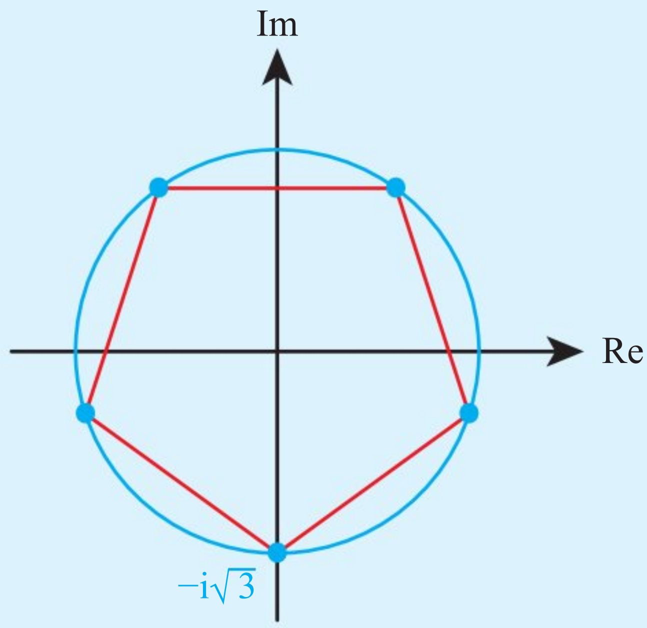
Give your answers in the form $z = re^{\mathrm{i}\theta}$, where $\pi < \theta \leqslant \pi$.

b Show these solutions on an Argand diagram.

15 The five vertices of the pentagon shown correspond to the solution of an equation of the form $z^n = w$, where w is a complex number.

$-\sqrt{3}\mathrm{i}$

Find the values of n and w.



16 Find, in Cartesian form,

$$\frac{8}{(1-\sqrt{3}\mathrm{i})^5}$$

17 Give that $w = 3-3\mathrm{i}$ and $z = \cos\frac{3\pi}{8}+\mathrm{i}\sin\frac{3\pi}{8}$, find, in Cartesian form, w^4z^6.

18 Let $z = 2\operatorname{cis}\frac{7\pi}{24}$.

Find the smallest positive integer value of n for which $z^n \in \mathbb{R}$.

19 Let $z = \operatorname{cis}\frac{5\pi}{18}$.

Find the smallest positive integer value of n for which $z^n = \mathrm{i}$.

20 Let $\omega = e^{\frac{2\pi\mathrm{i}}{7}}$.

a Express the seventh roots of unity in terms of ω.

b Find an integer k such that $\omega^k = -\omega$ or explain why such an integer doesn't exist.

c Write down the smallest positive integer p such that $\omega^{24} = \omega^p$.

d Write down an integer m such that $\omega^m = (\omega^2)^*$.

21 Let 1, ω, ω^2, ω^3 and ω^4 be the roots of the equation $z^5 = 1$.

a By expanding $(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4)$, show that $1+\omega+\omega^2+\omega^3+\omega^4 = 0$.

b Hence find the value of $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}$.

22 a Find, in Cartesian form, all the complex solutions of the equation $z^3 = 1$.

b Hence find the exact solutions of the equation $(z-1)^3 = (z+2)^3$.

23 Solve the equation $z^4 = 2+2\sqrt{3}\mathrm{i}$.

Give your answers in Euler form, where $-\pi < \theta \leqslant \pi$.

24 Solve the equation $z^5 = 16-16\sqrt{3}\mathrm{i}$.

Give your answers in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $0 \leqslant \theta < 2\pi$.

25 Solve the equation $z^6 = 4\sqrt{2}-4\sqrt{2}\mathrm{i}$.

Give your answers in the form $z = r\,\mathrm{cis}\,\theta$, where $-\pi < \theta \leqslant \pi$.

26 A square has vertices at (3, 3), (–3, 3), (–3, –3) and (3, –3).

The complex numbers corresponding to the vertices of the square are solutions of an equation of the form $z^n = w$, where $n \in \mathbb{N}$ and $w \in \mathbb{R}$.

Find the values of n and w.

27 Let $\omega = \mathrm{cis}\,\frac{2\pi}{5}$.

a Show that $1+\omega+\omega^2+\omega^3+\omega^4 = 0$.

b Show that $\mathrm{Re}(\omega)+\mathrm{Re}(\omega^2) = -\frac{1}{2}$.

c Hence find the exact value of $\cos\frac{2\pi}{5}$.

28 a Find, in polar form, the three solutions of the equation $z^3 = -1$.

b Expand $(x+2)^3$.

c Hence or otherwise solve the equation $z^3+6z^2+12z+9 = 0$, giving any complex solution in Cartesian form.

29 a Find all the solutions of the equation $z^4 = -4$.

b Hence solve the equation $z^4+4(z-1)^4 = 0$. Give your answers in Cartesian form.

30 a Solve the equation $z^3 = -32\sqrt{2}+32\sqrt{2}\mathrm{i}$.
Give your answers in the form $z = r\,\mathrm{cis}\,\theta$, where $0 \leqslant \theta < 2\pi$.

b Represent these solutions as points on the complex plane, labelling them A, B and C in increasing size of argument.

c The midpoint of A and B represents the complex number w.
Find, in Cartesian form, w^3.

4E Trigonometric identities

Complex numbers can be used to derive two types of trigonometric identity:

- multiple angle identities, e.g. for $\sin 3\theta$, $\cos 4\theta$
- identities for powers of trigonometric functions, e.g. for $\sin^5\theta$, $\cos^6\theta$.

You met double-angle identities, such as $\cos 2\theta = 2\cos^2\theta - 1$, in Mathematics: analysis and approaches SL Chapter 18 and Chapter 3 of this book.

WORKED EXAMPLE 4.25

Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

Let $z = \cos\theta + \mathrm{i}\sin\theta$

Then $z^4 = (\cos\theta + \mathrm{i}\sin\theta)^4$

Using De Moivre's theorem:

$$z^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$$

Applying De Moivre's theorem results in $\cos 4\theta$ being the real part

Using the binomial theorem:

$$z^4 = \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$$
$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + (4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\mathrm{i}$$

Applying the binomial theorem to expand the bracket gives another expression for the real part

Equating real parts:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

The two expressions for z^4 must have equal real parts

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

You want the answer in terms of $\cos\theta$ only, so use $\sin^2\theta = 1 - \cos^2\theta$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

You can use De Moivre's theorem to derive the following two useful results.

KEY POINT 4.16

If $z = \cos\theta + \mathrm{i}\sin\theta$, then

- $z^n + \dfrac{1}{z^n} = 2\cos n\theta$
- $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$

In turn, these results can be used to derive identities for powers of sin or cos in terms of multiple angles.

You can already do this for $\cos^2\theta$ and $\sin^2\theta$ by rearranging the $\cos 2\theta$ identity, for example, $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$.

WORKED EXAMPLE 4.26

Show that $\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

Using the binomial theorem:

$$\left(z - \frac{1}{z}\right)^4 = z^4 + 4z^3\left(-\frac{1}{z}\right) + 6z^2\left(-\frac{1}{z}\right)^2 + 4z\left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4$$

$$= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$

Group the terms to get expressions of the form $z^n + \frac{1}{z^n}$

$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{4}{z^2}\right) + 6$$

So,

Use $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$ on the LHS and $z^4 + \frac{1}{z^4} = 2\cos 4\theta$ and $z^2 + \frac{1}{z^2} = 2\cos 2\theta$ on the RHS

$$(2\mathrm{i}\sin\theta)^4 = 2\cos 4\theta - 4\cos 2\theta + 6$$

$$16\sin^4\theta = 2\cos 4\theta - 4\cos 2\theta + 6$$

$$\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

Tip

Trigonometric identities such as these are very useful when integrating powers of trigonometric functions.

TOK Links

Many of the identities you have met in this section can be verified using methods without complex numbers, but they are much easier to find using complex numbers. This was one of the reasons complex numbers grew to be accepted. In mathematics, is it more important for ideas to be true in the real world, or to be useful in the mathematical world?

TOOLKIT: Problem Solving

How might you plot a graph of e^z against z? Can you try to visualize what it looks like? What does this representation tell you about complex exponentials?

You are the Researcher

If you like plotting the complex function above, you might like to consider what its derivative looks like? The idea of calculus with complex numbers is very important, normally studied in a topic called complex analysis. It is not even obvious when a complex function has a derivative – the conditions for it to have a derivative are called the Cauchy–Riemann conditions.

Exercise 4E

1 a Find the real part of $(\cos\theta + i\sin\theta)^3$.

b Hence express $\cos 3\theta$ in terms of powers of $\cos\theta$.

2 a Find the imaginary part of $(\cos\theta + i\sin\theta)^4$.

b Hence show that $\sin 4\theta = 4\cos\theta(\sin\theta - 2\sin^3\theta)$.

3 Given that $e^{i\theta} = \cos\theta + i\sin\theta$, show that

a $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ b $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

4 a Express $\sin 5\theta$ in terms of powers of $\sin\theta$.

b Given that $4\sin^5\theta + \sin 5\theta = 0$, find the possible values of $\sin\theta$.

5 a Given that $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

b Show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

c Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4\theta \, d\theta$.

6 a Show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

b Show that $\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

c Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}$.

7 a Use De Moivre's theorem to show that

$32\cos^6\theta = \cos 6\theta + A\cos 4\theta + B\cos 2\theta + C$

where A, B and C are integers to be found.

b Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \frac{5\pi}{32}$.

8 a Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b Hence solve $16x^5 - 20x^3 + 5x + 1 = 0$.

9 a Expand $(z + z^{-1})^6$ and $(z - z^{-1})^6$.

b Hence show that $\cos^6\theta + \sin^6\theta = \dfrac{1}{8}(3\cos 4\theta + 5)$.

10 a Use the binomial expansion to write $(\text{cis}\,x)^5$ in Cartesian form.

b Hence show that $\sin 5x = 16\sin x\cos^4 x - 12\sin x\cos^2 x + \sin x$.

c Find $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{\sin x}$.

11 a Show that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

b Hence use the substitution $x = \cos\theta$ to solve $8x^3 - 6x - 1 = 0$. Give your answers in the form $x = \cos k\pi$, where k is a rational number.

12 a Find the real and imaginary parts of $(\cos\theta + i\sin\theta)^4$.

b Hence show that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

c Using the result in part **b**, solve the equation

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

giving your answers in the form $\tan k\pi$ where $k \in \mathbb{Q}$.

You are the Researcher

The method shown in question 11 is part of the Tschirnhaus–Vieta approach to solving cubic equations. You might want to find out more about how it works, and how it can be used to find the equivalent of the discriminant for cubic equations.

Checklist

- You should be able to work with the imaginary number i
  $\mathrm{i} = \sqrt{-1}$
- You should be able to find sums, products and quotients of complex numbers in Cartesian form:
  - A complex number z can be written in Cartesian form as
    $z = x + \mathrm{i}y$
    where $x, y \in \mathbb{R}$
  - Its complex conjugate, z^*, is
    $z^* = x - \mathrm{i}y$
  - The product of a complex number with its conjugate is real
    $zz^* = x^2 + y^2$
- You should be able to represent complex numbers geometrically on the complex plane (Argand diagram)
- You should be able to find the modulus, r, and argument, θ, of a complex number

  If $z = x + \mathrm{i}y$, then
  - $r = \sqrt{x^2 + y^2}$
  - $\tan\theta = \frac{y}{x}$
- You should be able to write complex numbers in modulus–argument (polar) form
  - $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\theta$
  - $z^* = r\,\mathrm{cis}(-\theta)$
- You should be able to find sums, products and quotients of complex numbers in modulus–argument form:
  - $|zw| = |z||w|$
  - $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
  - $\arg(zw) = \arg z + \arg w$
  - $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$
- You should be able to write complex numbers in Euler form
  - $z = r\mathrm{e}^{\mathrm{i}\theta}$ where $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$
  - $z^* = r\mathrm{e}^{-\mathrm{i}\theta}$
- You should be able to find sums, products and quotients of complex numbers in Euler form using the usual rules of algebra and exponents.
- You should be able to use the fact that roots of any polynomial with real coefficients are either real or occur in complex conjugate pairs.
  - A useful short-cut when working with complex conjugate roots is
  - $(x - z)(x - z^*) = x^2 - 2\mathrm{Re}(z)x + |z|^2$
- You should be able to use De Moivre's theorem to find powers of complex numbers
  - $(r(\cos\theta + \mathrm{i}\sin\theta))^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$ for $n \in \mathbb{Z}$
- You should be able to use De Moivre's theorem to find roots of complex numbers
  - The nth roots of unity are:
    — $1, \mathrm{cis}\frac{2\pi}{n}, \mathrm{cis}\frac{4\pi}{n}, \ldots, \mathrm{cis}\frac{2(n-1)\pi}{n}$
  - They form a regular n-gon on the Argand diagram.
  - They sum to zero:
    — $1 + \mathrm{cis}\frac{2\pi}{n} + \mathrm{cis}\frac{4\pi}{n} + \ldots + \mathrm{cis}\frac{2(n-1)\pi}{n} = 0$
  - The solutions of $z^n = w$ form a regular n-gon with vertices on a circle of radius $|z|$ centred at the origin.
- You should be able to use De Moivre's theorem to find trigonometric identities
  - If $z = \cos\theta + \mathrm{i}\sin\theta$, then
    — $z^n + \frac{1}{z^n} = 2\cos n\theta$
    — $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$

Mixed Practice

1 Plot and label the following points on a single Argand diagram.

a $z_1 = \frac{1}{\mathrm{i}}$ **b** $z_2 = (1+\mathrm{i})(2-\mathrm{i})$ **c** $z_3 = z_2^*$

2 Solve $x^2 - 2x + 2 = 0$.

3 Solve $x^2 - 6x + 12 = 0$.

4 If $z = \frac{1+\mathrm{i}}{1+2\mathrm{i}}$, find z^* in the form $a + \mathrm{i}b$.

5 One root of the equation $x^2 + bx + c = 0$ is $1+2\mathrm{i}$. Find the values of a and b, given that they are real.

6 Solve $\frac{z}{z+\mathrm{i}} = 1 + 2\mathrm{i}$.

7 Solve $z + \mathrm{i} = 2z^*$.

8 Solve $z + 4\mathrm{i} = \mathrm{i}z$.

9 **a** If $z = 1 + \mathrm{i}$, find the modulus and argument of z.

b If $w = \cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}$, find and simplify $z^6 w^5$.

10 If $z = 2 - 2\mathrm{i}$, find the modulus and argument of $\left(z^*\right)^3$.

11 If $\mathrm{i}p + 4q = 2 + 3\mathrm{i}$, find the values of p and q if they are

a real **b** conjugate complex numbers.

12 Find the three cube roots of -1 in the form $a + b\mathrm{i}$.

13 **a** Write down, in the form $\mathrm{e}^{\mathrm{i}\theta}$, the solutions of the equation $z^5 = 1$.

b Represent the solutions on an Argand diagram.

14 Write in Cartesian form

a $\mathrm{cis}\frac{\pi}{2} \div \mathrm{cis}\frac{\pi}{6}$ **b** $\mathrm{cis}\frac{\pi}{2} - \mathrm{cis}\frac{\pi}{6}$

15 Write $\frac{1}{a+\mathrm{i}}$ in Cartesian form, where $a \in \mathbb{R}$.

16 Solve $\frac{1}{z+\mathrm{i}} + \frac{2}{z-\mathrm{i}} = 0$.

17 Solve the simultaneous equations

$$z + z^* = 8$$
$$z - z^* = 6\mathrm{i}$$

18 One root of $x^3 - x^2 + x - 1 = 0$ is an integer. Find all three roots, including complex roots.

19 One root of $x^3 - 5x^2 + 7x + 13 = 0$ is an integer. Find all three roots, including complex roots.

20 The complex numbers z and w both have arguments between 0 and π. Given that $zw = -\sqrt{3} + \mathrm{i}$ and $\frac{z}{w} = -\frac{\mathrm{i}}{2}$, find the modulus and argument of z.

21 If both b and $\frac{2}{2+\mathrm{i}} - \frac{1}{b+\mathrm{i}}$ are real numbers, find the possible values of b.

22 Find the possible values of z if $\mathrm{Re}(z) = 2$ and $\mathrm{Re}(z^2) = 3$.

23 Solve $z + |z| = 18 + 12\mathrm{i}$.

24 **a** If $|z - 4| = 2|z - 1|$, find $|z|$.

b Sketch the solutions to the equation in part **a** on an Argand diagram.

25 Given that one root of the equation $z^3 - 11z^2 + 43z - 65 = 0$ is $3 - 2\mathrm{i}$, find the other two roots.

26 **a** Use a counterexample to prove that it is not always true that $\mathrm{Re}(z^2) \equiv \mathrm{Re}(z)^2$.

b If $\mathrm{Re}(z^2) = \mathrm{Re}(z)^2$, find $\mathrm{Im}(z)$.

27 **a** Find the three distinct roots of the equation $z^3 + 8 = 0$, giving your answers in

i modulus–argument form **ii** Cartesian form.

b The roots are represented by the vertices of a triangle in an Argand diagram. Find the area of the triangle.

28 If $z = x + \mathrm{i}y$ and $2|z| = |z + 3|$, find the relationship between x and y.

29 In this question $z = 1 + \mathrm{i}$ and $w = 2 - \mathrm{i}$. Find the minimum value of $|z + pw|$ as p changes.

30 Sketch on an Argand diagram all the points which satisfy $|z - 1| = |z - \mathrm{i}|$.

31 **a** Use De Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b Hence solve the equation $\cos 5\theta = 5\cos\theta$ for $\theta \in [0, 2\pi]$.

32 Let $z = 1 + \mathrm{i}$ and $\mathrm{w} = 1 + \sqrt{3}\mathrm{i}$.

a Write zw in Cartesian form.

b Write z and w in modulus– argument form. Hence find the modulus and argument of zw.

c Use your answers from parts **a** and **b** to find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

33 **a** Find the modulus and argument of $\sqrt{3} + \mathrm{i}$.

b Simplify $(\sqrt{3} + \mathrm{i})^7 + (\sqrt{3} - \mathrm{i})^7$.

34 **a** Find all complex roots of the equation $z^3 = 8$.

b Hence solve the equation $(z + 2)^3 = 8z^3$, giving your answers in Cartesian form.

35 **a** Show that $\cos\theta = \dfrac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}$.

b Expand $\left(x + \dfrac{1}{x}\right)^4$.

c Hence show that $\cos^4\theta \equiv a\cos 4\theta + b\cos 2\theta + c$ where a, b and c are constants to be determined.

d Evaluate $\displaystyle\int_0^{\pi} \cos^4(2x)\mathrm{d}x$.

36 The complex numbers $z_1 = 2 - 2\mathrm{i}$ and $z_2 = 1 - \sqrt{3}\mathrm{i}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

a find AB, giving your answer in the form $a\sqrt{b\ \ \sqrt{3}}$, where $a,\ b \in \mathbb{Z}^+$

b calculate $A\hat{O}B$ in terms of π.

Mathematics HL May 2011 Paper 1 TZ2 Q4

37 If ω is a non – real cube root of unity and $x, y \in \mathbb{R}$ evaluate

a $1 + \omega + \omega^2$ **b** $(\omega x + \omega^2 y)(\omega^2 x + \omega y)$.

38 **a** Prove that $z + z^* = 2\mathrm{Re}(z)$. **b** Simplify $(zw^*)^*$.

c Hence prove that $zw^* + wz^*$ is real.

39 Show that if x is real then $\cos(\mathrm{i}x)$ is real.

40 Find all complex numbers z which satisfy the equation $|z|z^3 = -81$. Give your answers in Cartesian form.

41 **a** Prove that $zz^* = |z|^2$.

b Prove that $|z-w|^2 + |z+w|^2 = 2|z|^2 + 2|w|^2$.

42 If $z|z| + \frac{2}{z^*} = 3z$, find the possible values of $|z|$.

43 If $z = \cos 2\theta + \mathrm{i}\sin 2\theta$ with $0 < \theta < \frac{\pi}{2}$,

a show that $|z+1| = 2\cos\theta$.

b show that $\arg(z+1) = \theta$.

44 **a** Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Let $\omega = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$.

b **i** Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

ii Hence deduce the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$.

c Show that $\cos\frac{2\pi}{7}$ is a root of the equation $8t^3 + 4t^2 - 4t - 1 = 0$.

45 **a** **i** Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus–argument form.

ii Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

iii Show that $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$.

b **i** State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus–argument form.

ii If z is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

iii Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients.

Mathematics HL May 2013 Paper 1 TZ2 Q13

46 **a** **i** Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^5$.

ii Hence use De Moivre's theorem to prove $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

iii State a similar expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^5 - 1 = 0$ which has the smallest positive argument.

b Find the value of r and the value of α.

c Using **a ii** and your answer from **b** show that $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$.

d Hence express $\sin 72°$ in the form $\frac{\sqrt{a + b\sqrt{c}}}{d}$ where $a, b, c, d \in \mathbb{Z}$.

Mathematics HL May 2015 Paper 2 TZ1 Q12

47 **a** Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$.

b Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

c Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

d Hence solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

e By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$.

Mathematics HL May 2011 Paper 1 TZ1 Q13

5 Mathematical proof

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalencies and make generalizations.

In this chapter you will learn…

- how to use mathematical induction to prove statements about patterns involving integers
- how to use proof by contradiction
- how to use counterexamples to disprove a statement.

CONCEPTS

The following concepts will be addressed in this chapter:

- Formulae are a **generalization** made on the basis of specific examples, which can then be extended to new examples.
- Proof serves to **validate** mathematical formulae and the **equivalence** of identities.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Find the first four terms of the following sequences.

a $u_n = n^2 + 1$ **b** $u_1 = 3,\ u_{n+1} = 2u_n - 1$

2 Evaluate $\sum_{r=1}^{3}(3r-1)$.

3 Show that $\frac{n(n+1)}{3} + \frac{(n+1)(2n-1)}{2} \equiv \frac{(n+1)(8n-3)}{6}$.

4 Prove that the sum of two multiples of 3 is also a multiple of 3.

5 Given that $\sin A = \frac{1}{3}$ and that $0 < A < \frac{\pi}{2}$, find the exact value of

a $\cos A$ **b** $\sin 2A$ **c** $\cos 2A$ **d** $\sin 3A$.

6 Write $\left(3 \text{ cis}\frac{\pi}{3}\right)\left(2 \text{ cis}\frac{\pi}{4}\right)$ in polar form.

7 Given that $y = xe^{3x}$ write $\frac{dy}{dx}$ in the form $(ax+b)e^{3x}$.

■ **Figure 5.1** Can we prove that the sun will rise tomorrow?

You have already met deductive proof, in which you start from a statement you know to be true and proceed with a sequence of valid steps to arrive at a conclusion. But some statements are difficult to prove in this direct way and require a different method of proof.

In this chapter, you will meet three new methods. Proof by induction can be used to prove some statements about integers. Proof by contradiction starts by assuming that a statement is false and shows that this is impossible. You will also see how to prove that a statement is false by using a counterexample.

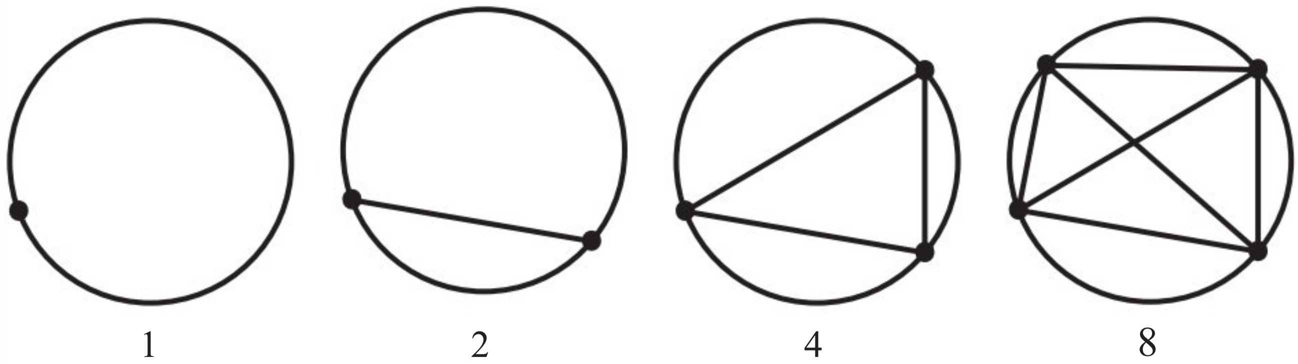
Starter Activity

Look at the pictures in Figure 5.1. What comes next in the sequence? How do you know? How certain can you be?

Now look at this problem:

n points are chosen on the circumference of a circle. Every point is joined to every other point by a chord. What is the maximum possible number of regions created?

The diagram shows the number of regions created when $n = 1, 2, 3$ and 4.



LEARNER PROFILE – Communicators

Is mathematics more about getting the right answer or communicating that your answer must be right? How does your mathematical communication vary depending upon the person reading the solution?

5A Proof by induction

Consider adding up consecutive odd numbers:

One odd number: sum $= 1 = 1^2$

Two odd numbers: sum $= 1+3 = 4 = 2^2$

Three odd numbers: sum $= 1+3+5 = 9 = 3^2$

Four odd numbers: sum $= 1+3+5+7 = 16 = 4^2$

It appears that the sum of the first n odd numbers is n^2. But how can you be certain that the observed pattern continues?

Suppose that you have confirmed (by direct calculation) that the pattern holds up to the 15th odd number; so you know that the sum of the first 15 odd numbers is

$$1+3+5+\ldots+29 = 225 = 15^2$$

To check that the pattern continues, you do not have to start adding from 1 again. You can use the result you already have, so

$$1+3+5+\ldots+29+31 = 225+31 = 256 = 16^2$$

You can then repeat the same procedure to continue the pattern, from $n = 16$ to $n = 17$, from $n = 17$ to $n = 18$, and so on.

Tip

The kth odd number is $2k-1$.

In general, suppose that you have checked that the pattern holds for the first k odd numbers. This means that

$$1+3+5+\ldots+(2k-1) = k^2$$

You can then prove that it continues up to $n = k+1$, because:

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2k+1) &= k^2+(2k+1) \\ &= (k+1)^2 \end{aligned}$$

Therefore, the pattern still holds for the first $k+1$ odd numbers.

Building upon the previous result like this, rather than starting all over again, is called an **inductive step**.

Does this prove that the pattern continues forever? You know that it holds for $n = 1$; because it holds for $n = 1$ it follows that it holds for $n = 2$; because it holds for $n = 2$ it follows that it holds for $n = 3$; and so on. You can continue this process to reach any number n, however large. Therefore, the pattern holds for all positive integers.

This is the first documented example of mathematical induction, used by the Italian mathematician Francesco Maurolico in 1575.

KEY POINT 5.1

The principle of mathematical induction:

Suppose that you have a statement (or rule) about a positive integer n.

1. If you can show that the statement is true for $n = 1$, and
2. if you assume that the statement is true for $n = k$, then you can prove that it is also true for $n = k+1$,

then the statement is true for all positive integers n.

In Step 2 you need to make a link between one proposition and the next – the inductive step. The exact way to do this depends upon the type of problem. In the following examples you will see how to apply the principle of mathematical induction in various contexts, and how to present your proofs correctly.

TOK Links

For a long time, it was thought that $1+1706n^2$ was never a square number. If you tried the first billion values of n, you would find that none of these are squares. The first example of a square is found when n is 30693385322765657197397207. Just trying lots of examples does not work; this is why methods such as proof by induction are so important in mathematics.

Induction and series

When applying mathematical induction to series, the link between $n = k$ and $n = k+1$ is simply adding the next term of the series.

WORKED EXAMPLE 5.1

Use the principle of mathematical induction to prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geqslant 1$.

Prove that the statement is true for $n = 1$ → When $n = 1$:

$$\text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{1(2)(3)}{6} = 1$$

So, the statement is true for $n = 1$.

Assume that the statement is true for $n = k$ and write down what this means → Assume it is true for $n = k$:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now let $n = k+1$: add the next term of the series → When $n = k+1$:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Keep rearranging, knowing what you want the final form to look like →

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

You need the right hand side to be

$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

As you want the answer to have a factor of $(k+1)$, write everything as a single fraction and factorize the expression. →

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

This is the expression you wanted. → So, the statement is true for $n = k+1$.

Write a conclusion → The statement is true for $n = 1$, and if true for $n = k$ it is also true for $n = k+1$. Therefore, the statement is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Induction and divisibility

Number theory is an important area of pure mathematics that is concerned with the properties of natural numbers. One of the important tasks in number theory is studying divisibility.

Tip

Note that in this example the first value of n is $n = 0$. Induction does not have to start from $n = 1$.

Consider the expression $f(n) = 7^n - 1$ for $n = 0, 1, 2\ldots$ Looking at the first few values of n:

$$f(0) = 7^0 - 1 = 0$$
$$f(1) = 7^1 - 1 = 6$$
$$f(2) = 7^2 - 1 = 48$$

It looks as if f(n) is divisible by 6 for all values of n. You can prove this using the principle of mathematical induction.

Tip

If a number is divisible by 6, it can be written as $6A$ for some integer A.

WORKED EXAMPLE 5.2

Prove that $f(n) = 7^n - 1$ is divisible by 6 for all integers $n \geqslant 0$.

Prove that the statement is true for $n = 0$:

$$f(0) = 7^0 - 1 = 0 = 6 \times 0$$

So, f(0) is divisible by 6.

Assume that the statement is true for $n = k$ and write down what this means:

Assume that f(k) is divisible by 6. Then $7^k - 1 = 6A$ for some $A \in \mathbb{Z}$.

Now let $n = k + 1$:

When $n = k + 1$:

$$f(k+1) = 7^{k+1} - 1$$

Relate $f(k + 1)$ to $f(k)$:

$$= 7 \times 7^k - 1$$

Using the result for $n = k$, $7^k = 6A + 1$:

$$= 7 \times (6A + 1) - 1$$

You are trying to prove that this is a multiple of 6, so simplify and factorize:

$$= 42A + 7 - 1$$
$$= 42A + 6$$
$$= 6 \times (7A + 1)$$

You have written $f(n+1)$ as a multiple of 6:

So, $f(k + 1)$ is divisible by 6.

Write a conclusion:

f(0) is divisible by 6, and if f(k) is divisible by 6 then so is f(k + 1) Therefore, f(n) is divisible by 6 for all $n \in \mathbb{N}$ by the principle of mathematical induction.

You are the Researcher

In number theory there are other methods of proving divisibility. In particular, deciding whether extremely large numbers are prime needs some very clever tests for divisibility. These are very important in code breaking and there are huge financial rewards for finding large prime numbers. You might like to research modular arithmetic and Fermat's little theorem.

Other examples of induction

Proof by induction can be used in many other situations involving a sequence of related statements. Examples include powers of complex numbers and repeated differentiation.

The following proof often appears in examination questions.

Proof 5.1

Prove De Moivre's theorem for all positive integers n:
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Prove the result for $n = 1$ ······ When $n = 1$:

$$\text{LHS} = (\cos\theta + i\sin\theta)^1$$
$$\text{RHS} = \cos(\theta) + i\sin(\theta)$$

So, the statement is true for $n = 1$.

Assume that the result is true for $n = k$ and write down what that means ······ Assume that the statement is true for $n = k$, so

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$$

Now let $n = k+1$. Make a link between powers k and $k+1$ ······ When $n = k+1$:

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$$

You want the RHS to equal $\cos((k+1)\theta) + i\sin((k+1)\theta)$ so expand the brackets and group real and imaginary terms ······

$$= (\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta) + i(\cos(k\theta)\sin\theta + \sin(k\theta)\cos\theta)$$

Recognise the expressions in brackets as $\cos(A+B)$ and $\sin(A+B)$ ······

$$= \cos((k+1)\theta) + i\sin((k+1)\theta)$$

So, the statement is true for $n = k+1$.

The statement is true for $n = 1$, and, if true for $n = k$, it is also true for $n = k+1$.

Write a conclusion ······ Therefore, the statement is true for all $n \in \mathbb{N}^+$ by the principle of mathematical induction.

Be the Examiner 5.1

Prove by induction that $1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$.

Find all the mistakes in this proof.

Let $n = k$:

$1 + 2 + \ldots + k = \frac{1}{2}k(k+1)$

When $n = k+1$:

$1 + 2 + \ldots + k + (k+1) = \frac{1}{2}(k+1)(k+2)$

$\frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$

Multiply by 2 and divide by $(k+1)$:

$k + 2 = k + 2$

So it is true when $n = k+1$.
The statement is true for $n = k$ and it is also true for $n = k+1$. So it is true for all n by induction.

TOK Links

Inductive reasoning is also used in science, where a conjecture is made on the basis of observed examples. How is this different from mathematical induction?

TOOLKIT: Problem Solving

There are 7 rational pirates, each senior to the next, who have found 10 coins.

The most senior pirate proposes a way of sharing out the coins.

They all vote. If the majority accepts the proposal, the coins are distributed and the process ends. In the case of a tie, the most senior pirate has the casting vote.

If the majority reject the proposal, the pirate who proposed it is thrown overboard and the process starts again with the next most senior pirate in charge.

You know the following facts about all the pirates:

- Each pirate wants to survive.
- Given survival, each pirate wants to maximize their number of coins.
- Each pirate would prefer to throw another overboard, if all else is equal.

How many coins can the most senior pirate get?

TOOLKIT: Proof

What is wrong with the following famous, flawed, proof by induction, proving that all horses are the same colour?

Let P(n) be the proposition that in any group of size n, all horses in that group are the same colour.

P(1) is clearly true – in a group of size one all horses will be the same colour.

Assume P(k) is true: in any group of size k all horses are the same colour.

Now in a group of size $k+1$ we can arbitrarily put the horses into order. The first k must all be the same colour since P(k) is assumed to be true. The last k must all be the same colour since P(k) is assumed to be true. Therefore, the first horse is the same colour as the overlapping group which is the same colour as the last horse so all $k+1$ horses are the same colour.

Therefore, since P(1) is true and P(k) implies P($k+1$), the proposition is true for all $n \geqslant 1$.

Exercise 5A

1 Show that $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

2 Use the principle of mathematical induction to prove that $1\times 3 + 2\times 4 + \ldots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$ for all $n \in \mathbb{Z}^+$.

3 Prove by induction that $\displaystyle\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$.

4 A sequence is defined by $u_n = 2\times 3^{n-1}$. Use the principle of mathematical induction to prove that $u_1 + u_2 + \ldots + u_n = 3^n - 1$.

5 Prove by induction that $\dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \dfrac{1}{5\times 7} + \ldots + \dfrac{1}{(2n-1)\times(2n+1)} = \dfrac{n}{2n+1}$.

6 Use induction to show that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.

7 Prove by induction that $\sum_{r=1}^{n} 3r(r-1) = n(n^2-1)$.

8 Show that $5^n - 1$ is divisible by four for all $n \in \mathbb{N}$.

9 Show that $4^n - 1$ is divisible by three for all $n \geqslant 1$.

10 Show that $7^n - 3^n$ is divisible by four for all $n \in \mathbb{N}$.

11 Use induction to prove that $30^n - 6^n$ is divisible by 12 for all integers $n \geqslant 0$.

12 Show using induction that $n^3 - n$ is divisible by six for all integers $n \geqslant 1$.

13 Using the principle of mathematical induction, prove that $n(n^2+5)$ is divisible by six for all integers $n \geqslant 1$.

14 Use induction to show that $7^n - 4^n - 3n$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

15 Prove, using the principle of mathematical induction, that $3^{2n+2} - 8n - 9$ is divisible by 64 for all positive integers n.

16 A sequence is given by the recurrence relation $u_1 = 7$, $u_{n+1} = 2u_n + 3$ for $n \geqslant 1$. Prove that the general formula for the sequence is $u_n = 5 \times 2^n - 3$.

17 Given that $u_{n+1} = 5u_n - 8$, $u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

18 A sequence has first term 1 and subsequent terms defined by the recurrence relation $u_{n+1} = 3u_n + 1$.
Prove by induction that $u_n = \frac{3^n - 1}{2}$.

19 Given that $y = \frac{1}{1-x}$, prove that $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$.

20 Given that $y = \frac{1}{1-3x}$, prove that $\frac{d^n y}{dx^n} = \frac{3^n\, n!}{(1-3x)^{n+1}}$.

21 Use induction to prove that $\frac{d^n}{dx^n}(xe^{2x}) = (2^n x + n2^{n-1})e^{2x}$.

22 Prove by induction that $\frac{d^{2n}}{dx^{2n}}(x \sin x) = (-1)^n (x \sin x - 2n \cos x)$.

23 Prove that $\frac{d^n}{dx^n}\left(x^2 e^x\right) = \left(x^2 + 2nx + n(n-1)\right)e^x$ for $n \geqslant 2$.

24 **a** Show that for any two complex numbers z and w, $(zw)^* = z^* w^*$.
b Prove by induction that $(z^n)^* = (z^*)^n$ for all positive integers n.

25 A sequence is given by the recurrence relation $u_1 = 5$ and $u_2 = 13$, $u_{n+2} = 5u_{n+1} - 6u_n$ for $n \geqslant 1$. Prove that the general formula for the sequence is $u_n = 2^n + 3^n$.

26 Given that $u_1 = 3$, $u_2 = 36$, $u_{n+2} = 6u_{n+1} - 9u_n$ prove by induction that $u_n = (3n-2)3^n$.

27 Prove that $1 \times 1! + 2 \times 2! + 3 \times 3! \ldots + n \times n! = (n+1)! - 1$.

28 Use the principle of mathematical induction to show that $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$.

29 Prove that $(n+1) + (n+2) + (n+3) + \ldots + (2n) = \frac{1}{2} n(3n+1)$.

30 Prove that $\sum_{k=1}^{n} k\, 2^k = (n-1)2^{n+1} + 2$.

31 Show that the sum of the cubes of any three consecutive integers is divisible by nine.

32 Prove by induction that $2 \times 6 \times 10 \times \ldots \times (4n-2) = \frac{(2n)!}{n!}$.

33 A sequence has first term 1 and the subsequent terms are given by the recurrence relation $u_{n+1} = \frac{u_n}{u_n + 1}$.
Show that the nth term of the sequence is given by $u_n = \frac{1}{n}$.

34 Use mathematical induction to prove that $2^n > 1 + n$ for all $n > 1$.

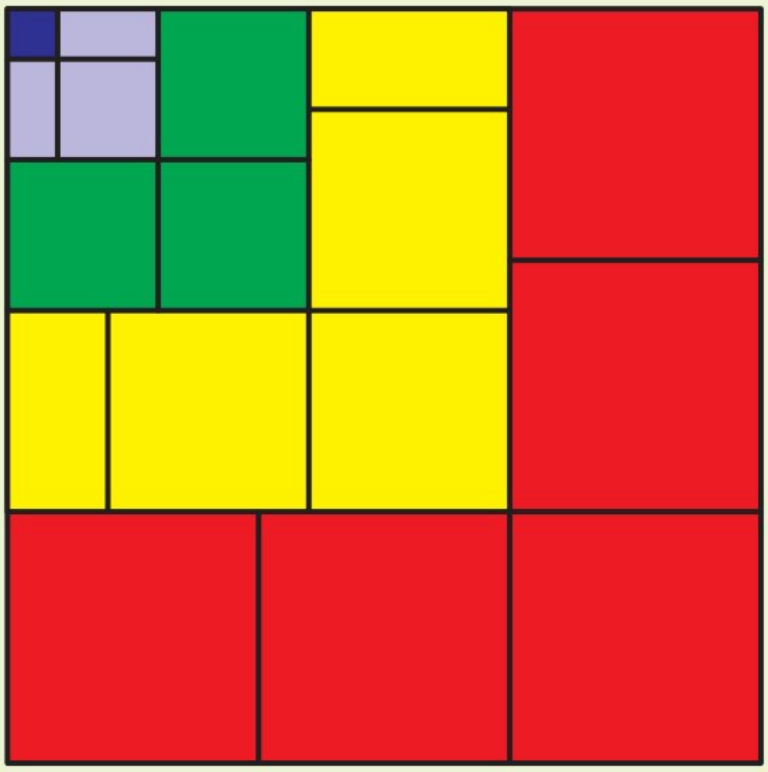
35 Prove by induction that $2^n > n^2$ for all $n \geqslant 4$.

CONCEPTS – GENERALIZATION AND VALIDITY

The result of question 1 in Exercise 5A suggests the following surprising **general** result:

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r\right)^2$$

The following 'proof without words' also suggests the same result. Can you explain why? Can you generalize this diagram?



Is a 'proof without words' a **valid** proof? What makes a proof valid?

5B Proof by contradiction

Proof by contradiction is an indirect method of proof, where you start by assuming the opposite of the proposition you want to prove and show that this assumption leads to an impossible or contradictory conclusion.

Proof by contradiction was used as early as 300 BC, when Euclid used it to prove that there are infinitely many prime numbers.

As an example, given the value of n^2 is odd, how can we prove that n must also be an odd number? To use proof by contradiction, you assume the opposite – that is, that n is an even number, given that n^2 is odd. So if n is even, then $n = 2k$, and $n^2 = (2k)^2 = 4k^2$, which is an even number. But this contradicts the initial statement that n^2 is odd. So, it must be that the assumption – that n is even – was false.

One of the most famous examples of proof by contradiction is the proof that $\sqrt{2}$ is an irrational number.

Tip

An irrational number cannot be written as a fraction.

WORKED EXAMPLE 5.3

Prove that $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$.

Try writing $\sqrt{2}$ as a fraction and see if this leads to a contradictory conclusion

Suppose we can write $\sqrt{2} = \frac{p}{q}$, where p and q are integers

Also assume that the fraction is in its simplest form

which have no common factors.

Now try to find an equation relating p and q which involves only whole numbers

Then

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

Use the fact that, if n^2 is an even number, so is n. (You can prove this fact by contradiction, in a similar way to the example in the text above)

This means that p^2 is an even number, so p is also an even number.
Write $p = 2k$:

$$(2k)^2 = 2q^2$$

$$4k^2 = 2q^2$$

$$2k^2 = q^2$$

The last equation shows that q^2 is a multiple of 2

This means that q^2 is an even number, so q is also an even number.

Summarise what you have proved so far. Does this contradict any assumptions you previously made?

We have proved that p and q are both even. But this contradicts the assumption that the fraction $\frac{p}{q}$ was in its simplest form.

Write a conclusion

Hence, the assumption that $\sqrt{2} = \frac{p}{q}$ must be false, meaning that $\sqrt{2}$ cannot be written as a fraction.

TOK Links

Proof by contradiction relies on the law of the excluded middle, which says that either a statement or its negative must be true. Can this principle be applied in other areas of knowledge?

You are the Researcher

You might like to research the German mathematician Georg Cantor, who used proof by contradiction to show that it is impossible to put all real numbers into an infinite list.

Exercise 5B

1 For an integer n, given that n^2 is an even number, prove that n is also an even number.

2 For integers a and b, given that ab is an even integer, prove that at least one of a and b must be even.

3 Prove that $\sqrt{5}$ is an irrational number.

4 Prove that $\sqrt[3]{2}$ is irrational.

5 The mean height of three children is 126 cm. Show that at least one of them must be at least 126 cm tall.

6 a Show by direct calculation that the difference of two rational numbers is rational.

b Use proof by contradiction to show that the sum of a rational number and an irrational number is irrational.

7 Prove that $\log_2 3$ is an irrational number.

8 Prove that $\log_3 7$ is irrational.

9 Prove that there is no largest even integer.

10 Prove that there is no smallest positive real number.

11 a Let p_1, p_2 and p_3 be three different prime numbers. Show that the number $p_1 p_2 p_3 + 1$ is not divisible by any of p_1, p_2 and p_3.

b Prove that there are infinitely many prime numbers.

12 a By sketching a suitable graph, show that the equation $x^3 + x - 1 = 0$ has exactly one real root.

b Prove by contradiction that this root is irrational.

5C Disproof by counterexample

Proving mathematical statements can be difficult, and you have now met several methods of proof. On the other hand, to disprove a statement you only need to find one **counterexample**.

Tip

The symbol ≡ means that the statement is an identity, true for all values of x.

WORKED EXAMPLE 5.4

Use counterexample to disprove the statement: $\sqrt{x^2 + 25} \equiv x + 5$.

Find a value of x for which $\sqrt{x^2+25} \neq x + 5$ — Let $x = 2$:

Show clearly that the two sides are not equal — LHS $= \sqrt{2^2 + 25} = \sqrt{29} \approx 5.39$
RHS $= 2 + 5 = 7$

Write a conclusion — So, $x = 2$ is a counterexample.

Tip

Not every value of x can be used as a counterexample. For example, when $x = 0$, $\sqrt{x^2 + 25}$ does equal $x + 5$.

Notice that it is not enough to just guess a counterexample – you need to clearly communicate how you know that this forms a counterexample.

You are the Researcher

Austrian mathematician Kurt Gödel proved in 1931 that there are some true mathematical statements that can be neither proved nor disproved. This result, that you might like to research further, is known as Gödel's incompleteness theorem.

Exercise 5C

1. Use a counterexample to disprove the statement $\sqrt{x^2 - 1} \equiv x - 1$.
2. Use a counterexample to disprove the statement $(x - y)^3 \equiv x^3 - y^3$.
3. Use a counterexample to disprove the statement $\ln(a + b) \equiv \ln a + \ln b$.
4. Use a counterexample to show that if $\frac{dy}{dx} = 2x$, then it is not necessarily true that $y = x^2$.
5. Use a counterexample to show that the following statement is not true:
   $\sin 2x = 1 \Rightarrow x = 45°$.
6. Consider the statement: If $\frac{a}{b} = \frac{c}{d}$, then $a = b$ and $c = d$. Use a counterexample to disprove this statement.
7. Renzhi says that a quadrilateral with four equal sides must be a square. Use a counterexample to disprove his statement.
8. Katie thinks that $\sqrt{x^2} = x$ for all real numbers x. Give a counterexample to disprove her statement.
9. Use a counterexample to disprove the statement: If ab is an integer, then a and b are both integers.
10. Let $f(n) = n^2 + n + 11$.
    a. Show that $f(n)$ is a prime number for $n = 1, 2$ and 3.
    b. Ilya says that $f(n)$ is a prime number for all integer values of n. Use a counterexample to disprove Ilya's statement.
11. Use a counterexample to disprove this statement: The sum of two irrational numbers is an irrational number.
12. Use a counterexample to disprove the statement: If $x^2 > 100$, then $x > 10$.
13. Find a counterexample to disprove the statement: If $z^4 = 1$, then $z = 1$ or -1.
14. Angela says that an irrational number raised to an irrational power is always irrational. By considering $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$, disprove Angela's statement.

Checklist

- You should be able to use the principle of mathematical induction to prove statements about patterns involving integers:
  - If you can show that:
    - the statement is true for $n = 1$, and
    - when the statement is true for $n = k$, it is also true for $n = k + 1$, then the statement is true for all positive integers n.
  - The proof can be adapted for a starting value other than $n = 1$.
- You should be able to use proof by contradiction:
  - Assume that the statement you are trying to prove is false and show that this leads to an impossible or contradictory conclusion.
- You should be able to use counterexamples to disprove a statement.
  - One counterexample is sufficient to prove that a statement is false.

Mixed Practice

1. Prove by induction that $1 \times 2 + 2 \times 3 + 3 \times 4 \ldots + n(n + 1) = \frac{n}{3}(n + 1)(n + 2)$.
2. Use induction to prove that $\sum_{r=1}^{n} r(3r - 5) = n(n + 1)(n - 2)$.
3. Prove by contradiction that $\sqrt{3}$ is an irrational number.
4. Use a counterexample to disprove the statement: $(x + 2)^2 \equiv x^2 + 4$.
5. Use a counterexample to disprove the statement: $a^x + a^y \equiv a^{x+y}$.

6 Use a counterexample to disprove the statement: If $a + b$ is an integer, then a and b are both integers.

7 Marek thinks that all prime numbers are odd. Use a counterexample to show that he is wrong.

8 Use induction to prove that $1 \times 2^2 + 2 \times 3^2 + \ldots + n(n+1)^2 = \dfrac{n(n+1)(n+2)(3n+5)}{12}$.

9 Prove by induction that $\displaystyle\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)} = \frac{2n}{2n+1}$.

10 Prove by induction that $12^n - 1$ is divisible by 11 for all integers $n \geqslant 0$.

11 Use induction to prove that $3^{2n} + 7$ is divisible by 8 when $n \in \mathbb{N}$.

12 Use the method of mathematical induction to prove that $5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

Mathematics HL May 2013 Paper 2 TZ2 Q8

13 Consider a function f, defined by $f(x) = \dfrac{x}{2-x}$ for $0 \leqslant x \leqslant 1$.

a Find an expression for $(f \circ f)(x)$.

Let $F_n(x) = \dfrac{x}{2^n - (2^n - 1)x}$, where $0 \leqslant x \leqslant 1$.

b Use mathematical induction to show that, for any $n \in \mathbb{Z}^+$,

$$\underbrace{(f \circ f \ldots \circ f)}_{n\text{ times}}(x) = F_n(x)$$

c Show that $F_{-n}(x)$ is an expression for the inverse of F_n.

Mathematics HL November 2012 Paper 1 Q12, parts (a)–(c)

14 **a** **i** Express the sum of the first n positive odd integers using sigma notation.

ii Show that the sum stated above is n^2.

iii Deduce the value of the difference between the sum of the first 47 positive odd integers and the sum of the first 14 positive odd integers.

b A number of distinct points are marked on the circumference of a circle, forming a polygon. Diagonals are drawn by joining all pairs of non-adjacent points.

i Show on a diagram all diagonals if there are 5 points.

ii Show that the number of diagonals is $\dfrac{n(n-3)}{2}$ if there are n points, where $n > 2$.

iii Given that there are more than one million diagonals, determine the least number of points for which this is possible.

Mathematics HL May 2013 Paper 2 TZ2 Q11, parts (a) and (b)

15 Prove with induction that the sum of the first n terms of a geometric series with first term a and common ratio r is $\dfrac{a(r^n - 1)}{r - 1}$.

16 Use mathematical induction to prove that, for all positive integers n, $9^n - 2^n$ is divisible by 7.

17 Use the principle of mathematical induction to show that $15^n - 2^n$ is a multiple of 13 for all $n \in \mathbb{N}$.

18 Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for $n \geqslant 0$, $n \in \mathbb{N}$.

19 Use proof by contradiction to show that $\log_{10} 3$ is an irrational number.

20 **a** Alia says: 'If a and b are integers such that ab is odd, then both a and b must be odd'. Use proof by contradiction to show that Alia is correct.

b Bahar says: 'If a and b are integers such that ab is even, then both a and b must be even'. Use a counterexample to disprove Bahar's statement.

21 Prove by contradiction that there are infinitely many odd numbers.

22 Gabriella says: 'If two straight lines do not intersect, then they are parallel'. Give a counterexample to disprove her statement.

23 Prove by induction that

$$\sum_{r=1}^{r=n} \frac{r}{2^r} = 2 - \left(\frac{1}{2}\right)^n (n+2), \quad n \in \mathbb{N}.$$

24 Using the principle of mathematical induction, show that $2^n > 11n$ for all integers $n \geqslant 7$.

25 **a** Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geqslant 0$, $n \in \mathbb{Z}$.

b Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$.

c Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geqslant 2$, $n \in \mathbb{Z}$.

Mathematics HL May 2015 Paper 1 TZ2 Q13

26 Use mathematical induction to prove that $(2n)! \geqslant 2^n(n!)^2$, $n \in \mathbb{Z}^+$.

Mathematics HL November 2014 Paper 1 Q8

27 **a** Use induction to prove De Moivre's theorem for all positive integers:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

b Use induction to prove De Moivre's theorem for all negative integers.

28 Use induction to show that the sum of the squares of the first n odd numbers can be written as $\frac{n}{3}(an^2 - 1)$, where a is an integer to be found.

29 Prove by induction that, for any positive integer n,

$$2 \times 4 \times 6 \ldots (4n-2) = \frac{(2n)!}{n!}$$

30 Prove, using induction, that for positive integer n,

$$\cos x \times \cos 2x \times \cos 4x \times \ldots \times \cos(2^n x) = \frac{\sin(2^n x)}{2^n \sin x}$$

31 **a** Prove, using induction, that

$$\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}, \quad n \in \mathbb{Z}^+$$

b Hence find the exact value of $\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} \ldots + \sin\frac{13\pi}{7}$.

32 Prove that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} \ldots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$.

33 **a** By using the formula for ${}^nC_r$ show that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 \leqslant r \leqslant n-1$.

b Prove by induction that ${}^nC_1 + {}^nC_2 + \ldots {}^nC_{n-1} = 2^n - 2$.

34 **a** Prove by induction that for all positive integers $n^5 - n$ is divisible by five.

b By factorizing prove that $n^5 - n$ is also divisible by six.

c Is $n^5 - n$ always divisible by 60? Either prove that it is, or give a counter example to disprove it.

35 There are n lines in a plane such that no two are parallel and no three pass through the same point.

Use induction to show that the number of intersection points created by these lines is $\frac{n(n-1)}{2}$.

Prove also that the number regions formed is $\frac{n(n+1)}{2} + 1$.

6 Polynomials

ESSENTIAL UNDERSTANDINGS

- Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

In this chapter you will learn…

- how to sketch graphs of polynomial functions
- how to find a remainder when two polynomials are divided
- check for factors of a polynomial
- about a relationship between roots and coefficients of a polynomial.

CONCEPTS

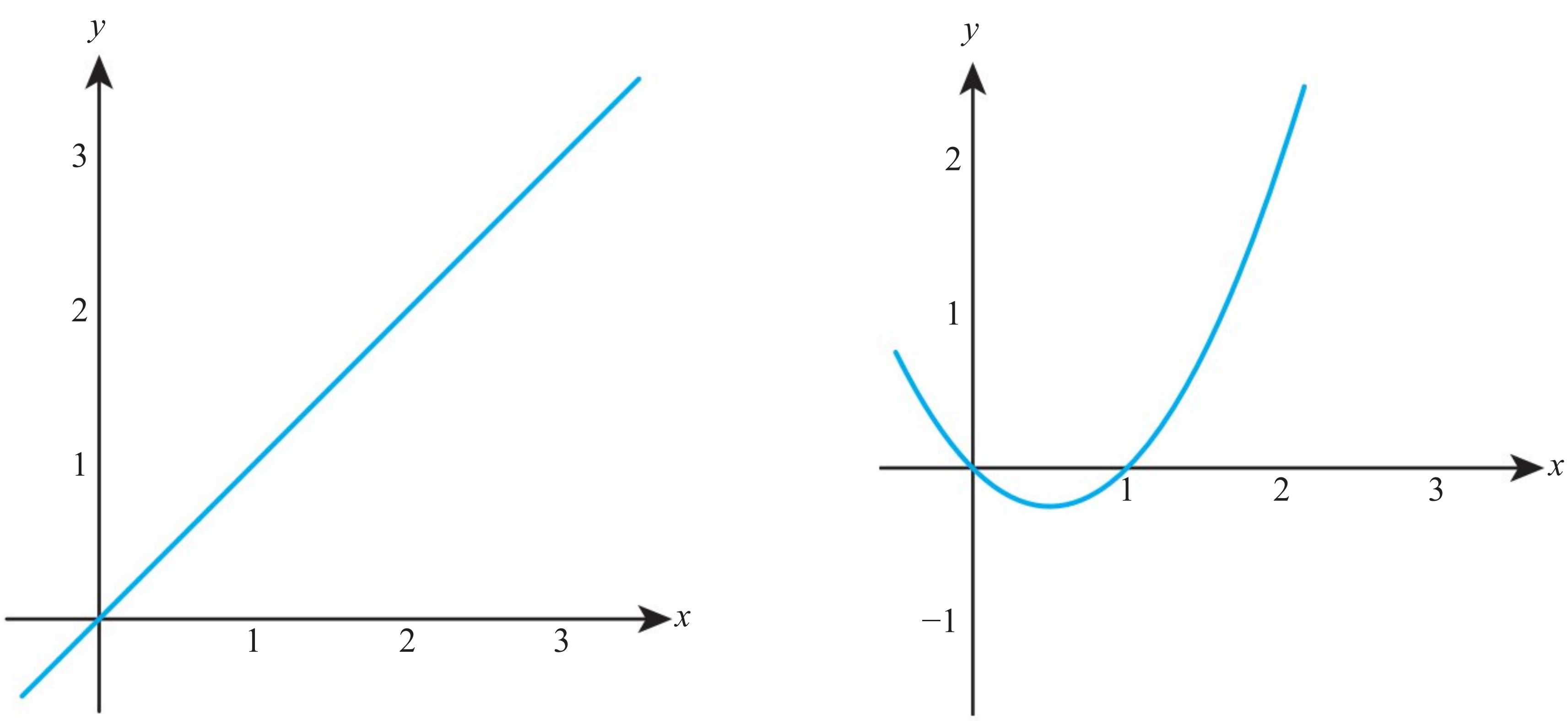
The following concepts will be addressed in this chapter:

- Different **representations** of functions, symbolically and visually as graphs, equations and tables, provide different ways to communicate mathematical **relationships**.
- The parameters in a function or equation correspond to geometrical features of a graph and can **represent** physical quantities in **spatial** dimensions.
- Moving between different forms to **represent** functions allows for deeper understanding and provides different approaches to problem solving.
- Extending results from a specific case to general form can allow us to apply them to a larger **system**.

LEARNER PROFILE – Principled

Does 'fair' mean the same thing as 'equal'? Would it be fair for everybody to get equal results in an exam? How can mathematics be used to define 'fairness'?

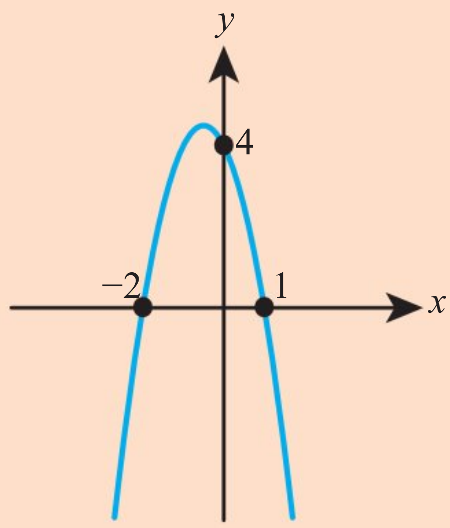
Figure 6.1 What comes next in this sequence?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Find the x-intercepts of the graph of $y = 3x^2 + x - 2$.
2. Find the equation of this quadratic graph:
3. Given that $x = 2$ is one root of the equation $x^3 - 8x^2 + 22x - 10$, find the other two roots.



Polynomial equations are used in modelling many real-life situations, for example those involving length, area and volume. You already know how to factorize, solve equations and draw graphs involving quadratic polynomials. Those ideas are now extended to look at expressions involving higher powers of x.

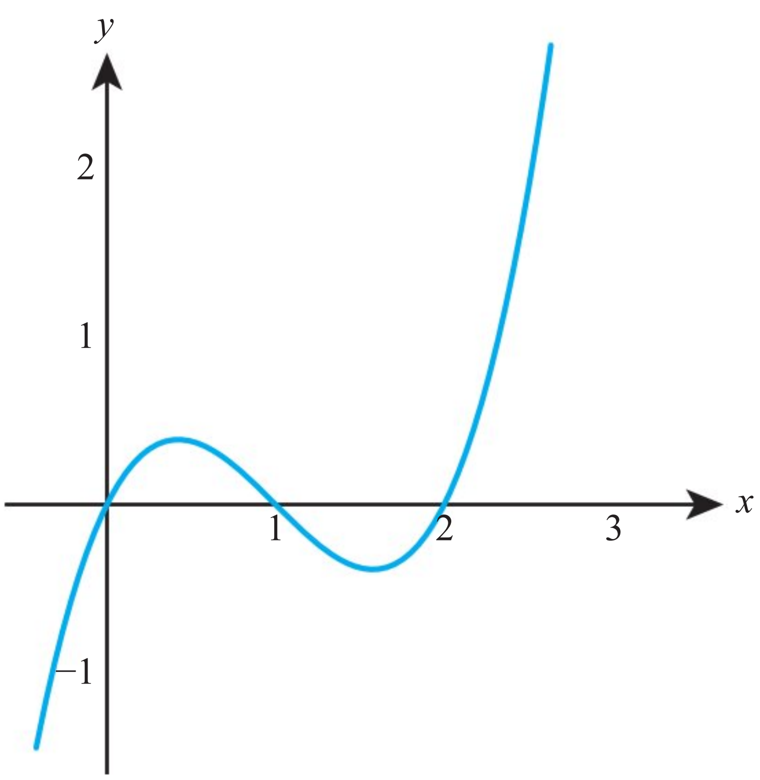
Starter Activity

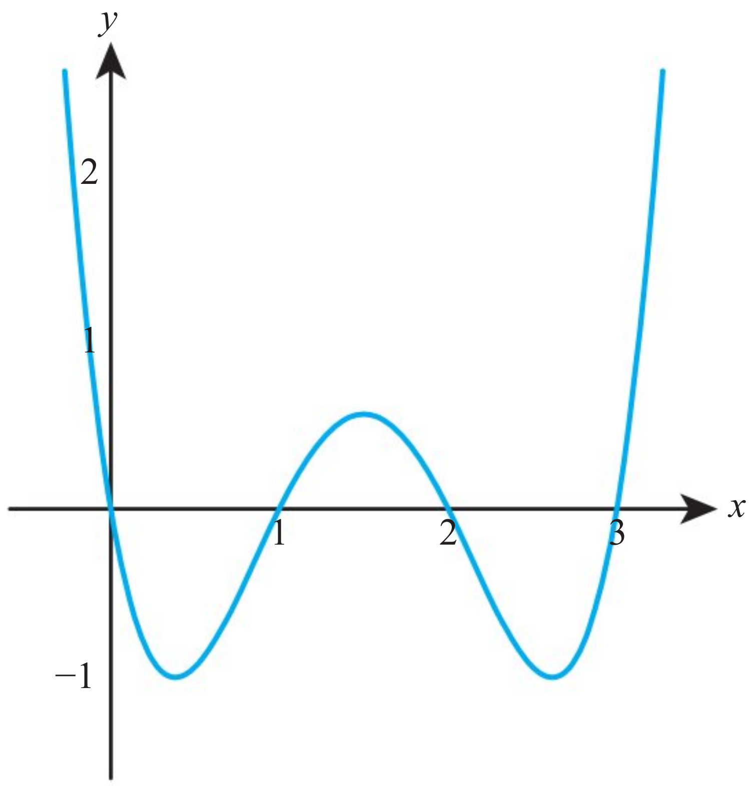
Look at the graphs shown in Figure 6.1. Suggest the next two graphs in the sequence. In small groups, discuss the relationship between the number of zeros and the number of turning points of the graphs.

Now look at this problem:

The table below shows twelve functions. Divide the functions into groups based on a particular criterion. How many different criteria can you think of to categorize them? You should consider both algebraic and graphical representations of the functions.

$f_1(x) = 2(x-1)(x-2)(x-4)$	$f_5(x) = x^4 - 16$	$f_9(x) = -4x^2 + 20x - 16$
$f_2(x) = x^2 - 5x + 4$	$f_6(x) = 2(x-1)^3 + 16$	$f_{10}(x) = (x-2)(x+2)(x^2+4)$
$f_3(x) = 4(x-1)(4-x)$	$f_7(x) = 2x^3 - 14x^2 + 30 - 16$	$f_{11}(x) = -x^4 + 10x^3 - 33x^2 + 40x - 16$
$f_4(x) = (x-2.5)^2 - 2.25$	$f_8(x) = -(x-4)^2(x-1)^2$	$f_{12}(x) = 2x^3 - 6x^2 + 6x + 14$

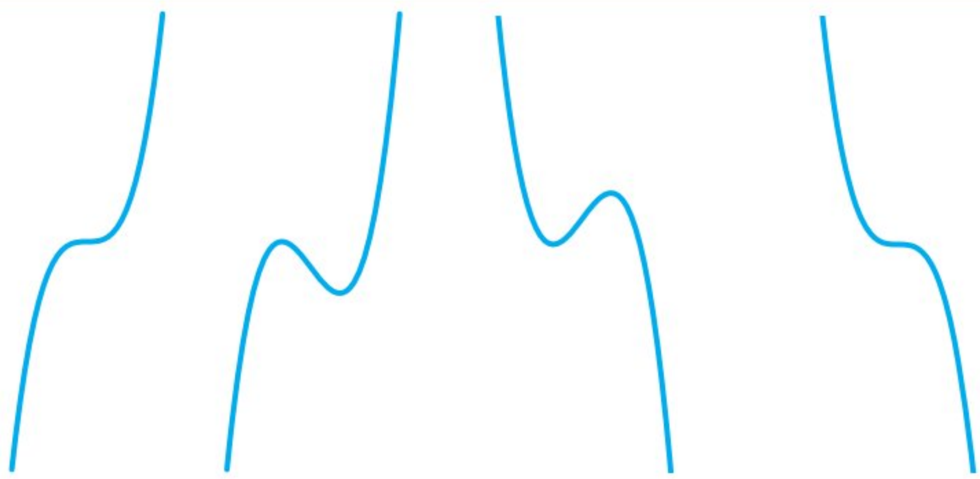
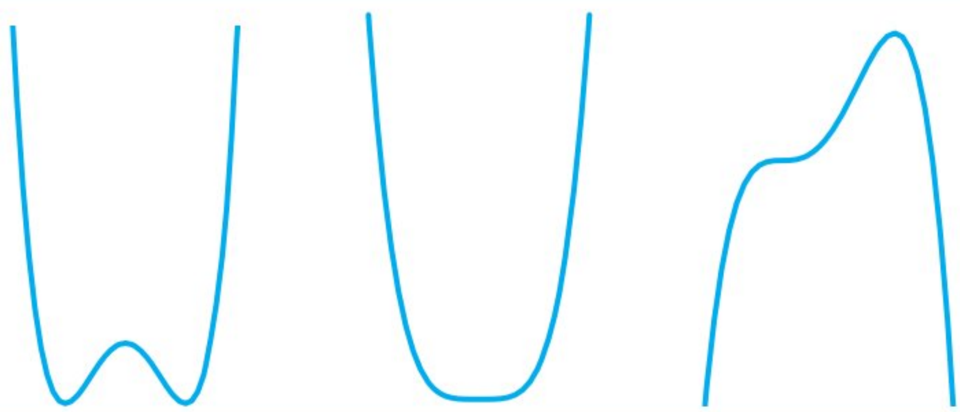




6A Graphs and equations of polynomial functions

A **polynomial** is an expression that can be written as a sum of terms involving only non-negative integer powers of x. For example, $5x^3 - x + 4$ is a polynomial, but $x^2 + \frac{3}{x}$ is not. The highest power of x in the expression is called the **degree** or **order** of the polynomial.

Possible shapes of a polynomial graph depend on its degree. The table shows some possible shapes of graphs of polynomials with degree 2, 3 and 4.

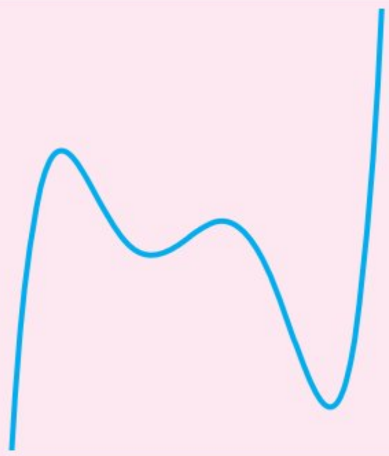
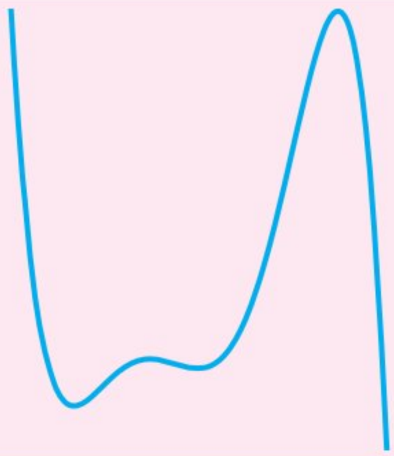
degree	name	examples of possible graphs
2	quadratic	
3	cubic	
4	quartic	

The general shape depends on whether the degree is even or odd, and whether the term with the highest power has a positive or negative coefficient.

The linear expression $mx + c$ that you studied in Chapter 4 of the Mathematics: analysis and approaches SL book can be considered a polynomial of degree 1.

KEY POINT 6.1

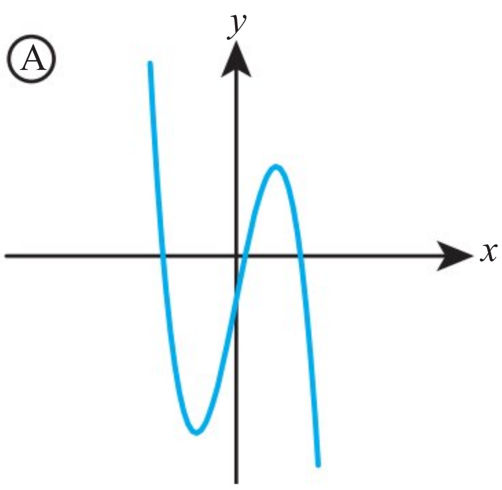
General shape of a polynomial graph with the highest order term ax^n:

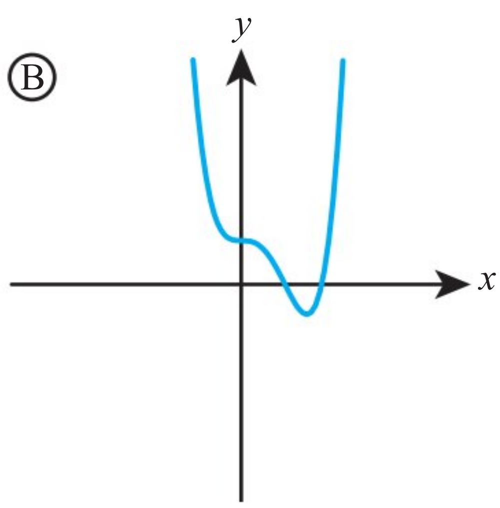
	$a > 0$	$a < 0$
n even		
n odd		

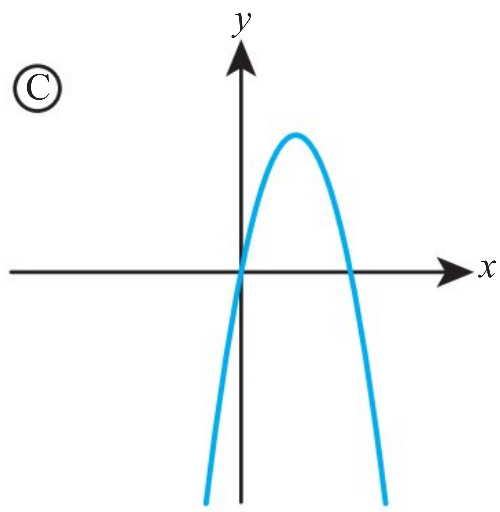
WORKED EXAMPLE 6.1

Match each equation with the corresponding graph, giving reasons for your choice.

a $y = -2x^2 + 5x$ **b** $y = -2x^3 + 5x - 1$ **c** $y = x^4 - 2x^3 + 1$







Graphs B and C show polynomials of even degree because their long-term behaviour is the same at very positive and very negative x-values. B has a positive highest order coefficient

Graph B has equation (c)
Graph C has equation (a)

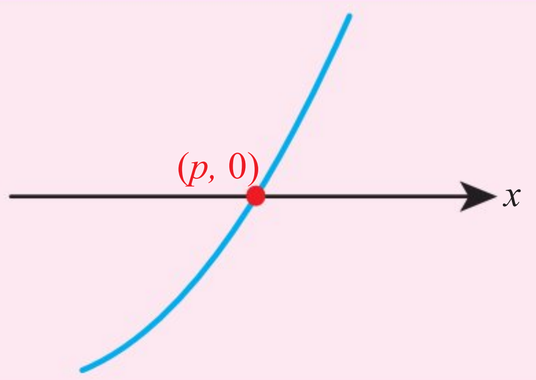
Graph A shows a polynomial of odd degree, because at very negative x-values it is very positive, and at very positive x-values it is very negative. It therefore also has a negative highest order coefficient

Graph A has equation (b)

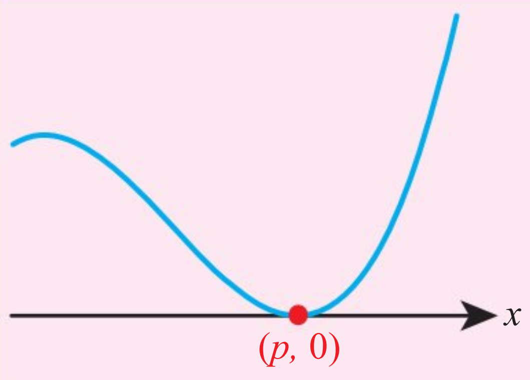
Factors and zeroes

Factors of a polynomial tell you where the graph crosses the x-axis.

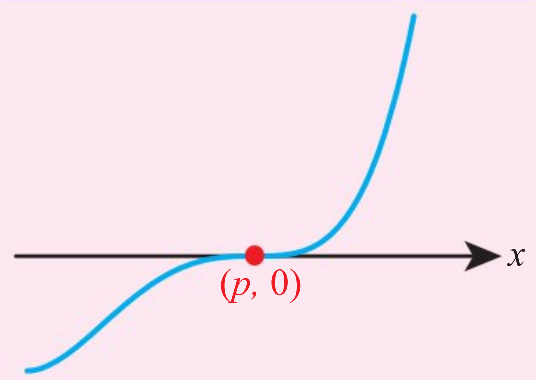
KEY POINT 6.2



If a polynomial has a factor $(x-p)$, then the graph crosses the x-axis at $(p, 0)$.



If a polynomial has a factor $(x-p)^2$, then the graph touches the x-axis at $(p, 0)$.



If a polynomial has a factor $(x-p)^3$, then the graph crosses the x-axis and has zero gradient at $(p, 0)$.

CONCEPTS – SPACE

Spatially, if the function has a factor of $(x-a)^2$, then the part of the graph close to $x = a$ looks like a quadratic. If the function has a factor of $(x-a)^3$, then close to $x = a$ it looks like a cubic. This idea of approximating more complex functions with simpler ones over a small region is hugely important in advanced mathematics, physics and economics.

WORKED EXAMPLE 6.2

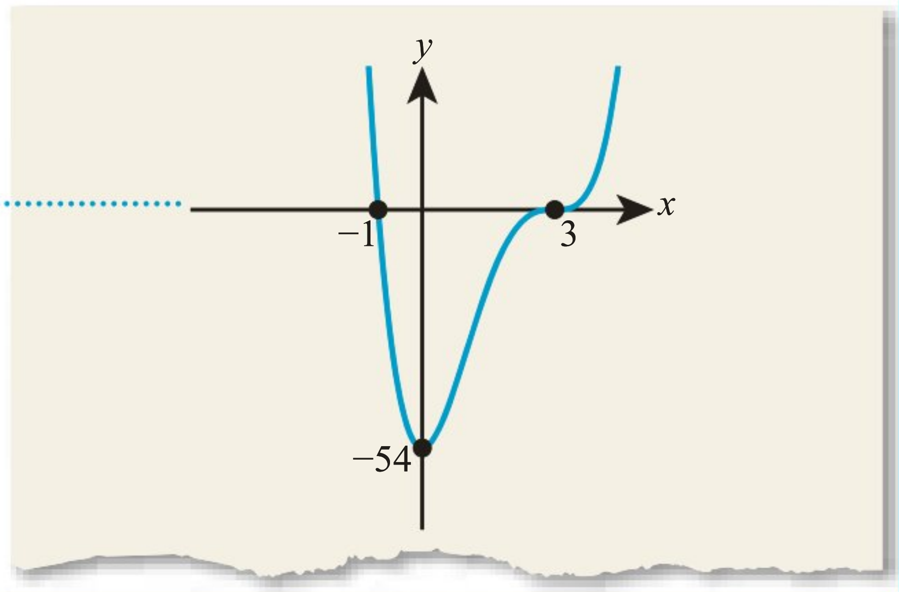
Sketch the graph of $y = 2(x+1)(x-3)^3$.

If the brackets were expanded, the term with the highest power would be $2x^4$

From the factors, the x-intercepts are -1 and 3

Since $(x-3)$ is cubed, the graph has zero gradient at the intercept $(3, 0)$

The y-intercept is $2(0+1)(0-3)^3 = -54$



Tip

A graph sketch should show the shape and the intercepts.

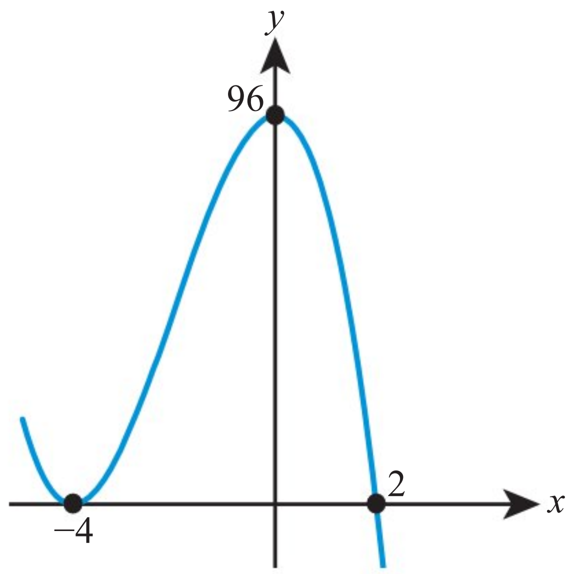
TOK Links

A function is known to have $f(0)=0$, $f(1)=1$, $f(2)=2$.
This could be modelled by the function $f(x)=x$ or $f(x)=x^3-3x^2+3x$ or $f(x)=2x^3-6x^2+5x$.
In the absence of any other information it is difficult to know which one to choose, but there is an idea from philosophy called Occam's razor that says that generally the simpler solution is more likely to be correct. Is this a valid way of choosing between models in mathematics? How about in science?

You can also use the graph to find the equation, by writing it in the factorized form first. Remember that, as well as using the x-intercepts to find factors, you also need to find the coefficient of the highest order term. This can be done by using any other point on the graph (often the y-intercept).

WORKED EXAMPLE 6.3

Find a possible equation of this graph.



The graph looks like a cubic with a negative x^3 term

The factors are $(x + 4)$ and $(x - 2)$. The graph touches the x-axis at $(-4, 0)$ so the factor $(x + 4)$ is squared

$y = -a(x + 4)^2(x - 2)$

To find a, use the y-intercept

When $x = 0,\ y = 96$:

$-a(4)^2(-2) = 96$

$32a = 96 \Rightarrow a = 3$

So, the equation is $y = -3(x + 4)^2(x - 2)$

Be the Examiner 6.1

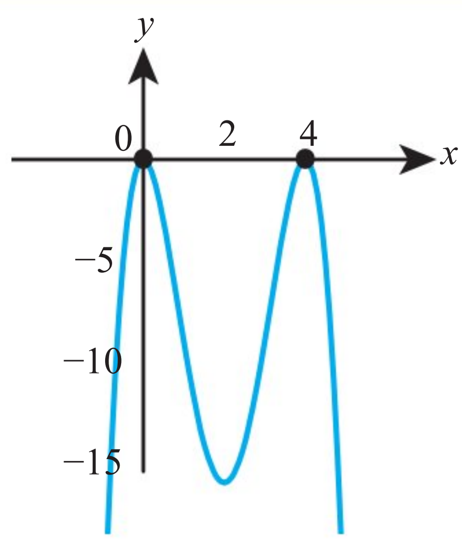
Which of the following is a possible equation of this graph?

A $y = x(4 - x)$

B $y = x^2(4 - x)^2$

C $y = -x^2(x - 4)^2$

Which is the correct solution? Identify the errors made in the incorrect solutions.



Solution 1	Solution 2	Solution 3
The x-intercepts are (0, 0) and (4, 0). The graph is upside down so the x^2 term is negative. The answer is A.	The graph touches both x-intercepts, so both factors are squared. The graph is upside down so the x^4 term is negative. The answer is B.	The graph touches both x-intercepts, so both factors are squared. The graph is upside down so the x^4 term is negative. The answer is C.

TOOLKIT: Modelling

Plot the following data using technology:

x	y
0	0
1	28
2	36
3	30

What is the best quadratic you can fit to this data set? What is the best cubic?

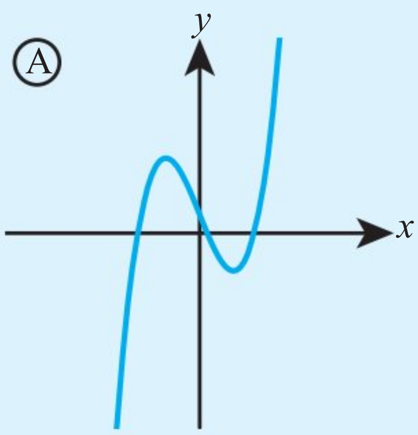
How many points does it take to define a linear function? A quadratic function? A cubic function?

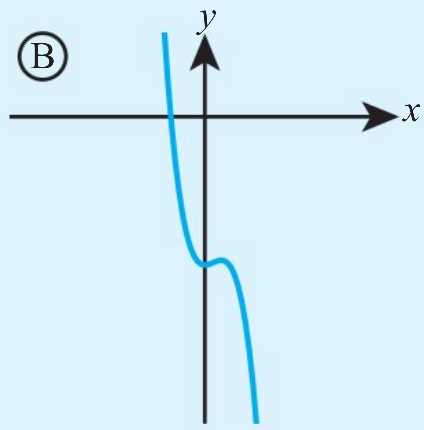
The process of fitting a polynomial to a data set is called polynomial regression, and there are many interesting methods of doing this. Many spreadsheets provide this option. You might like to investigate further how they do it.

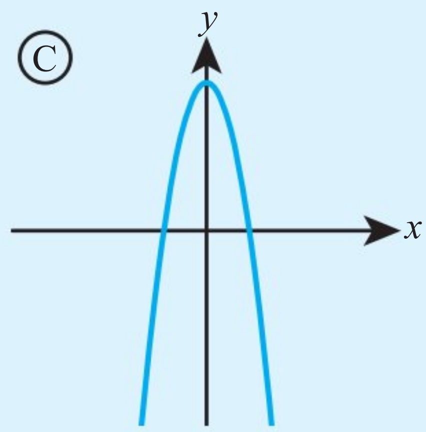
Exercise 6A

For questions 1 to 3, use the method demonstrated in Worked Example 6.1 to match each equation with its graph.

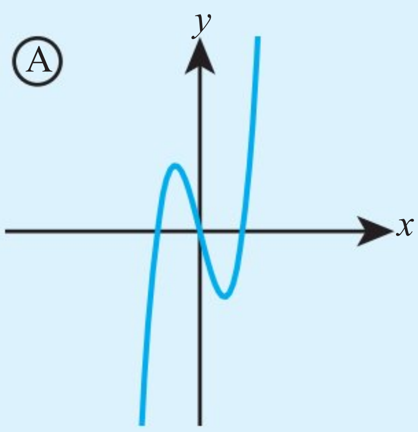
1 a i $y = 2x^3 - 5x + 1$ **ii** $y = 4 - 3x^2$ **iii** $y = -3x^3 + 2x^2 - 4$

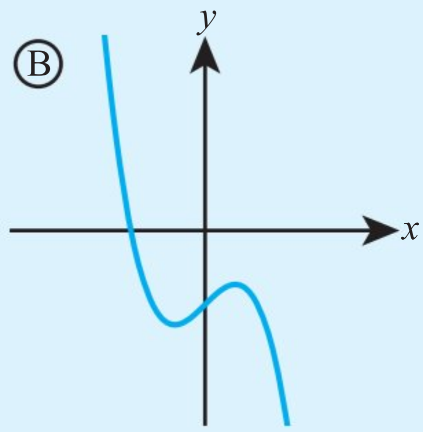


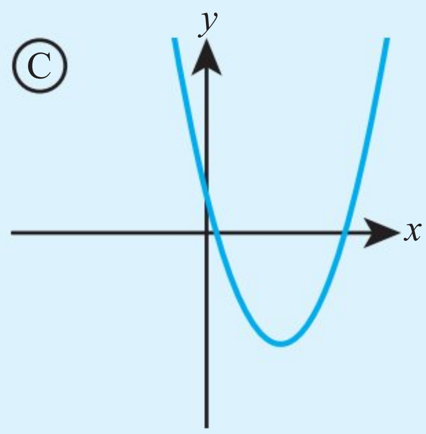




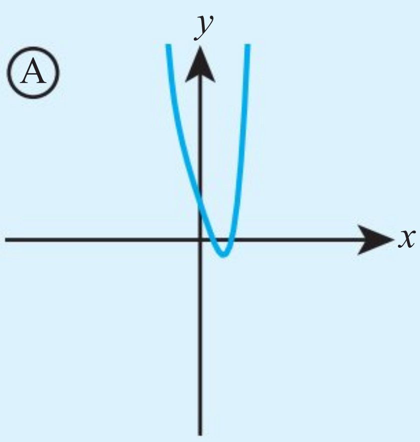
b i $y = -x^3 + 2x - 4$ **ii** $y = x^2 - 4x + 1$ **iii** $y = 3x^3 - 4x$

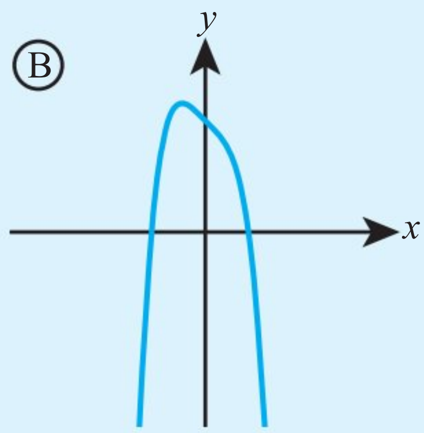


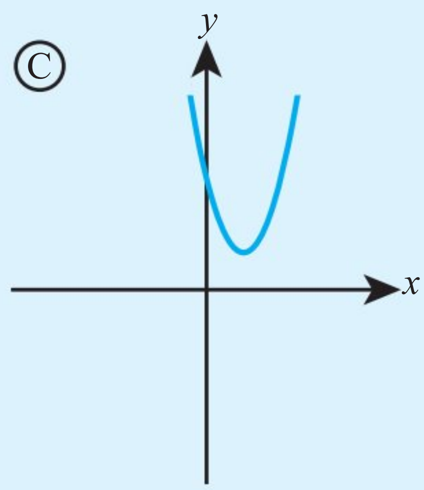




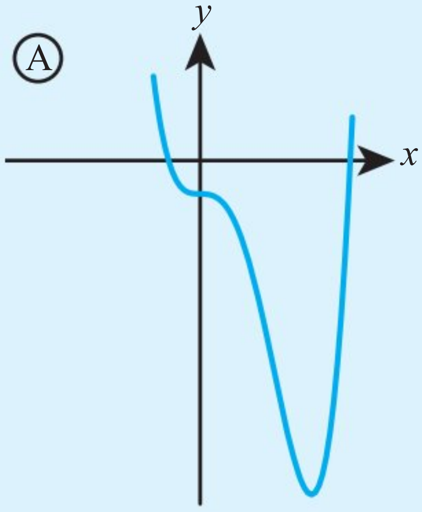
2 a i $y = x^4 - 3x + 1$ **ii** $y = -x^4 - x + 3$ **iii** $y = 2x^2 - 4x + 3$

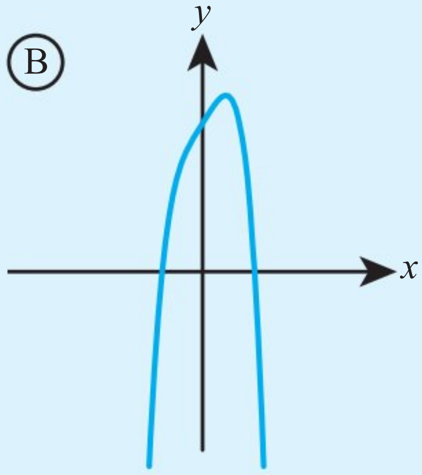


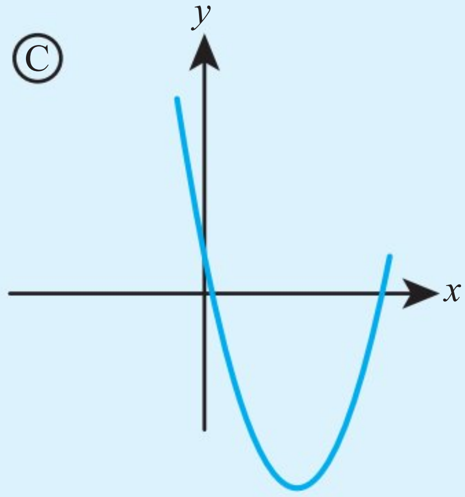




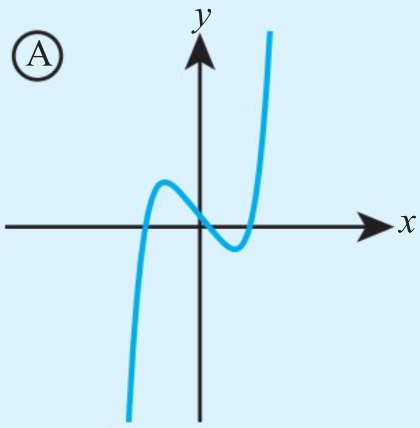
b **i** $y = x^2 - 5x + 1$ **ii** $y = -2x^4 + 2x + 5$ **iii** $y = x^4 - 4x^3 - 3$

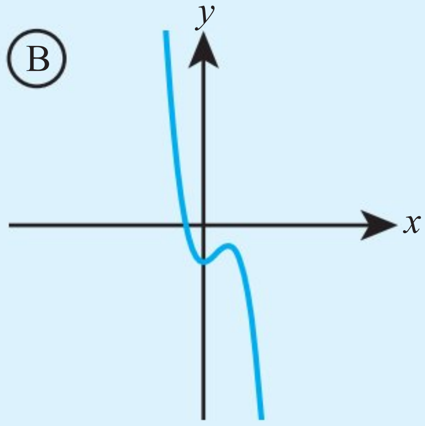


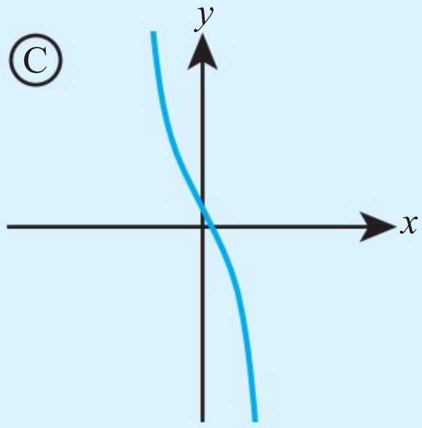




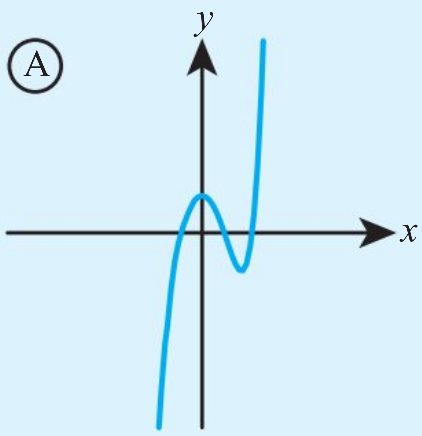
3 **a** **i** $y = -3x^3 + 3x^2 - 1$ **ii** $x^5 - 4x + 1$ **iii** $-x^5 - 4x + 1$

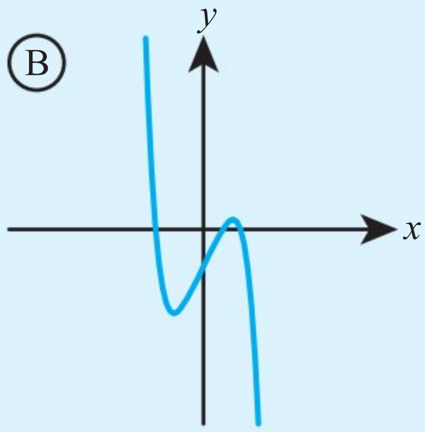


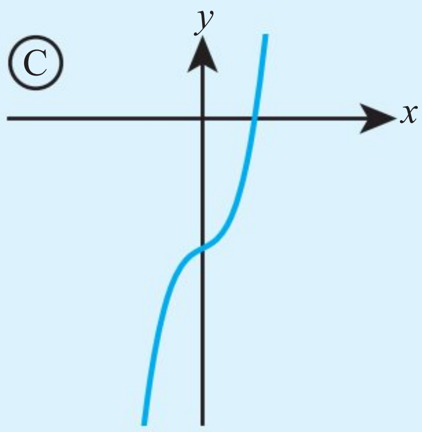




b **i** $y = 2x^3 + x - 7$ **ii** $y = x^5 - 3x^2 + 1$ **iii** $y = -x^5 + 2x - 1$







For questions 4 to 10, use the method demonstrated in Worked Example 6.2 to sketch each graph, showing all the axis intercepts.

4 **a** $y = 2(x - 1)(x - 4)(x + 2)$
b $y = 6x(x + 2)(x - 3)$

5 **a** $y = -5x(x - 1)(x + 2)$
b $y = -(x - 1)(x + 2)(x + 4)$

6 **a** $y = (x + 1)(x - 2)^2$
b $y = (x + 1)^2(x - 2)$

7 **a** $y = x(x + 2)(x + 3)(x - 1)$
b $y = 2(x - 3)(x - 4)(x + 1)(x - 2)$

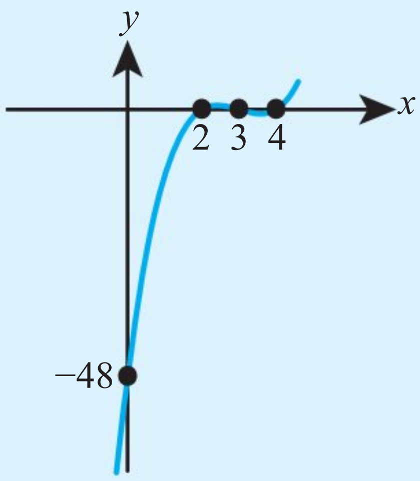
8 **a** $y = -3x^2(x - 1)(x - 3)$
b $y = -5x(x - 2)^2(2x + 1)$

9 **a** $y = 2(x - 3)^2(x + 1)^2$
b $y = 2(x - 3)^2(x - 1)^2$

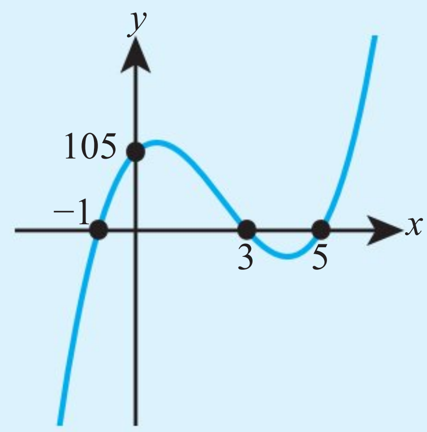
10 **a** $y = 2(x - 1)(x - 3)^3$
b $y = 3(x + 1)(x - 2)^3$

For questions 11 to 16, use the method demonstrated in Worked Example 6.3 to find a possible polynomial equation for each graph.

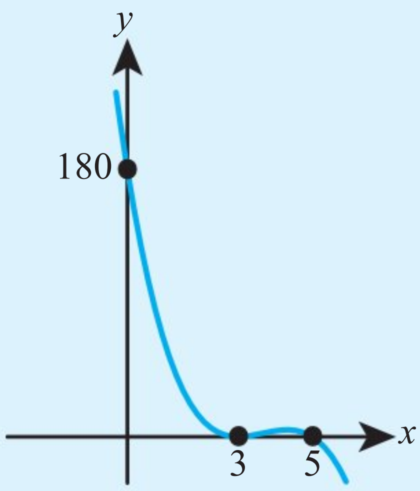
11 a



b



12 a



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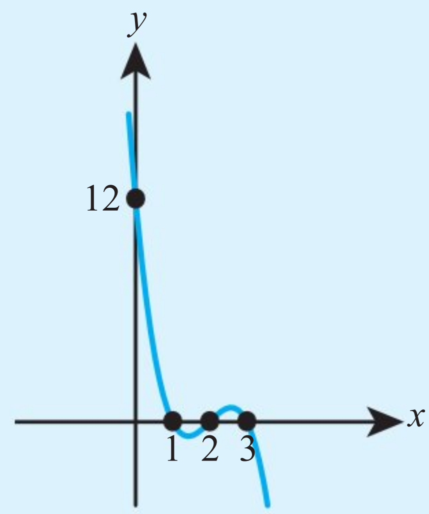
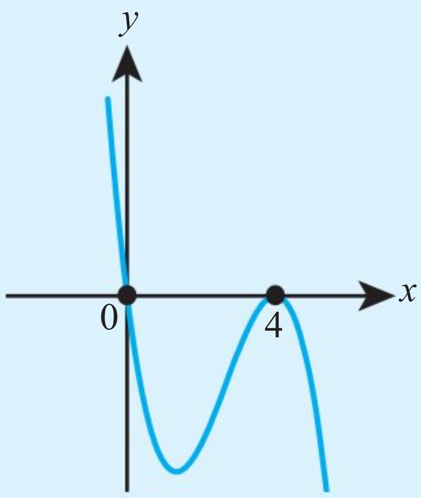


13 a



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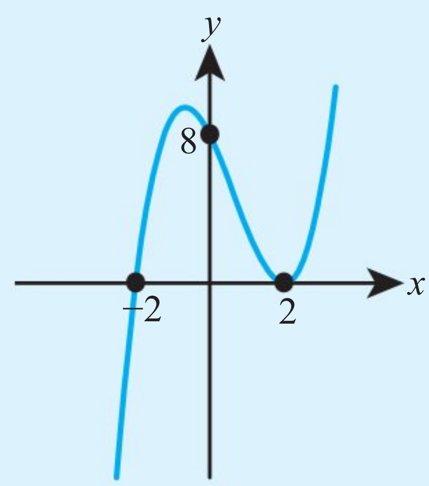
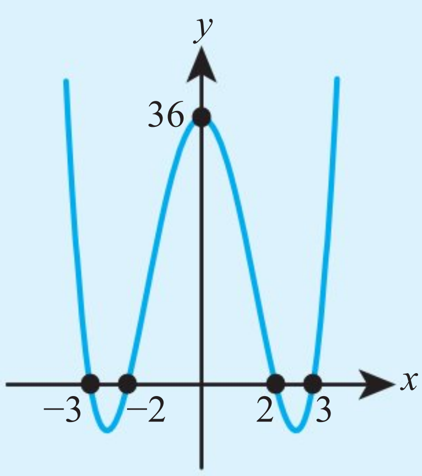
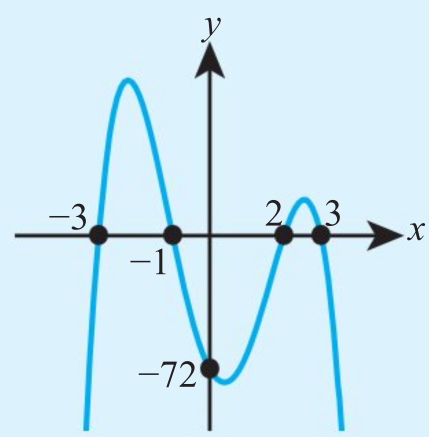


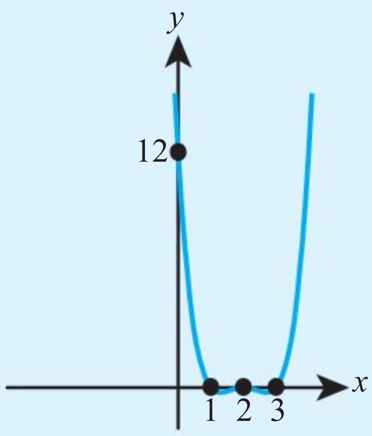
14 a



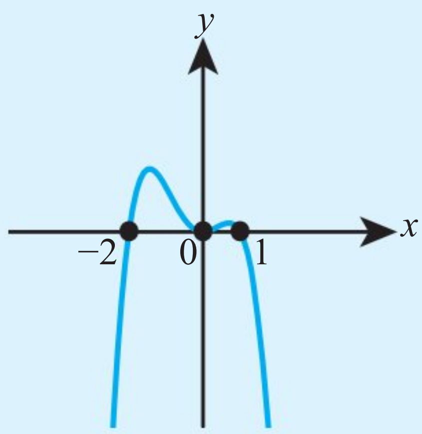
b



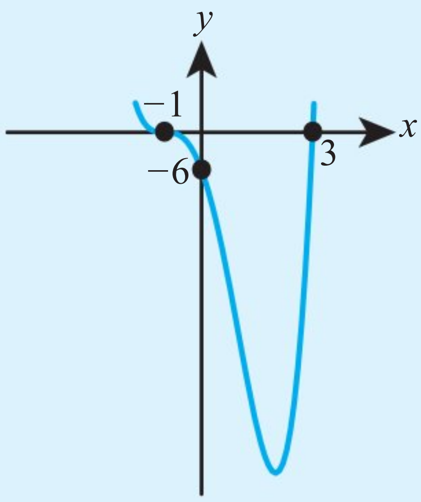
15 **a**



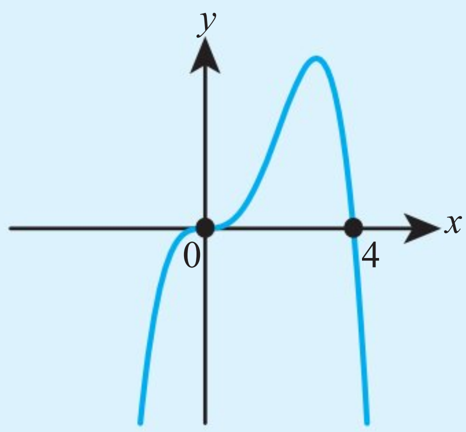
b



16 **a**



b



17 Sketch the graph of $y = 3x(x - 1)(x + 2)$, showing all the axis intercepts.

18 Sketch the graph of $y = -(x + 1)(x - 1)(x - 3)$, showing all the axis intercepts.

19 Given that $f(x) = 3x^3 - 12x$:

- **a** Factorize $f(x)$ fully.
- **b** Hence sketch the graph of $y = f(x)$, showing all the axis intercepts.

20 Given that $f(x) = 5x - 5x^3$:

- **a** Factorize $f(x)$ fully.
- **b** Hence sketch the graph of $y = f(x)$, showing all the axis intercepts.

21 Sketch the graph of

- **a** $y = (x + 1)^2(2 - x)$
- **b** $y = (x + 1)(2 - x)^2$.

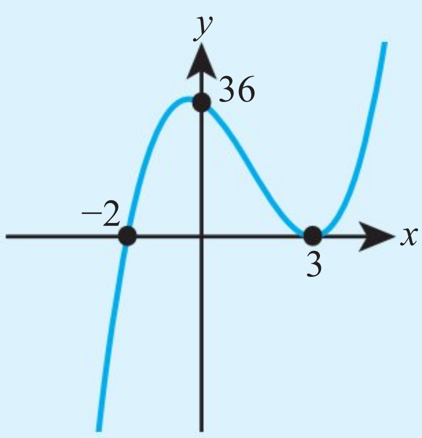
22 Sketch the graph of $y = -2(x - 1)^2(x + 2)^2$, showing all the axis intercepts.

23 Given that $f(x) = 4x^3 - x^4$,

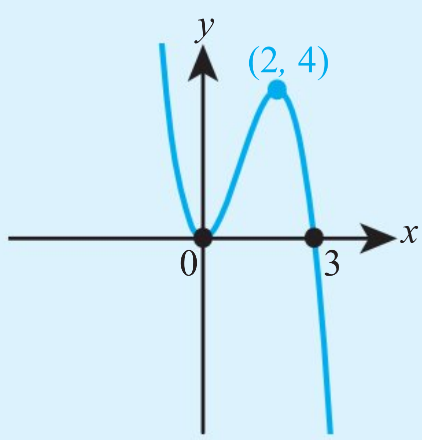
- **a** factorize $f(x)$ fully.
- **b** Hence sketch the graph of $y = f(x)$, showing all the axis intercepts.

24 The two graphs below each have equations of the form $y = px^3 + qx^2 + rx + s$. Find the values of p, q, r, s for each graph.

a



b



25 **a** Sketch the graph of $y = (x - p)^2(x - q)$ where $p < q$.

- **b** How many solutions does the equation $(x - p)^2(x - q) = k$ have when $k > 0$?

6B The factor and remainder theorems

You already know how to factorize polynomials when you are given some of the factors. But sometimes a polynomial cannot be divided exactly and there is a remainder. For example, you can check that $3x^2 + x + 1 = (x+2)(3x-5) + 11$, so when $3x^2 + x + 1$ is divided by $(x+2)$, the quotient is $(3x-5)$ and the remainder is 11.

This can also be written as division: $\frac{3x^2+x+1}{x+2} = 3x - 5 + \frac{11}{x+2}$. This form will be useful when working with rational functions in Chapter 7.

This is similar to dividing numbers; for example, $20 = 3 \times 6 + 2$ so when 20 is divided by 3, the quotient is 6 and the remainder is 2, which can also be written as $\frac{20}{3} = 6 + \frac{2}{3}$.

To find a reminder, you can use the method of comparing coefficients, as you did when factorizing polynomials.

You learnt how to factorize polynomials in Chapter 4.

Tip

Whenever a polynomial is divided by a linear factor, the remainder will be a constant.

WORKED EXAMPLE 6.4

Find the quotient and remainder when $3x^3 - 2x^2 + x + 5$ is divided by $(x+2)$.

Since you are dividing a cubic by a linear polynomial, the quotient will be quadratic and the remainder will be a constant. You know that the first term of the quadratic is $3x^2$ because $3x^3 = x \times 3x^2$

$$3x^3 - 2x^2 + x + 5 = (x+2)(3x^2 + Bx + C) + R$$

Expand the brackets…

$$= 3x^3 + Bx^2 + Cx + 6x^2 + 2Bx + 2C +$$

…and compare coefficients

Coefficient of x^2: $B + 6 = -2 \Rightarrow B = -8$

Coefficient of x: $C + 2(-8) = 1 \Rightarrow C = 17$

Constant term: $2(17) + R = 5 \Rightarrow R = -29$

The quotient is $3x^2 + Bx + C$ and the remainder is R

Hence, the quotient is $3x^2 - 8x + 17$ and the remainder is -29.

This method of finding the remainder is quite long. Furthermore, sometimes we just want to know the remainder, without finding the quotient. A useful shortcut is given by the following result.

KEY POINT 6.3

The remainder theorem:

If a polynomial function $f(x)$ is divided by $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$.

Proof 6.1

Prove that if a polynomial function f(x) is divided by $(ax - b)$, the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$.

When f(x) is divided by $(ax - b)$, the quotient is a polynomial q(x) and the remainder is a number R

Let the quotient be q(x) and the remainder be R.
Then $\mathrm{f}(x) = (ax - b)\mathrm{q}(x) + R$.

Substitute a suitable value for x to make the bracket $(ax - b)$ equal to zero.

When $x = \frac{b}{a}$:

$$\mathrm{f}\left(\frac{b}{a}\right) = \left(a\left(\frac{b}{a}\right) - b\right)\mathrm{q}\left(\frac{b}{a}\right) + R$$

$$= (b - b)\mathrm{q}\left(\frac{b}{a}\right) + R$$

$$\therefore\ \mathrm{f}\left(\frac{b}{a}\right) = R, \text{ as required}$$

WORKED EXAMPLE 6.5

Find the remainder when $\mathrm{f}(x) = x^3 + 3x + 5$ is divided by $3x + 4$.

Rewrite the quotient in the form $(ax - b)$

$$(3x + 4) = (3x - (-4))$$

Use the remainder theorem

$$\mathrm{f}\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 + 3 \times \left(-\frac{4}{3}\right) + 5$$

$$= -\frac{37}{27}$$

So, the remainder is $-\frac{37}{27}$.

Be the Examiner 6.2

When $x^3 - x + k$ is divided by $3x - 2$ the remainder is 3. Find the value of k.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathrm{f}(2) = 3$: $2^3 - 2 + k = 3$, $k = -3$	$\mathrm{f}\left(\frac{2}{3}\right) = 3$: $\left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right) + k = 3$, $k = \frac{91}{27}$	$\mathrm{f}\left(\frac{3}{2}\right) = 3$: $\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right) + k = 3$, $k = \frac{9}{8}$

If the remainder is zero, the denominator divides the numerator exactly – it is a factor.

Tip

The converse is also true: If $(ax - b)$ is a factor, then $\mathrm{f}\left(\frac{b}{a}\right) = 0$.

KEY POINT 6.4

The factor theorem:

If $\mathrm{f}\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of f(x).

WORKED EXAMPLE 6.6

Find the value of a such that $(x - 2)$ is a factor of $f(x) = 2x^3 + x^2 - 7x + a$.

By the factor theorem, $(x - 2)$ is a factor if $f(2) = 0$

By the factor theorem, $f(2) = 0$, so:

$$2 \times (2)^3 + (2)^2 - 7 \times (2) + a = 0$$
$$16 + 4 - 14 + a = 0$$
$$a = 6$$

Once you have found a factor, you can use the method of comparing coefficients to factorize the polynomial.

WORKED EXAMPLE 6.7

a Show that $(x + 3)$ is a factor of $f(x) = 2x^3 + x^2 - 9x + 18$.

b Hence, show that the equation $f(x) = 0$ has exactly one real root.

By the factor theorem, $(x + 3)$ is a factor if $f(-3) = 0$

a $f(-3) = 2 \times (-3)^3 + (-3)^2 - 9 \times (-3) + 18$
$= -54 + 9 + 27 + 18$
$= 0$

Therefore, $(x + 3)$ is a factor of $f(x)$.

To solve the equation $f(x) = 0$ you need to factorize $f(x)$. Start by taking out a factor $(x + 3)$. The second factor is quadratic, with first term $2x^2$ (because $2x^3 = x \times 2x^2$) and last term 6 (because $18 = 3 \times 6$)

b $2x^3 + x^2 - 9x + 18 = (x + 3)(2x^2 + Ax + 6)$

Expand the brackets

$= 2x^3 + Ax^2 + 6x + 6x^2 + 3Ax + 18$

Coefficient of x^2:

and compare the coefficient of x^2

$$Ax^2 + 6x^2 = x^2$$
$$A = -5$$

So, $f(x) = (x + 3)(2x^2 - 5x + 6)$

If $f(x) = 0$, then one of the factors must be zero

$f(x) = 0 \Leftrightarrow (x + 3) = 0$ or $2x^2 - 5x + 6 = 0$

To check whether $2x^2 - 5x + 6$ has any real roots, you can use the discriminant

$(-5)^2 - 4(2)(6) = -23 < 0$

So, $2x^2 - 5x + 6 = 0$ has no real roots.

The only real root comes from the factor $(x + 3)$

Hence, the only real root is $x = -3$.

You already know how to use the quadratic formula to find roots of a quadratic equation when it is not easy to factorize. A similar formula exists for cubic equations – find out about Cardano's formula. It turns out that the method can also be adapted to solve equations of order four, but that it is not possible to find a formula for polynomial of order five! This was discovered due to the work of the French mathematician Evariste Galois (1811–1832) and the Norwegian mathematician Niels Abel (1802–1829).

CONCEPTS – SYSTEMS

Polynomials are one particular type of mathematical function. Classifying different mathematical functions is an important part of a **systematic** understanding of mathematics, because the different types of function have different properties and rules associated with them. The factor and remainder theorems only work in general with polynomial functions.

Exercise 6B

For questions 1 to 4, use the method of comparing coefficients demonstrated in Worked Example 6.4 to find the quotient and remainder.

1 **a** $x^2 + 3x + 5$ divided by $x + 1$
  b $x^2 + x - 4$ divided by $x + 2$

2 **a** $x^3 - 6x^2 + 4x + 8$ divided by $x - 3$
  b $x^3 - 7x^2 + 11x + 3$ divided by $x - 1$

3 **a** $6x^4 + 7x^3 - 5x^2 + 5x + 10$ divided by $2x + 3$
  b $12x^4 - 10x^3 + 11x^2 - 5$ divided by $3x - 1$

4 **a** x^3 divided by $x + 2$
  b $3x^4$ divided by $x - 1$

For questions 5 to 8, use the method demonstrated in Worked Example 6.5 to find the remainder when f(x) is divided by the given linear polynomial.

5 **a** $f(x) = 2x^3 - x + 3$ by $(x - 2)$
  b $f(x) = 2x^3 + x - 5$ by $(x - 4)$

6 **a** $f(x) = x^4 + 2x^2 + 4$ by $(x + 2)$
  b $f(x) = x^4 + x^3 - x + 1$ by $(x + 1)$

7 **a** $f(x) = x^3 + 4x^2 - 3$ by $(3x - 2)$
  b $f(x) = x^3 - 3x + 1$ by $(2x - 1)$

8 **a** $f(x) = 3x^3 + 2x + 3$ by $(2x + 3)$
  b $f(x) = 2x^3 - x + 5$ by $(3x + 1)$

For questions 9 to 12, use the method demonstrated in Worked Example 6.6 to find the value of a such that f(x) has the given factor.

9 **a** $f(x) = 2x^3 - x + a$, factor $(x - 2)$
  b $f(x) = 2x^3 + x + a$, factor $(x - 4)$

10 **a** $f(x) = x^4 + ax^2 + 4$, factor $(x + 2)$
  b $f(x) = x^4 + ax^3 - x + 1$, factor $(x + 1)$

11 **a** $f(x) = x^3 + 4x^2 + a$, factor $(3x - 2)$
  b $f(x) = x^3 - 3x + a$, factor $(2x - 1)$

12 **a** $f(x) = ax^3 + 2x + 4$, factor $(2x + 3)$
  b $f(x) = ax^3 - x + 5$, factor $(3x + 1)$

For questions 13 to 16, use the method demonstrated in Worked Example 6.7 to show that the given linear polynomial is a factor of f(x), factorize f(x) and state the number of distinct real roots of the equation $f(x) = 0$.

13 **a** $f(x) = x^3 + 2x^2 - x - 2$, factor $(x - 1)$
  b $f(x) = x^3 + x^2 - 4x - 4$, factor $(x - 2)$

14 **a** $f(x) = x^3 - x^2 - 8x + 12$, factor $(x + 3)$
  b $f(x) = x^3 - 5x^2 + 3x + 9$, factor $(x + 1)$

15 **a** $f(x) = x^3 - 3x^2 + 12x - 10$, factor $(x - 1)$
  b $f(x) = x^3 - 2x^2 + 2x - 15$, factor $(x - 3)$

16 **a** $f(x) = 6x^3 - 11x^2 + 6x - 1$, factor $(3x - 1)$
  b $f(x) = 12x^3 + 13x^2 - 37x - 30$, factor $(3x - 5)$

17 When $x^3 + ax + 7$ is divided by $(x + 2)$ the remainder is –5. Find the value of a.

18 When $x^3 - 6x^2 + 4x + a$ is divided by $(x - 3)$ the remainder is 2. Find the value of a.

19 The polynomial $x^2 + kx - 8k$ has a factor $(x - k)$. Find the possible values of k.

20 $6x^3 + ax^2 + bx + 8$ has a factor $(x + 2)$ and leaves a remainder of –3 when divided by $(x - 1)$. Find a and b.

21 $x^3 + 8x^2 + ax + b$ has a factor of $(x - 2)$ and leaves a remainder of 15 when divided by $(x - 3)$. Find a and b.

22 The polynomial $x^2 - (k+1)x - 3$ has a factor $(x - k + 1)$. Find the value of k.

23 $f(x) = x^3 - ax^2 - bx + 168$ has factors $(x - 7)$ and $(x - 3)$. Solve the equation $f(x) = 0$.

24 $f(x) = x^3 + ax^2 + 9x + b$ has a factor of $(x - 11)$ and leaves a remainder of -52 when divided by $(x + 2)$. Find the remainder when $f(x)$ is divided by $(x - 2)$.

25 a Show that $(x - 4)$ is a factor of $x^3 - 2x^2 - 11x + 12$.

b Hence solve the equation $x^3 - 2x^2 - 11x + 12 = 0$.

26 a Show that $(x - 2)$ is a factor of $x^3 - 5x^2 + 7x - 2$.

b Hence solve the equation $x^3 - 5x^2 + 7x - 2 = 0$.

27 Let $f(x) = 6x^3 - x^2 + k$.

a Given that $(2x + 1)$ is a factor of $f(x)$, find the value of k.

b For this value of k, show that the equation $f(x) = 0$ has only one real root.

28 Let $f(x) = 2x^3 - 7x^2 - 3x + 3$.

a Show that $(2x - 1)$ is a factor of $f(x)$.

b Find the number of real roots of the equation $f(x) = 0$.

29 a Show that, for all real values of p, $(x - p)$ is a factor of $f(x) = 2x^3 - px^2 - 2p^2x + p^3$.

b Hence find, in terms of p, the roots of the equation $f(x) = 0$.

30 a Show that, for all real values of k, $(x - 1)$ is a factor of $f(x) = x^3 - x^2 + k^2x - k^2$.

b Show that, for all real values of k, the equation $f(x) = 0$ has only one real root.

31 Given that $(ax + b)$ is a factor of $x^2 + bx + a$, find an expression for b in terms of a.

32 Given that $(2x + 1)$ and $(x - 2)$ are factors of $f(x) = 2x^4 - 3x^3 + 16x^2 - 27x - 18$, find all the roots of the equation $f(x) = 0$.

33 Let $f(x) = x^3 + ax^2 + 3x + b$.

The remainder when $f(x)$ is divided by $(x + 1)$ is 6. Find the remainder when $f(x)$ is divided by $(x - 1)$.

34 The polynomial $x^2 - 5x + 6$ is a factor of $2x^3 - 15x^2 + ax + b$. Find the values of a and b.

35 The roots of the equation $x^3 + bx^2 + cx + d = 0$ for an arithmetic sequence with middle term 3. Show that $3c + d = 54$.

6C Sum and product of roots of polynomial equations

Quadratic equations

When you first learned to solve quadratic equations, you were probably told to look for two numbers which add up to the middle coefficient and multiply to give the constant term. For example, $x^2 - 7x + 10$ factorizes as $(x - 2)(x - 5)$ because $(-2) + (-5) = (-7)$ and $2 \times 5 = 10$. Hence, the roots of the equation $x^2 - 7x + 10 = 0$ are 2 and 5.

This equation is equivalent to, for example, $3x^2 - 21x + 30 = 0$. The roots are still 2 and 5, but now they do not add up to 21; instead, they add up to $\frac{21}{3}$ and multiply to give $\frac{30}{3}$.

Tip

These results apply to both real and complex roots.

KEY POINT 6.5

If p and q are roots of a quadratic equation $ax^2 + bx + c = 0$, then

$$p + q = -\frac{b}{a} \quad \text{and} \quad pq = \frac{c}{a}$$

Proof 6.2

Prove that if p and q are roots of a quadratic equation $ax^2 + bx + c = 0$, then $p + q = -\frac{b}{a}$ and $pq = \frac{c}{a}$.

If you know the roots, you can write the equation in factorized form. Remember that the coefficient of x^2 may not be 1 → The equation is $a(x - p)(x - q) = 0$

Expand the brackets to write the equation in the form $ax^2 + bx + c = 0$ →

$$a(x^2 - px - qx + pq) = 0$$
$$ax^2 - a(p + q)x + apq = 0$$

Compare coefficients → Hence, $-a(p + q) = b \Leftrightarrow p + q = -\frac{b}{a}$

$$apq = c \Leftrightarrow pq = \frac{c}{a}$$

These results can be used to find other combinations of roots of the equation.

WORKED EXAMPLE 6.8

The equation $4x^2 + 7x - 1 = 0$ has roots p and q. Without solving the equation, find the value of

a $(p + 2)(q + 2)$ **b** $p^2 + q^2$.

Write the expression in a form that involves $p + q$ and pq → **a** $(p + 2)(q + 2) = pq + 2p + 2q + 4$

$$= pq + 2(p + q) + 4$$

You know that $p + q = -\frac{7}{4}$ and $pq = \frac{-1}{4}$ →

$$= -\frac{1}{4} + 2\left(-\frac{7}{4}\right) + 4$$
$$= \frac{1}{4}$$

You can get p^2 and q^2 by squaring $(p + q)^2$ → **b** $(p + q)^2 = p^2 + 2pq + q^2$

$$\Rightarrow p^2 + q^2 = (p + q)^2 - 2pq$$

Now use the values of $p + q$ and pq →

$$= \left(-\frac{7}{4}\right)^2 - 2\left(\frac{-1}{4}\right)$$
$$= \frac{57}{16}$$

Find an equation with given roots

You can also use the result from Key Point 6.5 to find an equation with given roots.

WORKED EXAMPLE 6.9

Find a quadratic equation with integer coefficients and roots $\frac{5}{3}+\frac{7}{2}\text{i}$ and $\frac{5}{3}-\frac{7}{2}\text{i}$.

If the equation is $ax^2+bx+c=0$, you can use the roots to find the coefficient

If the equation is $ax^2+bx+c=0$, then

$$-\frac{b}{a}=\left(\frac{5}{3}+\frac{7}{2}\text{i}\right)+\left(\frac{5}{3}-\frac{7}{2}\text{i}\right)=\frac{10}{3}$$

$$\Rightarrow b=-\frac{10}{3}a$$

and

$$\frac{c}{a}=\left(\frac{5}{3}+\frac{7}{2}\text{i}\right)\left(\frac{5}{3}-\frac{7}{2}\text{i}\right)$$

Use the difference of two squares

$(x+\text{i}y)(x-\text{i}y)=x^2-(\text{i}y)^2$
$=x^2+y^2$

$$=\left(\frac{5}{3}\right)^2+\left(\frac{7}{2}\right)^2=\frac{541}{36}$$

$$\Rightarrow c=\frac{541}{36}a$$

You could pick any value for a and then find b and c. If you want b and c to be integers, it seems sensible to take $a=36$

Let $a=36$. Then $b=-120$ and $c=541$.
The equation is $36x^2-120x+541=0$.

Tip

You could also find the equation by expanding $\left(x-\left(\frac{5}{3}+\frac{7}{2}\text{i}\right)\right)\left(x-\left(\frac{5}{3}-\frac{7}{2}\text{i}\right)\right)$, but this involves the same calculations and is slightly longer.

You can also find an equation whose roots are related to the roots of a given equation.

WORKED EXAMPLE 6.10

The equation $3x^2-x+4=0$ has roots α and β. Find a quadratic equation with integer coefficients and roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Find the sum and product of the roots of the new equation

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

Use $\alpha+\beta=\frac{1}{3}$ and $\alpha\beta=\frac{4}{3}$

$$=\frac{1/3}{4/3}=\frac{1}{4}$$

$$\frac{1}{\alpha}\frac{1}{\beta}=\frac{1}{\alpha\beta}$$

$$=\frac{1}{4/3}=\frac{3}{4}$$

Now find an equation whose sum of the roots is $\frac{1}{4}$ and the product of the roots is $\frac{3}{4}$

For the new equation, $ax^2+bx+c=0$:

$$-\frac{b}{a}=\frac{1}{4}\Rightarrow b=-\frac{1}{4}a$$

$$\frac{c}{a}=\frac{3}{4}\Rightarrow c=\frac{3}{4}a$$

To make the coefficients integers, take $a=4$

Let $a=4$. Then $b=-1$ and $c=3$.
So, the equation is $4x^2-x+3=0$.

CONCEPTS – RELATIONSHIPS

This section shows that the **relationship** between equations and their roots works in both directions. Given a quadratic equation, you can work out the roots and given the roots you can work out a possible quadratic equation. You have seen many instances when you start with the equation and use the roots. Can you think of any instance when you might start with the roots and want to know the equation?

Higher order polynomials

Similar relationships between roots and coefficients can be found for higher order polynomials.

KEY POINT 6.6

For a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$:

- the sum of the roots is $-\frac{a_{n-1}}{a_n}$
- the product of the roots is $(-1)^n \frac{a_0}{a_n}$.

If some of the roots are repeated, they should appear more than once in the sum and the product.

WORKED EXAMPLE 6.11

The equation $ax^4 - 28x^3 + 65x^2 - 46x + d = 0$ has roots $\frac{1}{2}, \frac{1}{2}, 3 + \mathrm{i}$ and $3 - \mathrm{i}$. Find the values of a and d.

The sum of the roots is $-\left(-\frac{28}{a}\right)$

$$\frac{1}{2} + \frac{1}{2} + 3 + \mathrm{i} + 3 - \mathrm{i} = \frac{28}{a}$$

$$7 = \frac{28}{a}$$

$$a = 4$$

The product of the roots is $(-1)^4 \frac{d}{a}$

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(3 + \mathrm{i})(3 - \mathrm{i}) = \frac{d}{4}$$

Use $(x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$

$$\frac{1}{4}(3^2 + 1^2) = \frac{d}{4}$$

$$d = 10$$

TOOLKIT: Problem Solving

The equation $x^4 + bx^3 + cx^2 + d = 0$ has roots p, q, r and s.

Find expressions for b and c in terms of p, q, r and s.

Hence, find a quartic equation with roots $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ and $\frac{1}{s}$.

Exercise 6C

For questions 1 to 6, p and q are the roots of the given equation. Use the method demonstrated in Worked Example 6.8 to find the value of the required expression.

1 Find $p+q+5$ for these equations

- **a** $5x^2+2x+4=0$
- **b** $4x^2+3x+5=0$

2 Find $3pq$ for these equations

- **a** $2x^2+x+4=0$
- **b** $3x^2-2x+5=0$

3 Find $(p+3)(q+3)$ for these equations

- **a** $2x^2-x+3=0$
- **b** $4x^2-3x+1=0$

4 Find $\frac{1}{p}+\frac{1}{q}$ for these equations

- **a** $x^2+4x+1=0$
- **b** $x^2+3x+2=0$

5 Find p^2+q^2 for these equations

- **a** $3x^2-x-9=0$
- **b** $2x^2-3x-8=0$

6 Find p^2q+pq^2 for these equations

- **a** $x^2+x-3=0$
- **b** $x^2+x-6=0$

For questions 7 to 10, use the method demonstrated in Worked Example 6.9 to find a quadratic equation with integer coefficients and given roots.

7 **a** $-3, 2$ **b** $-5, 1$

8 **a** $\frac{2}{3}, \frac{1}{2}$ **b** $\frac{3}{4}, \frac{2}{5}$

9 **a** $4+3\text{i}, 4-3\text{i}$ **b** $2+5\text{i}, 2-5\text{i}$

10 **a** $\frac{1}{2}+\frac{2}{3}\text{i}, \frac{1}{2}-\frac{2}{3}\text{i}$ **b** $\frac{3}{4}+\frac{1}{3}\text{i}, \frac{3}{4}-\frac{1}{3}\text{i}$

For questions 11 to 13, α and β are the roots of the equation $2x^2-3x+6=0$. Use the method demonstrated in Worked Example 6.10 to find a quadratic equation with integers coefficients and given roots.

11 **a** $\alpha+2$ and $\beta+2$ **b** $\alpha-3$ and $\beta-3$

12 **a** 5α and 5β **b** $\frac{\alpha}{2}$ and $\frac{\beta}{2}$

13 **a** $\frac{2}{\alpha}$ and $\frac{2}{\beta}$ **b** $\frac{5}{\alpha}$ and $\frac{5}{\beta}$

For questions 14 to 20, you are given a polynomial equation and its roots. Use the method demonstrated in Worked Example 6.11 to find the values of a and k.

14 **a** $x^3+ax^2+3x+k=0$, roots $1, -2, -5$

b $x^3+ax^2-4x+k=0$, roots $2, -2, -3$

15 **a** $ax^3+5x^2-x+k=0$, roots $1, -2, -\frac{3}{2}$

b $ax^3+14x^2+13x+k=0$, roots $\frac{1}{3}, -2, -3$

16 **a** $2x^3+ax^2+14x+k=0$, roots $\frac{1}{2}, 2+\text{i}, 2-\text{i}$

b $2x^3+ax^2+8x+k=0$, roots $-\frac{1}{2}, 1+2\text{i}, 1-2\text{i}$

17 **a** $x^4+ax^3+7x^2-16x+k=0$, roots $1, 3, 2\text{i}, -2\text{i}$

b $x^4+ax^3+7x^2-9x+k=0$, roots $-1, 2, 3\text{i}, -3\text{i}$

18 **a** $ax^4-2x^3-21x^2+12x+k=0$, roots $2, 2, -1 -\frac{7}{3}$

b $ax^4-34x^3+68x^2-30x+k=0$, roots $1, 3, 3, -\frac{1}{5}$

19 **a** $ax^5-2x^3+2x^2-3x+k=0$, roots $1, 1, -2, \text{i}, -\text{i}$

b $ax^5-5x^4+9x^3-9x^2+8x+k=0$, roots $1, 2, 2, \text{i}, -\text{i}$

20 **a** $x^6+ax^5-x^4-16x^2+k=0$, roots $1, -1, 2, -2, 2\text{i}, -2\text{i}$

b $x^6+ax^5-12x^4+23x^2+k=0$, roots $2, -2, 3, -3, \text{i}, -\text{i}$

21 The equation $2x^2+x+3=0$ has roots p and q. Without solving the equation, find the exact value of $(p-4)(q-4)$.

22 The equation $x^2-ax+3a=0$ has roots p and q. Find, in terms of a,

a $5pq$ **b** $(p+q)^2$.

23 The equation $kx^2 - 3x + 5k = 0$ has roots α and β.
Find, in terms of k,

a $\alpha + \beta + 2$

b $\frac{1}{\alpha} + \frac{1}{\beta}$.

24 The equation $ax^2 + 3x - a^2 = 0$ has roots p and q. Express $(p - q)^2$ in terms of a.

25 The equation $x^2 - kx + 2k = 0$ has roots p and q. Express $\frac{3}{p} + \frac{3}{q}$ in terms of k.

26 Given that p, q and r are the roots of the equation $3x^3 + 6x^2 + 12x - 4 = 0$, find the value of $p^2qr + pq^2r + pqr^2$.

27 Given that α, β, γ are the roots of the polynomial $2x^3 - 5x^2 + 3 = 0$, find the exact value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$.

28 Let p and q be the roots of the equation $5x^2 - 3x + 2 = 0$.

a Find the value of $(p + 3)(q + 3)$.

b The quadratic equation $5x^2 + bx + c = 0$ has roots $p + 3$ and $q + 3$. Find the values of a and b.

29 The quartic equation $x^4 - ax^3 + bx^2 - cx + d = 0$ has real coefficients, and two of its roots are 3i and $3 - \text{i}$.

a Write down the other two roots.

b Hence find the values of a and d.

30 When two resistors of resistances R_1 and R_2 are connected in series in an electric circuit, the total resistance in the circuit is $R = R_1 + R_2$. When they are connected in parallel, the total resistance satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

Two resistors have resistances equal to the two roots of the quadratic equation $3R^2 - aR + 2a = 0$. Find the total resistance in the circuit if the two resistors are connected

a in series

b in parallel.

31 A random number generator selects one of the roots of the equation $x^4 - 11x^3 + 35x^2 - 11x + 6 = 0$, all with equal probability. If a large number of random numbers are generated, find their mean.

32 The cubic equation $3x^3 - 5x + 3 = 0$ has roots α, β and γ. Show that $\alpha + \beta = \frac{1}{\alpha\beta}$.

33 Let p and q be the roots of the equation $5x^2 - 3x + 2 = 0$.

a Find the values of pq and $p^2 + q^2$.

b Hence find a quadratic equation with integer coefficients and roots p^2 and q^2.

34 The equation $3x^2 + x - 8 = 0$ has roots p and q.

a Find the value of $p^2 + q^2$.

b Find a quadratic equation with integer coefficients and roots p^2 and q^2.

35 a Show that $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$.

b Given that p and q are the roots of the equation $4x^2 - x + 2 = 0$, find the exact value of $p^3 + q^3$.

c Find a quadratic equation with integer coefficients and roots p^3 and q^3.

36 a Expand and simplify $(\sqrt{\alpha} + \sqrt{\beta})^2$.

b Write $3x^2 - 30x + 73$ in the form $a(x - h)^2 - k$.

c Let α and β be the roots of the equation $3x^2 - 30x + 73 = 0$. By sketching a suitable graph, explain why α and β are real and positive.

d Find the exact value of $\sqrt{\alpha} + \sqrt{\beta}$.

37 The equation $x^4 + bx^3 + cx^2 + dx + e = 0$ has roots p, $2p$, $3p$ and $4p$. Show that $3b^4 = 1250e$.

38 Given that α and β are the roots of the equation $5x^2 + 2x + 3 = 0$, find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

39 The cubic equation $x^3 + 2ax^2 + 3a^2x + 2 = 0$, where $a \in \mathbb{R}$, has roots p, q and r.

a By expanding $(x - p)(x - q)(x - r)$ show that $pq + qr + rp = 3a^2$.

b Find an expression for $p^2 + q^2 + r^2$ in terms of a.

c Explain why this implies that the roots are not all real.

40 **a** Show that

  i $p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp)$

  ii $p^2q^2 + q^2r^2 + r^2p^2 = (pq + qr + rp)^2 - 2pqr(p + q + r)$.

b The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots p, q and r.

  i Write down the values of $p + q + r$ and pqr.

  ii by expanding $a(x - p)(x - q)(x - r)$ show that $pq + qr + rp = \frac{c}{a}$.

c The equation $2x^3 - 7x + 4 = 0$ has roots α, β and γ.

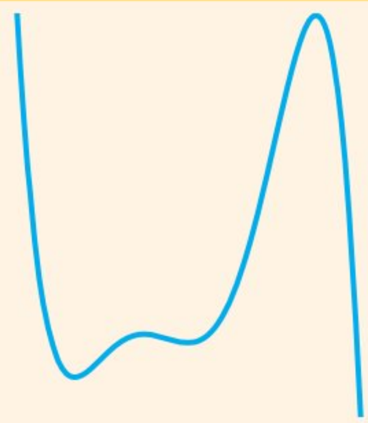
  i Show that $\alpha^2 + \beta^2 + \gamma^2 = 7$.

  ii Find the values of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ and $\alpha^2\beta^2\gamma^2$.

  iii Hence find a cubic equation with integer coefficients and roots α^2, β^2 and γ^2.

Checklist

- You should know the highest power of x in a polynomial expression is called the **degree** of the polynomial.
- You should know the shape of a polynomial graph depends on whether the degree (n) is odd or even, and on the sign of the coefficient of the highest power term (a):

	$a > 0$	$a < 0$
***n* even**		
***n* odd**		

- You should understand the factors of a polynomial tell you about the x-intercepts:
  - If a polynomial has a factor $(x - p)$, then the graph crosses the x-axis at $(p, 0)$.
  - If a polynomial has a factor $(x - p)^2$, then the graph touches the x-axis at $(p, 0)$.
  - If a polynomial has a factor $(x - p)^3$, then the graph crosses the x-axis and has zero gradient at $(p, 0)$.
- You should know the remainder theorem: If a polynomial function $f(x)$ is divided by $(ax - b)$ the remainder is $f\left(\frac{b}{a}\right)$.
- You should know the factor theorem: If $f\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor of $f(x)$.
- You should know that for a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$:
  - the sum of the roots is $-\frac{a_{n-1}}{a_n}$
  - the product of the roots is $(-1)^n \frac{a_0}{a_n}$
  - you can use these results to find a polynomial with given roots.

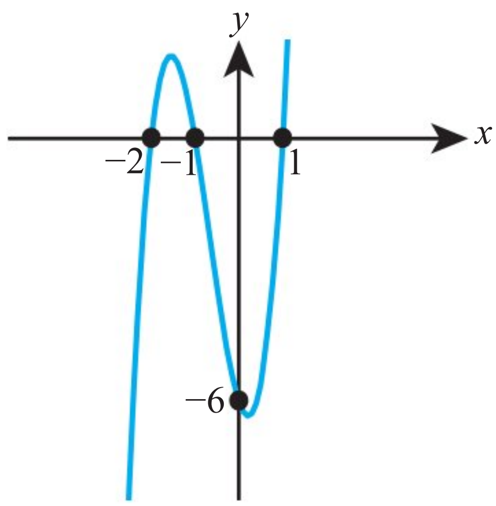
Mixed Practice

1 **a** Factorize $f(x) = 3x^3 - 2x^2 - x$.

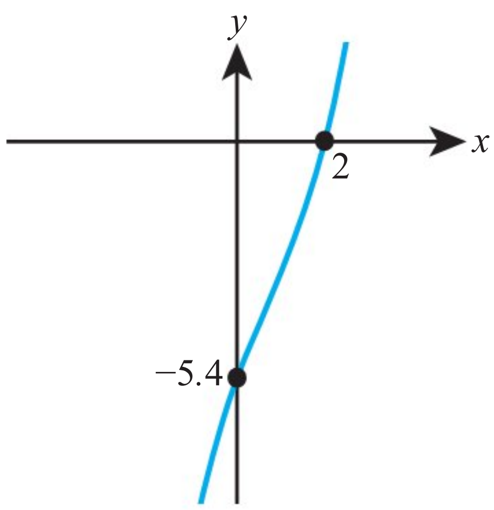
b Hence sketch the graph of $y = f(x)$, showing all the axis intercepts.

2 Sketch the graph of $y = (x - 3)(x + 1)(5 - 2x)$ showing all the axis intercepts.

3 The diagram shows the graph of $y = f(x)$, where f is a cubic polynomial. Find an expression for $f(x)$, giving your answer in a fully expanded form.



4 The diagram shows the graph of $y = 0.3(x - p)(x^2 + k^2)$, where $k > 0$. Find the values of p and k.



5 A polynomial is given by $f(x) = (ax + b)^3$. The remainder when $f(x)$ is divided by $(x - 2)$ is 8 and the remainder when it is divided by $(x + 3)$ is -27. Find the values of a and b.

6 $f(x) = x^3 + 4x^2 + ax + b$ has a factor of $(x - 1)$ and leaves a remainder of 17 when divided by $(x - 2)$. Find the constants a and b.

7 When $f(x) = x^4 + px^2 - x + q$ is divided by $(x - 3)$ the remainder is 52 and $(x + 1)$ is a factor of $f(x)$. Find the values of p and q.

8 The quadratic equation $5x^2 + bx + c = 0$ has real coefficients and one of its roots is $4 + 7\text{i}$. Find the values of b and c.

9 The sum of the roots of the equation $4x^3 - ax^2 + 5x + 3 = 0$ is $\frac{3}{2}$. Find the value of a.

10 The mean of the roots of the equation $kx^4 + 8x^3 - 3x + 1 = 0$ is $-\frac{1}{2}$. Find the value of k.

11 The equation $4x^4 - 40ax^3 + 140a^2x^2 - 200a^3x + 1536 = 0$ has roots a, $2a$, $3a$ and $4a$. Find the possible values of a.

12 The equation $3x^3 + 2x^2 - x + 5 = 0$ has roots a, b and c.

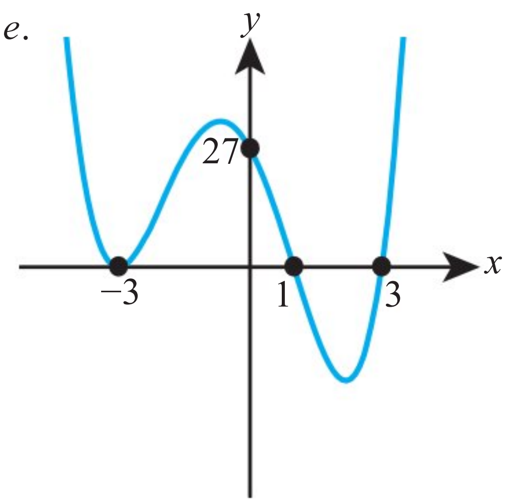
a Write down the value of abc.

b Find the value of $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$.

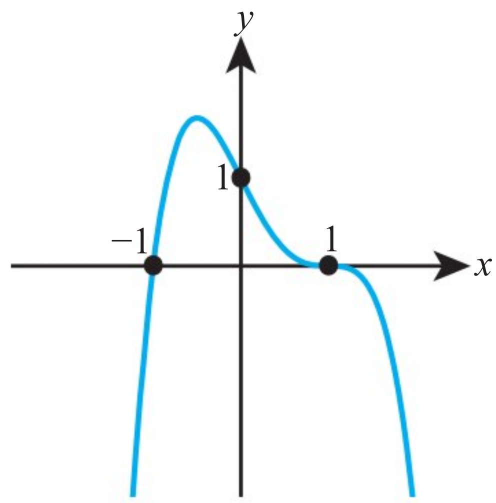
13 The same remainder is found when $2x^3 + kx^2 + 6x + 32$ and $x^4 - 6x^2 - k^2x + 9$ are divided by $x + 1$. Find the possible values of k.

Mathematics HL May 2012 Paper 1 TZ2 Q1

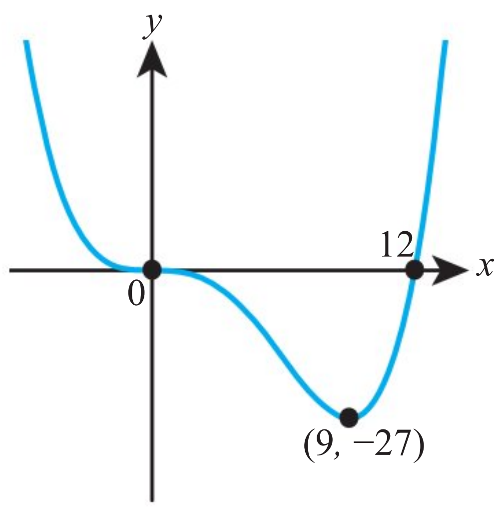
14 The diagram shows the graph with equation $y = ax^4 + bx^3 + cx^2 + dx + e$. Find the values of a, b, c, d and e.



15 The diagram shows the graph of $y = f(x)$, where $f(x)$ is a polynomial of order 4. Find an expression for $f(x)$, giving your answer in an expanded form.



16 The diagram shows a graph of a polynomial of order 4. Find the expression for the polynomial, giving your answer in an expanded form.



17 **a** Show that $(x - 2)$ is a factor of $f(x) = 2x^3 - 5x^2 + x + 2$.

b Factorize $f(x)$ completely.

c Hence sketch the graph $y = f(x)$, showing all the axis intercepts.

18 **a** Show that $(x + 1)$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$.

b Factorize $f(x)$.

c Sketch the graph of $y = f(x)$.

19 Sketch the graph of $y = (x - a)^2(x - b)(x - c)$ where $b < 0 < a < c$.

20 Given that one of the roots of the equation $x^3 + ax^2 - 7x + 15 = 0$ is -3, find all the roots.

21 One of the roots of the equation $ax^3 + bx^2 + 177x - 210 = 0$ is 2, and the sum of the other two roots is 12. Find the values of a and b.

22 One of the roots of the equation $3x^3 - 12x^2 + 16x - 8 = 0$ is an integer between 0 and 3 inclusive. Find the other two roots.

23 The roots p and q of the equation $ax^2 + bx + c = 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 0$. Show that $b + 3c = 0$.

24 Let $f(x) = (ax + b)^4$. The remainder when $f(x)$ is divided by $(x - 2)$ is 16 and the remainder when it is divided by $(x + 1)$ is 81. Find the possible values of a and b.

25 The quartic equation $x^4 + px^3 + 14x^2 - 18x + q = 0$ has real coefficients and two of its roots are 3i and 1 – 2i. Find the values of p and q.

26 The quadratic equation $3x^2 - 4x + 7 = 0$ has roots p and q.

a Find the value of $p^2 + q^2$.

b Find a quadratic equation with integer coefficients and roots p^2 and q^2.

27 Let $g(x) = 3x^5 - 6x^4 + 13x^2 - 2x + 18$.

a Write down the sum of the roots of the equation $g(x) = 0$.

b A new polynomial is defined by $h(x) = g(x - 4)$. Find the sum of the roots of the equation $h(x) = 0$.

28 Let $f(x) = 5x^4 + 2x^3 - x^2 - x + 3$.

a Write down the sum and the product of the roots of the equation $f(x) = 0$.

b Find the product of the roots of the equation $f(3x) = 0$.

29 The function $f(x) = 4x^3 + 2ax - 7a$, $a \in \mathbb{R}$, leaves a remainder of -10 when divided by $(x - a)$.

a Find the value of a.

b Show that for this value of a there is a unique real solution to the equation $f(x) = 0$.

Mathematics HL May 2011 Paper 2 TZ1 Q4

30 The equation $5x^3 + 48x^2 + 100x + 2 = a$ has roots r_1, r_2 and r_3.

Given that $r_1 + r_2 + r_3 + r_1 r_2 r_3 = 0$, find the value of a.

Mathematics HL May 2014 Paper 1 TZ1 Q4

31 **a** Find the exact solutions of the equation $x^2 - 4x + 5 = 0$.

b Given that $x^2 - 4x + 5$ is a factor of $x^4 - 4x^3 + 8x^2 + ax + b$, find the values of a and b.

32 The polynomial $x^2 - 4x + 3$ is a factor of the polynomial $x^3 + ax^2 + 27x + b$. Find the values of a and b.

33 **a** Given that a polynomial $f(x)$ can be written as $f(x) = (x - a)^2\, g(x)$, show that $f'(x)$ has a factor $(x - a)$.

b The polynomial $2x^4 + bx^3 + 11x^2 - 12x + e$ has a factor $(x - 2)^2$. Find the values of b and e.

34 The roots of the equation $6x^3 - 19x^2 + cx + d = 0$ form a geometric sequence with the second term equal to 1. Find the values of c and d.

35 The cubic equation $x^3 + px^2 + qx + c = 0$, has roots α, β, γ. By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that

a **i** $p = -(\alpha + \beta + \gamma)$

ii $q = \alpha\beta + \beta\gamma + \gamma\alpha$

iii $c = -\alpha\beta\gamma$.

It is now given that $p = -6$ and $q = 18$ for parts **b** and **c** below.

b **i** In the case that the three roots α, β, γ form an arithmetic sequence, show that one of the roots is 2.

ii Hence determine the value of c.

c In another case the three roots α, β, γ form a geometric sequence. Determine the value of c.

Mathematics HL May 2015 Paper 1 TZ2 Q12

7 Functions

ESSENTIAL UNDERSTANDINGS

- Creating different representations of functions to model relationships between variables, visually and symbolically, as graphs, equations and tables represents different ways to communicate mathematical ideas.

In this chapter you will learn…

- about rational functions of the form $f(x) = \frac{ax+b}{cx^2+dx+e}$ and $f(x) = \frac{ax^2+bx+c}{dx+e}$
- how to solve cubic inequalities
- how to solve other inequalities graphically
- how to sketch graphs of the functions $y = |f(x)|$ and $y = f(|x|)$
- how to solve modulus equations and inequalities
- how to sketch graphs of the form $y = \frac{1}{f(x)}$
- how to sketch graphs of the form $y = f(ax+b)$
- how to sketch graphs of the form $y = [f(x)]^2$
- about even and odd functions
- about restricting the domain so that the inverse function exists
- about self-inverse functions.

CONCEPTS

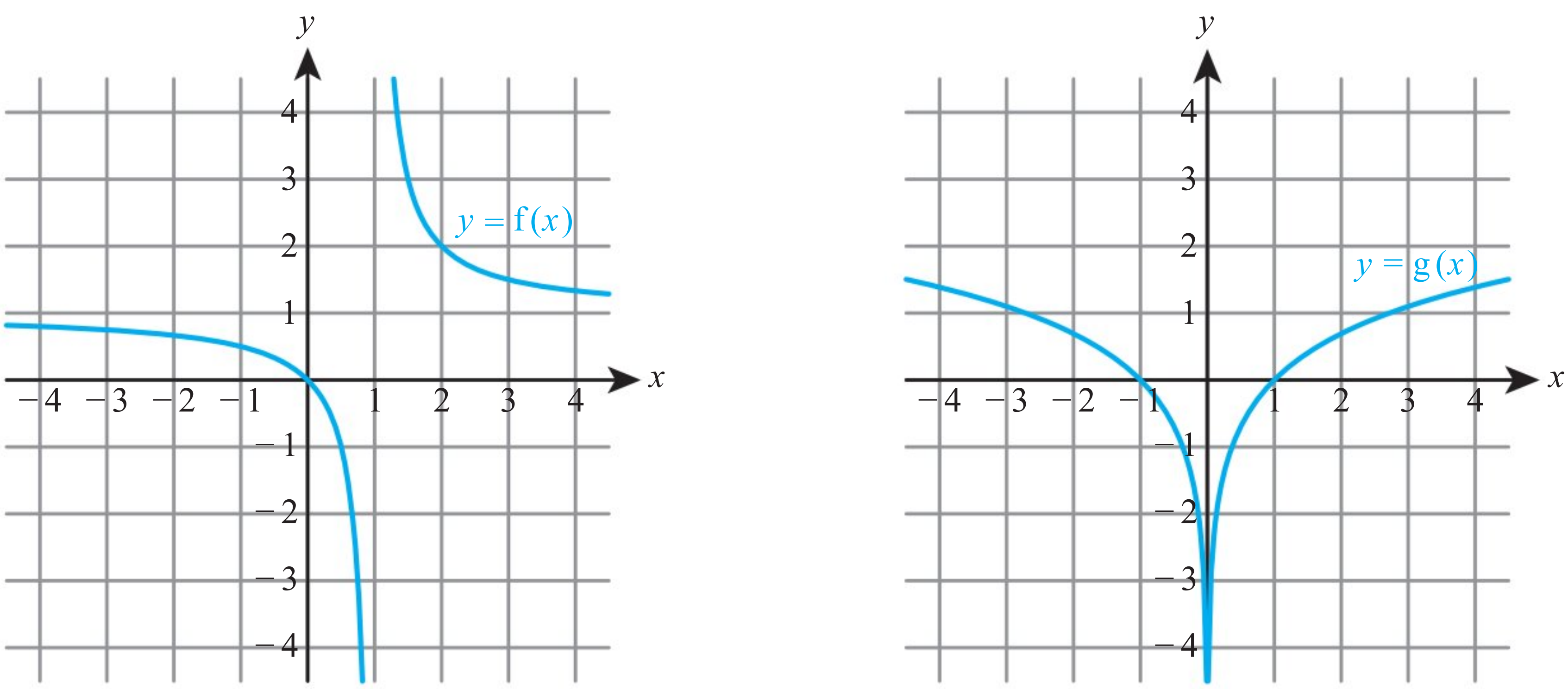
The following concepts will be addressed in this chapter:

- Different **representations** of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph.

LEARNER PROFILE – Risk-takers

Do you learn more from getting a problem right or wrong?

Figure 7.1 Graphs with different symmetries



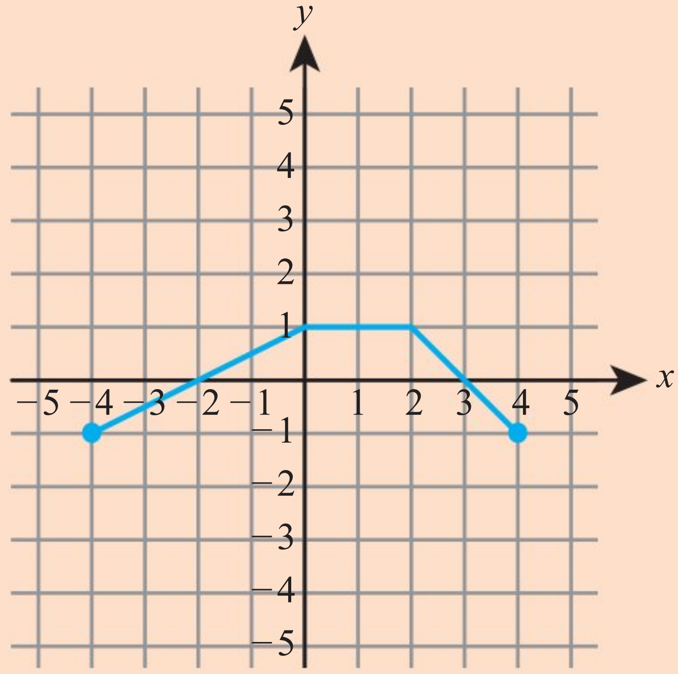
PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Sketch the graph of $y = \dfrac{2x-1}{x+3}$, labelling all asymptotes and axis intercepts.
2. Solve the inequality $x^2 + 2x - 8 < 0$.
3. Solve, to 3 significant figures, with x in radians, the equation $\ln x = \sin 2x$.
4. The graph of $y = f(x)$ is shown.

   Sketch the graph of:

   a $y = 2f(x) + 3$ **b** $y = -f(2x)$.



5. The function f is given by $f(x) = \dfrac{2x-1}{x+3}$, $x \neq -3$.

   Find f^{-1} and give its domain.

Starter Activity

Look at the pictures in Figure 7.1. In small groups, discuss any similarities you can identify between these functions.

Now look at this problem:

Use technology to investigate transformations of $f(x) = x^3 - 4x$.

a Draw the graph of $y = f(x)$.

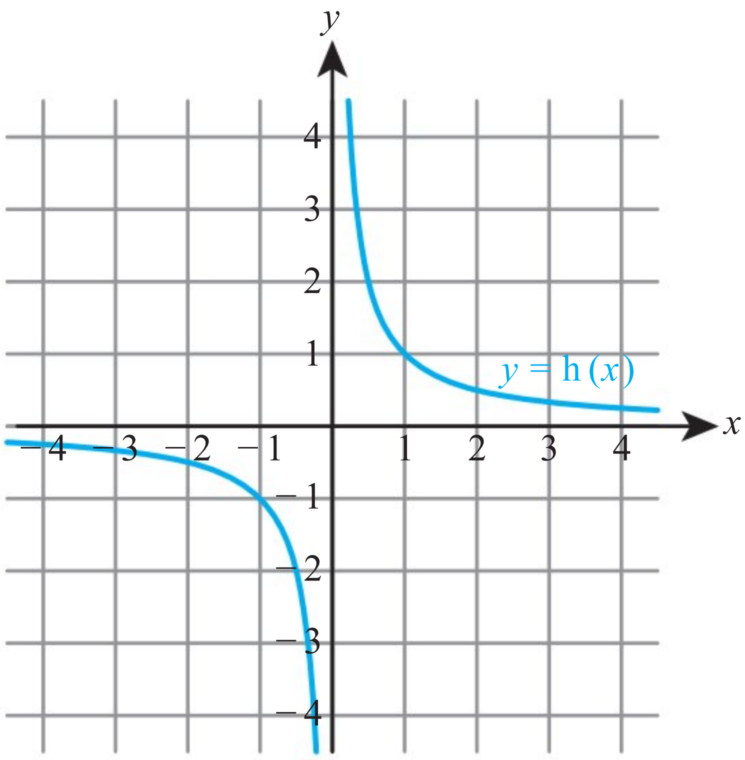
b Draw each of the following, and describe their relationship to $y = f(x)$.

  i $y = f(2x + 3)$ **ii** $y = f\left(\frac{1}{2}x - 3\right)$

c Draw $y = f(-x)$ and $y = -f(x)$. What do you notice?

You are already familiar with translations, stretches and reflections in the coordinate axes of graphs, but there are many other useful transformations that can be applied. Relating the algebraic representation of a function to possible symmetries of its graph also leads to different categorizations of functions.

It is also important to be able to apply sequences of transformations to graphs, including where there is more than one x-transformation.

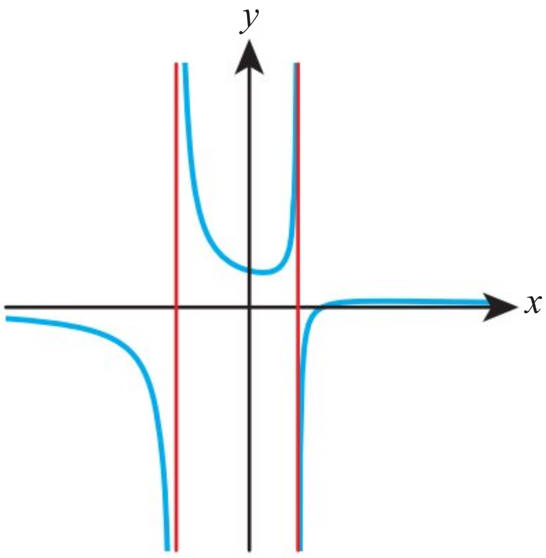


7A Rational functions of the form $f(x)=\dfrac{ax+b}{cx^2+dx+e}$ and $f(x)=\dfrac{ax^2+bx+c}{dx+e}$

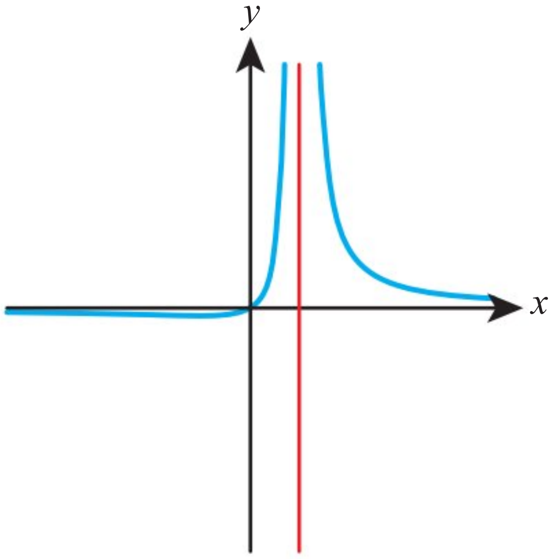
You saw how to sketch the graph of the hyperbola $y=\dfrac{ax+b}{cx+d}$ in Section 16B of Mathematics: analysis and approaches SL.

Whereas the hyperbola $y=\dfrac{ax+b}{cx+d}$ always has one vertical asymptote, the graph of the rational function $y=\dfrac{ax+b}{cx^2+dx+e}$ could have two vertical asymptotes, one vertical asymptote or no vertical asymptotes, depending on the number of real solutions to the quadratic in the denominator.

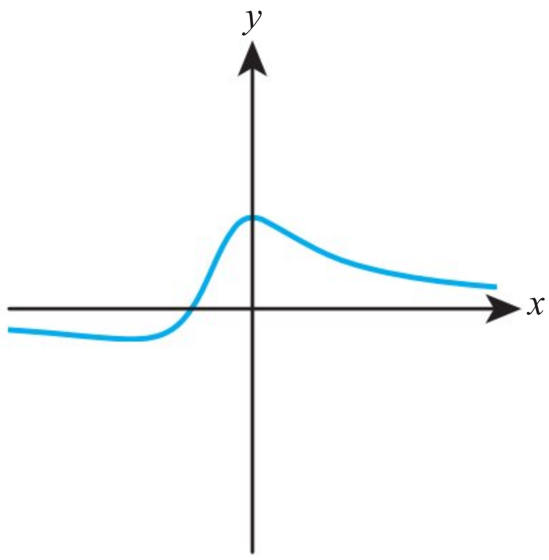
You also know that the horizontal asymptote will now always be $y=0$, since the x^2 term in the denominator dominates and causes $y\to 0$ as $x\to\pm\infty$.



2 vertical asymptotes



1 vertical asymptote



0 vertical asymptotes

KEY POINT 7.1

If $y=\dfrac{ax+b}{cx^2+dx+e}$ then

- the y-intercept is $\left(\dfrac{b}{e},\,0\right)$
- the x-intercept is $\left(0,\,-\dfrac{b}{a}\right)$
- the horizontal asymptote is at $y=0$
- any vertical asymptotes occur at solutions of $cx^2+dx+e=0$.

WORKED EXAMPLE 7.1

Sketch the graph of $y=\dfrac{x-3}{2x^2+x-3}$, labelling any axis intercepts and asymptotes.

x-intercepts occur when $y=0$

x-intercepts:

$x-3=0$

$x=3$

So, (3, 0)

y-intercepts occur when $x=0$

y-intercepts:

$$y=\frac{0-3}{2\times 0^2+0-3}=\frac{-3}{-3}=1$$

So, (0, 1)

Vertical asymptotes occur when the denominator is zero

Vertical asymptotes:

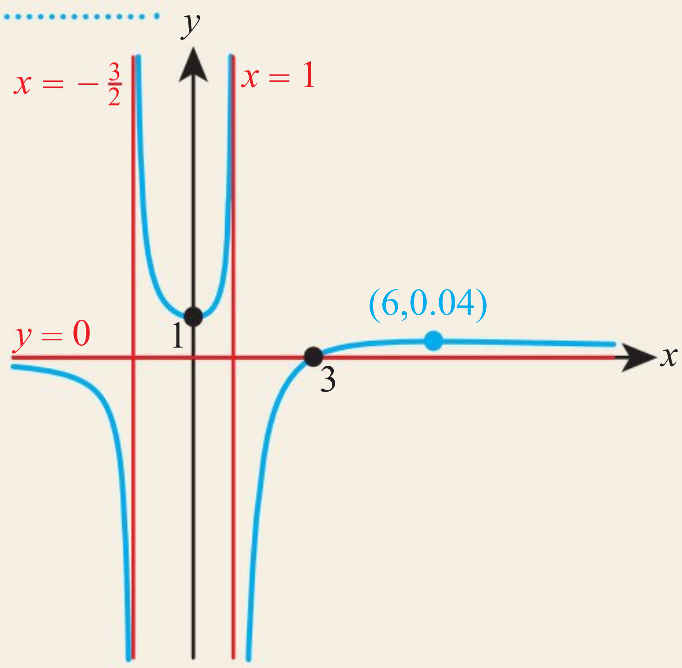
$$2x^2 + x - 3 = 0$$
$$(2x+3)(x-1) = 0$$
$$x = -\frac{3}{2} \text{ or } x = 1$$

Because the degree of the denominator is larger then the degree of the numerator, the function tends to zero as $x \to \pm\infty$

As $x \to \pm\infty$, $y \to 0$

So, horizontal asymptote at $y = 0$

Sketch the graph using all these features



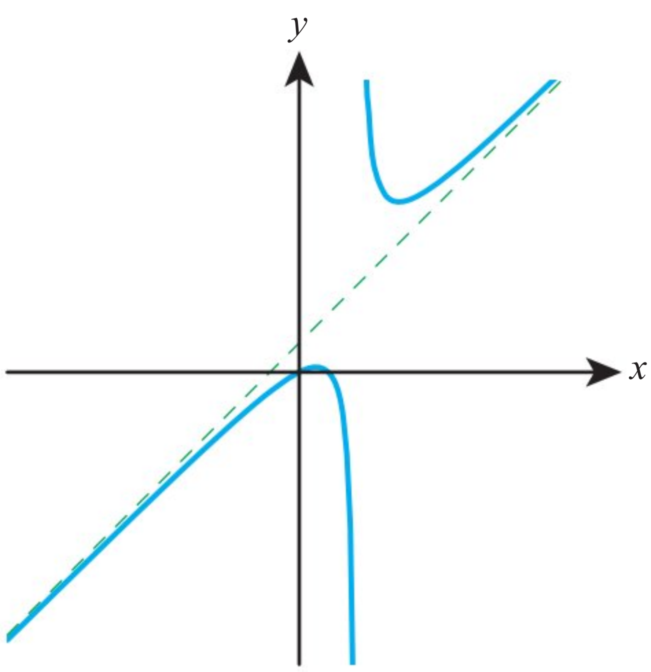
To the right of the $x = 1$ asymptote: you can check that $y < 0$ when $x = 2$, so the graph starts below the x-axis, crosses it at $x = 3$ and then approaches it again for large x. There is therefore a maximum point with $x > 3$

To the left of the $x = -\frac{3}{2}$ asymptote: you can check that $y < 0$ when $x = -2$, so the graph is below the x-axis

You can find the coordinates of the turning point using calculus or check using your calculator.

If $y = \dfrac{ax^2+bx+c}{dx+e}$, there will always be one vertical asymptote, and now when x becomes very large (either positive or negative) the x^2 term in the numerator dominates, so y tends to ∞ (or $-\infty$ if $\frac{a}{d} < 0$) rather than zero, i.e. there is no horizontal asymptote.

However, the graph does tend to a non-horizontal asymptote as x becomes large. This is called an **oblique asymptote**.



You used polynomial division or comparing coefficients in Section 4C and Section 6B.

To find the equation of the oblique asymptote, we need to use polynomial division (or comparing coefficients) to express the rational function in an appropriate form.

KEY POINT 7.2

If $y = \dfrac{ax^2 + bx + c}{dx + e}$, then

- the y-intercept is $\left(0, \dfrac{c}{e}\right)$
- any x-intercepts occur at solutions of $ax^2 + bx + c = 0$
- the vertical asymptote is at $x = -\dfrac{e}{d}$
- there will be an oblique asymptote of the form $y = px + q$.

WORKED EXAMPLE 7.2

Sketch the graph of $y = \dfrac{x^2 - x - 2}{x - 3}$, labelling any axis intercepts and asymptotes.

x-intercepts occur when $y = 0$

x-intercepts:

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, -1$$

So, (2, 0) and (−1, 0)

y-intercepts occur when $x = 0$

y-intercepts:

$$y = \frac{0^2 - 0 - 2}{0 - 3} = \frac{2}{3}$$

So, $\left(0, \dfrac{2}{3}\right)$

Vertical asymptotes occur when the denominator is zero

Vertical asymptote:

$$x - 3 = 0$$
$$x = 3$$

Because the degree of the numerator is larger than the degree of the denominator, we need to do polynomial division (or comparing coefficients)

As $x \to \pm\infty$:

$$x^2 - x - 2 = (x - 3)(x + 2) + 4$$

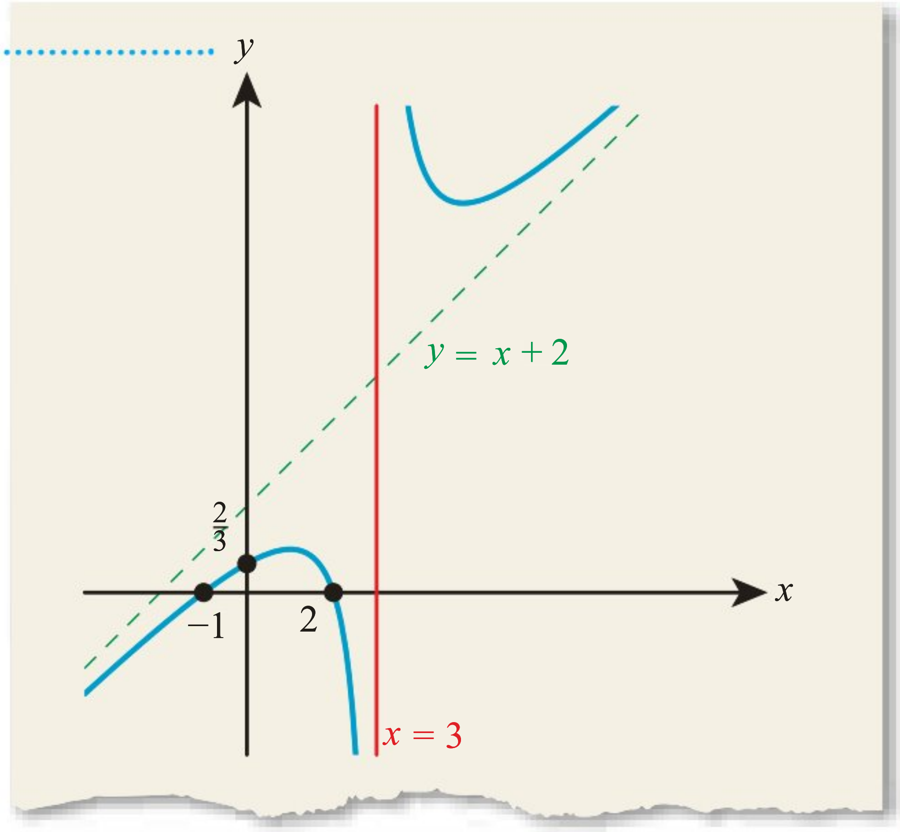
So,

$$\frac{x^2 - x - 2}{x - 3} = x + 2 + \frac{4}{x - 3}$$

We can now see that when $x \to \pm\infty$, $x + 2 + \dfrac{4}{x - 3} \to x + 2 + 0 = x + 2$

Therefore, oblique asymptote: $y = x + 2$.

Sketch the graph using all these features



Exercise 7A

For questions 1 to 4, use the method demonstrated in Worked Example 7.1 to sketch the graph. In each case label any vertical asymptotes and axis intercepts.

1 a $y = \frac{3x-2}{x^2+4x}$ b $y = \frac{5-2x}{x^2+x-2}$

2 a $y = \frac{1-x}{x^2-2x-3}$ b $y = \frac{x-3}{x^2-6x+8}$

3 a $y = \frac{4x}{x^2+2x+5}$ b $y = \frac{-3x}{x^2-x+2}$

4 a $y = \frac{x+3}{x^2+4x+4}$ b $y = \frac{x-2}{x^2-2x+1}$

For questions 5 to 8, use the method demonstrated in Worked Example 7.2 to sketch the graph. In each case label any vertical and oblique asymptotes and axis intercepts.

5 a $y = x - \frac{4}{x+3}$ b $y = 2x - \frac{6}{x-2}$

6 a $y = 3 - x + \frac{6}{x+2}$ b $y = 2x + 1 - \frac{9}{x-1}$

7 a $y = \frac{x^2+4x-5}{x-2}$ b $y = \frac{x^2-4}{x+3}$

8 a $y = \frac{x^2+4x}{x+1}$ b $y = \frac{x^2-3x-10}{x-4}$

9 For the graph of $y = \frac{4x+5}{4x^2-9}$

a find the equations of the vertical asymptotes

b sketch the graph, labelling the coordinates of any axis intercepts.

10 For the graph of $y = \frac{5x-10}{3x^2+2x-8}$

a find the equations of the vertical asymptotes

b sketch the graph, labelling the coordinates of any axis intercepts.

11 One of the asymptotes of the graph of $y = \frac{x+1}{2x^2+kx-12}$ is $x = -4$.

a Find the value of k.

b Find the equation of the other vertical asymptote.

c Sketch the graph, labelling any axis intercepts.

12 **a** On the same set of axes, sketch the graph of $y = \dfrac{3x}{x^2 - 2x + 1}$ and the graph of $y = x + 2$, labelling any vertical asymptotes and axis intercepts.

b Hence state the number of solutions of the equation $\dfrac{3x}{x^2 - 2x + 1} = x + 2$.

13 **a** On the same set of axes, sketch the graph of $y = \dfrac{x - 1}{2x^2 + 5x - 3}$ and the graph of $y = 2x - 1$, labelling any vertical asymptotes and axis intercepts.

b Hence state the number of solutions of the equation $\dfrac{x - 1}{2x^2 + 5x - 3} = 2x - 1$.

14 A curve has equation $y = \dfrac{2x - 3}{x^2 + 4}$.

a **i** If the line $y = k$ intersects the curve, show that $4k^2 + 3k - 1 \leqslant 0$.

ii Hence find the coordinates of the turning points of the curve.

b Sketch the curve.

15 **a** **i** Find the set of values of k for which the equation $ks^2 - (k + 1)x - 2k - 2 = 0$ has real roots.

ii Hence determine the range of the function $\text{f}: x \mapsto \dfrac{x + 2}{x^2 - x - 2}$.

b State the equations of the vertical asymptotes of $y = \text{f}(x)$ and the coordinates of any axis intercepts.

c Sketch the graph of $y = \text{f}(x)$.

16 The curve C has equation $y = \dfrac{x - a}{(x - b)(x - c)}$.

Sketch C when:

a $a < b < c$

b $b < a < c$.

17 Let $\text{f}(x) = \dfrac{x^2 - 6x + 10}{x - 3}$.

a Show that $y = \text{f}(x)$ has an oblique asymptote at $y = Ax + B$, where A and B are constants to be found.

b Find the turning points of $\text{f}(x)$.

c State the coordinates of any axis intercepts and the equation of the vertical asymptote of $y = \text{f}(x)$.

d Sketch the graph of $y = \text{f}(x)$.

18 **a** Show that the function $\text{f}(x) = \dfrac{2x^2 - x - 3}{2x - 5}$ can be written as $\text{f}(x) = Ax + B + \dfrac{C}{2x - 5}$, where A, B and C are constants to be found.

b Write down the equation of the oblique asymptote of $y = \text{f}(x)$.

c By finding a condition on k for there to be real solutions to the equation $\text{f}(x) = k$, find the range of f.

d State the axis intercepts and equation of the vertical asymptote of $y = \text{f}(x)$.

e Sketch the graph of $y = \text{f}(x)$.

19 The function f is defined by $\text{f}(x) = \dfrac{x + c}{x^2 - 3x - c}$.

The range of f is $\text{f}(x) \in \mathbb{R}$.

Find the possible values of c.

20 The function f is given by $\text{f}(x) = \dfrac{x^2 + 2ax + a^2 - 1}{x + a}$, $a > 0$.

a Find the equation of the oblique asymptote of $y = \text{f}(x)$.

b Show that f has no stationary points.

c Sketch the graph of $y = \text{f}(x)$, labeling all asymptotes and axis intercepts.

7B Solutions of $g(x) \geqslant f(x)$, both analytically and graphically

In Chapter 15 of Mathematics: analysis and approaches SL you solved quadratic inequalities by sketching the graph and identifying the relevant region. This same method can be used with cubic inequalities.

WORKED EXAMPLE 7.3

Solve the inequality $x^3 > x^2 + 6x$.

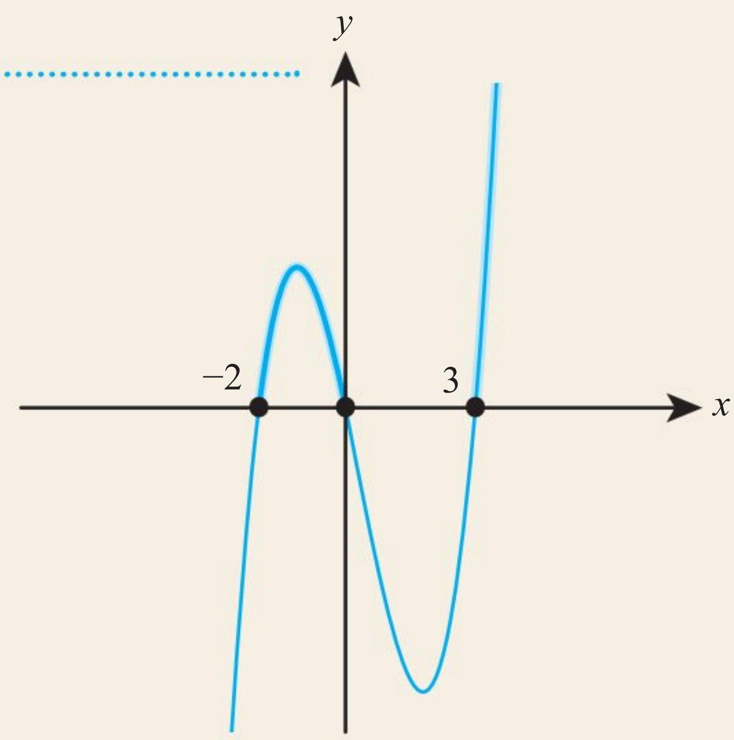
Rearrange in the form $f(x) > 0$

$$x^3 > x^2 + 6x$$
$$x^3 - x^2 - 6x > 0$$

Factorize in order to find roots of $f(x) = 0$

$$x(x^2 - x - 6) > 0$$
$$x(x + 2)(x - 3) > 0$$

Now sketch the graph of $y = f(x)$ and identify regions where the y-value is greater than zero



Describe the highlighted sections in terms of x

So, $-2 < x < 0$ or $x > 3$.

This method can be applied to inequalities involving other functions whose graphs you can sketch, such as rational functions.

WORKED EXAMPLE 7.4

Solve the inequality $\frac{7}{x^2+x-2} > 4$.

Sketch the graph of $y = \frac{7}{x^2+x-2}$ and add the line $y = 4$

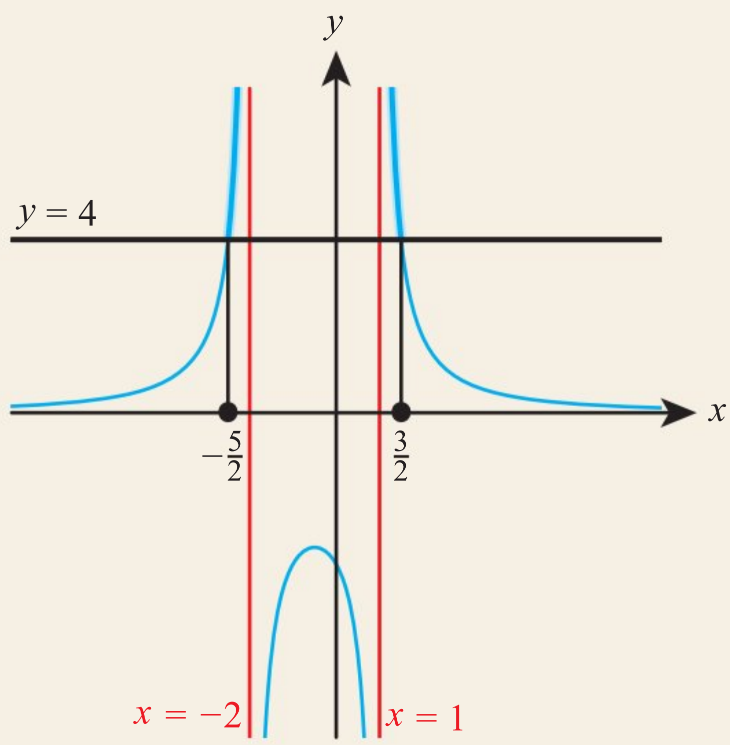
No x-intercept

y-intercept: $\left(-\frac{7}{2}, 0\right)$

Vertical asymptotes:

$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1, -2$$

Horizontal asymptote: $y = 0$



Find the points of intersection

When $y = 4$

$$\frac{7}{x^2+x-2} = 4$$
$$4x^2 + 4x - 8 = 7$$
$$4x^2 + 4x - 15 = 0$$
$$(2x-3)(2x+5) = 0$$
$$x = \frac{3}{2}, -\frac{5}{2}$$

Describe the part of the graph above the line $y = 4$ in terms of x

Note that x can not be equal to −2 or 1 due to the asymptotes there

So, $-\frac{5}{2} \leqslant x < -2$ or $1 < x \leqslant \frac{3}{2}$.

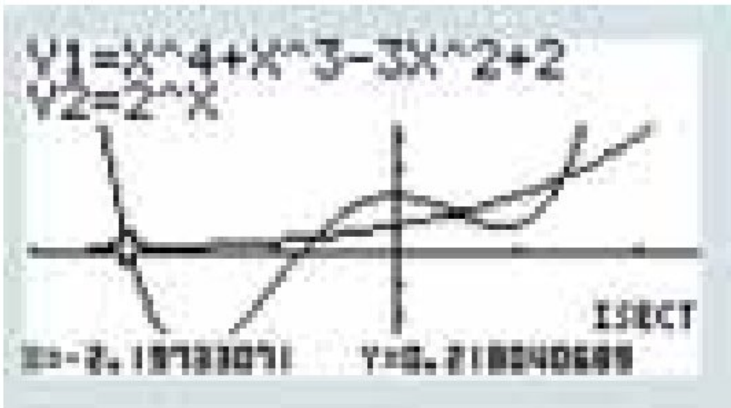
You can use your GDC to solve more complicated inequalities graphically.

WORKED EXAMPLE 7.5

Solve the inequality $x^4 + x^3 - 3x^2 + 2 < 2^x$.

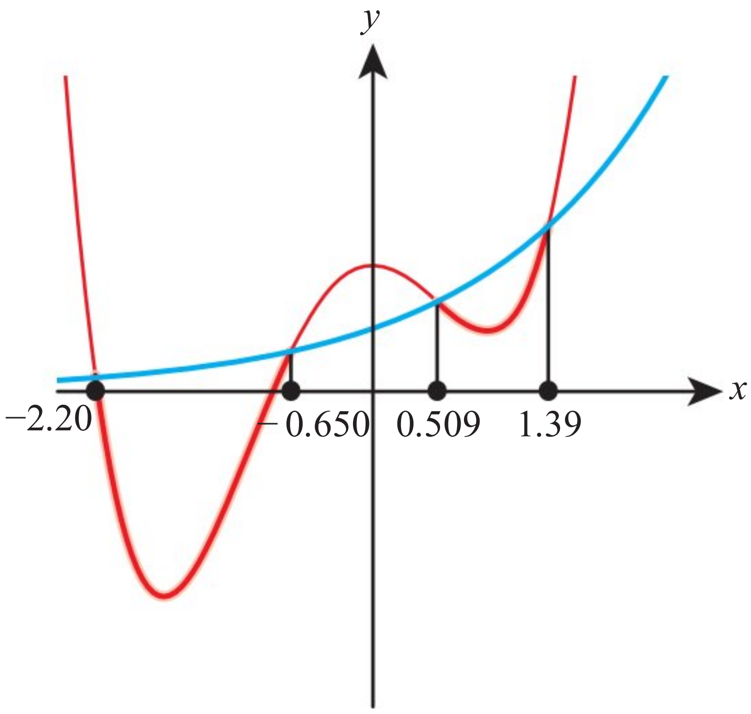
Use the GDC to find the intersection points of $y = x^4 + x^3 - 3x^2 + 2$ and $y = 2^x$

The expressions are equal when $x = -2.20, -0.650, 0.509, 1.39$



Now highlight the regions where $y = x^4 + x^3 - 3x^2 + 2$ is below $y = 2^x$

So, $-2.20 < x < -0.650$ or $0.509 < x < 1.39$.



Exercise 7B

For questions 1 to 5, use the method demonstrated in Worked Example 7.3 to solve the inequality.

1 a $x^3 + 2x > 3x^2$
  b $x^3 > 4x^2 + 5x$

2 a $x^3 < 6x^2 - 8x$
  b $x^3 + 18x < 9x^2$

3 a $(x+3)(x-2)(x-5) \leqslant 0$
  b $(x-1)(x-3)(x-4) \leqslant 0$

4 a $(4-x)(x-3)(x+2) \geqslant 0$
  b $(2-x)(x-1)(x-8) \geqslant 0$

5 a $(x+1)(x-2)^2 > 0$
  b $(x+3)^2(x-4) > 0$

For questions 6 to 10, use the method demonstrated in Worked Example 7.5 to solve the inequality.

6 a $x^3 + 3x^2 - 2 \leqslant 2^{-x}$

b $x^3 + 8x^2 + 20x + 16 \leqslant 3^{-x}$

7 a $2e^{-x} \geqslant x^2 - 3$

b $e^{-x+1} \geqslant x^2 - 1$

8 a $4\ln x > x - 2$

b $\ln(x-1) > 2x - 5$

9 a $x^4 + 2x^3 + 1 < 3x^2 + 4x$

b $3x^2 + 10x - 9 > x^4 + 4x^3$

10 a $\cos 4x \leqslant e^{x+1} - 2$

b $\sin 3x \geqslant e^{\frac{x}{3}} - 2$

11 Solve the inequality $2x^3 + x^2 > 6x$.

12 a Show that $(x-2)$ is a factor of $2x^3 + x^2 - 7x - 6$

b Hence solve the inequality $3x^3 + 2x^2 \leqslant x^3 + x^2 + 7x + 6$.

13 a Show that $(x+3)$ is a factor of $2x^3 + 11x^2 + 12x - 9$.

b Hence solve the inequality $11x^2 - 4 > 5 - 12x - 2x^3$.

14 Given that $a < b < c$, solve the inequality $(x-a)(x-b)(x-c) > 0$.

15 Given that $a < b$, solve the inequality $(x-a)(x-b)^2 < 0$.

16 Find the set of values of x for which $x^4 - 4x^2 + 3x + 1 \leqslant 0$.

17 Solve the inequality $2x^5 - 6x^4 + 8x^2 - 1 \geqslant 0$.

18 Find the set of values of x for which $3\ln(x^2+1) < x + 2$.

19 The solution of the inequality $x^3 + bx^2 + cx + d < 2$ is $x < 3$, $x \neq 1$. Find the values of the integers a, b and c.

20 The solution of the inequality $ax^3 + bx^2 + cx + d > 3$ is $x < -4$ or $-1 < x < \frac{3}{2}$. Find the values of the integers a, b, c and d where $|a|$ is as small as possible.

21 Solve the inequality $\dfrac{3x}{x^2 + x - 6} \geqslant 2^x$.

22 Solve the inequality $\dfrac{x-2}{3x^2 + 2x - 8} \leqslant \ln(x+4)$.

23 Find the set of values of x for which the function $f(x) = 2 + 8x^3 - x^4$ is decreasing.

24 Find the set of values of x for which the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x - 5$ is increasing.

25 a On the same axes draw the graphs of $y = \dfrac{3x+5}{x-2}$ and $y = 4$.

b Hence solve the inequality $\dfrac{3x+5}{x-2} \geqslant 4$.

26 a On the same axes draw the graphs of $y = \dfrac{2x-7}{x+1}$ and $y = x - 3$.

b Hence solve the inequality $\dfrac{2x-7}{x+1} < x - 3$.

27 A curve has equation $y = \dfrac{2x-a}{x+b}$, where $a, b > 0$.

a Sketch the curve, labelling all asymptotes and the coordinates of all axis intercepts.

b Hence solve the inequality $\dfrac{2x-a}{x+b} > 3$.

28 The solution of the inequality $\dfrac{16x+1}{px+1} > x + 4$ is $x < q$ or $r < x < 3$. Find p, q and r.

7C The graphs of the functions $y = |f(x)|$ and $y = f(|x|)$

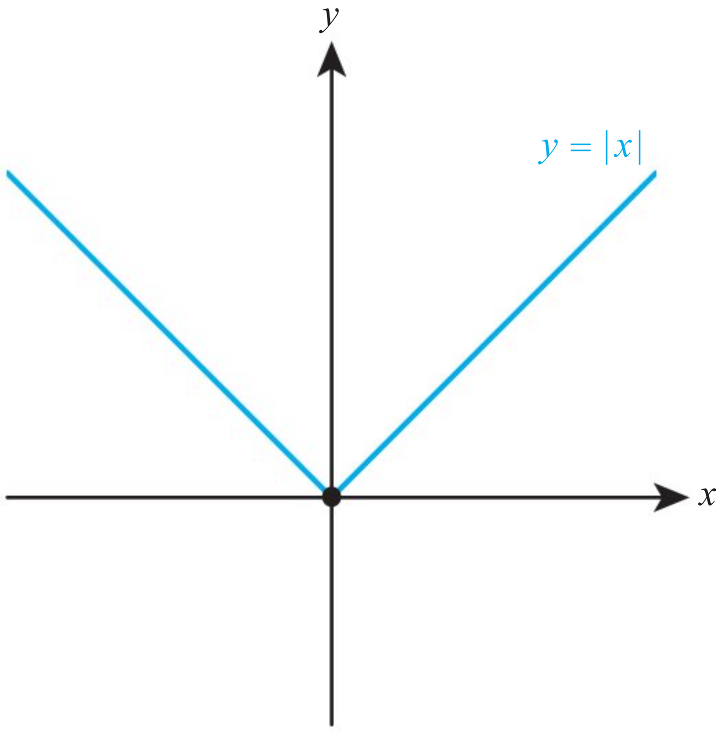
The modulus function leaves positive numbers unaffected but reverses the sign of negative numbers.

KEY POINT 7.3

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$

This means that the graph is of $y = |x|$ is given by $y = x$ when $x \geqslant 0$ and $y = -x$ when $y < 0$.

The domain of $|x|$ is all real numbers, whilst the range is all positive numbers and zero.



This idea can be applied to other functions involving the modulus function. The graph of $y = |f(x)|$ will be identical to that of $y = f(x)$ when $f(x) \geqslant 0$ but will be $y = -f(x)$ whenever $f(x) < 0$.

Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis this means that any part of $y = f(x)$ below the x-axis is just reflected in the x-axis.

KEY POINT 7.4

To sketch the graph of $y = |f(x)|$, start with the graph of $y = f(x)$ and reflect in the x-axis any parts that are below the x-axis.

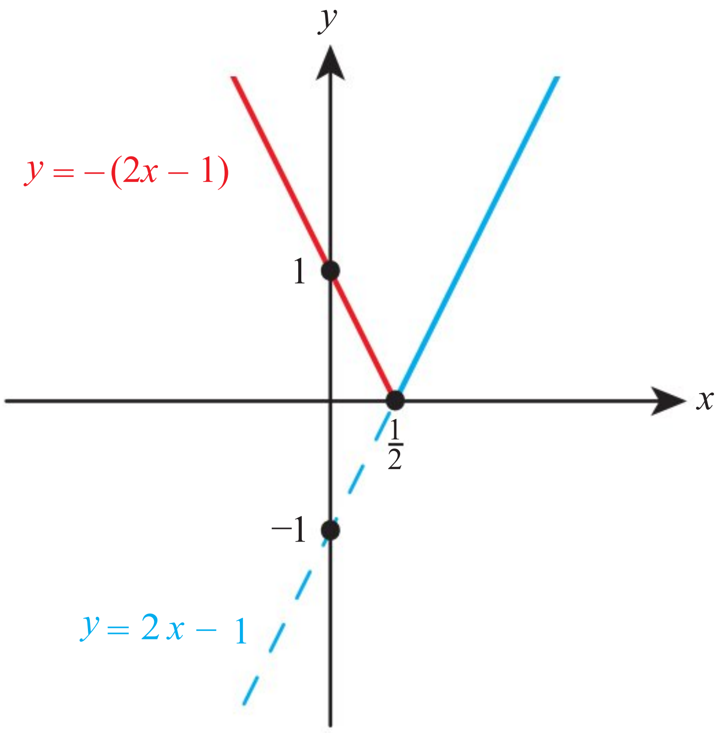
WORKED EXAMPLE 7.6

Sketch the graph of

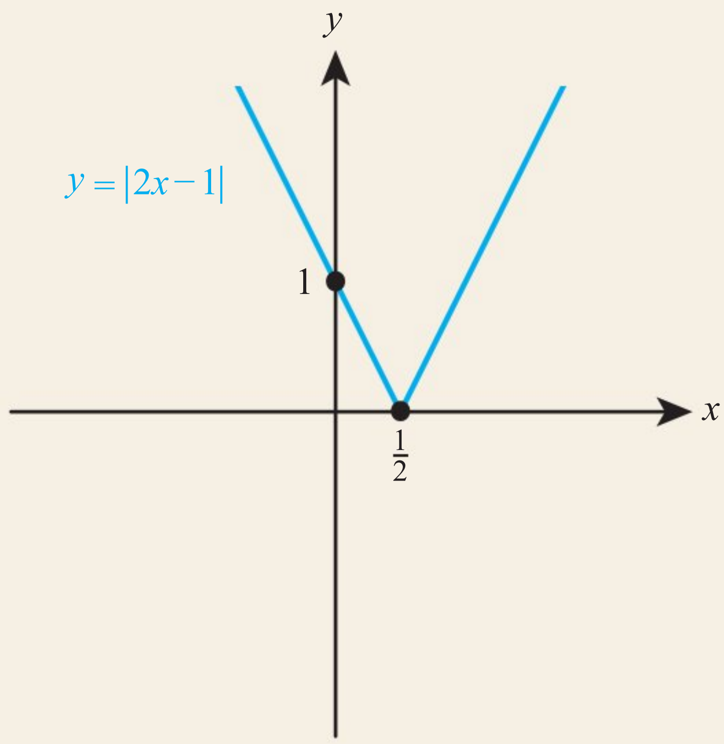
a $y = |2x - 1|$

b $y = |x^2 - x - 2|$.

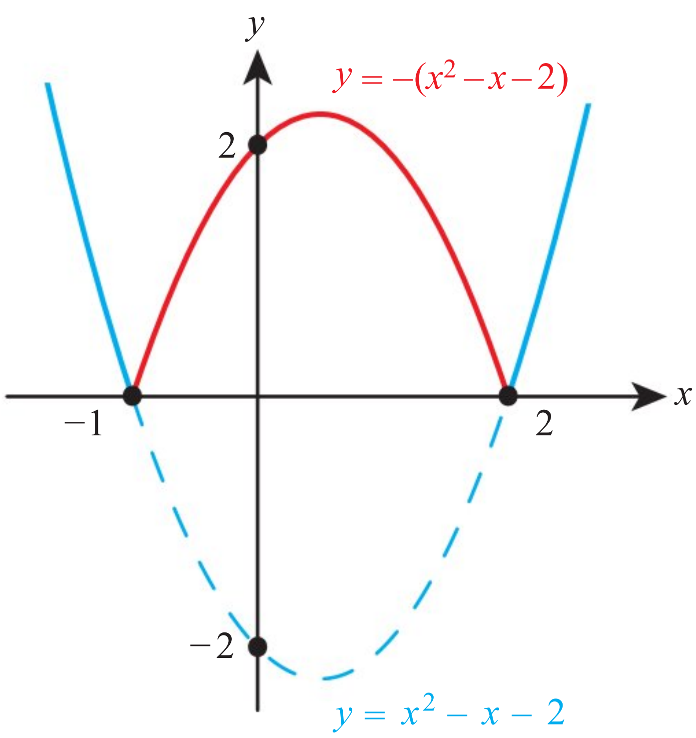
Sketch $y = 2x - 1$ and reflect the part below the x-axis to be above it



a



Sketch $y = x^2 - x - 2$ and reflect the part below the x-axis to be above it



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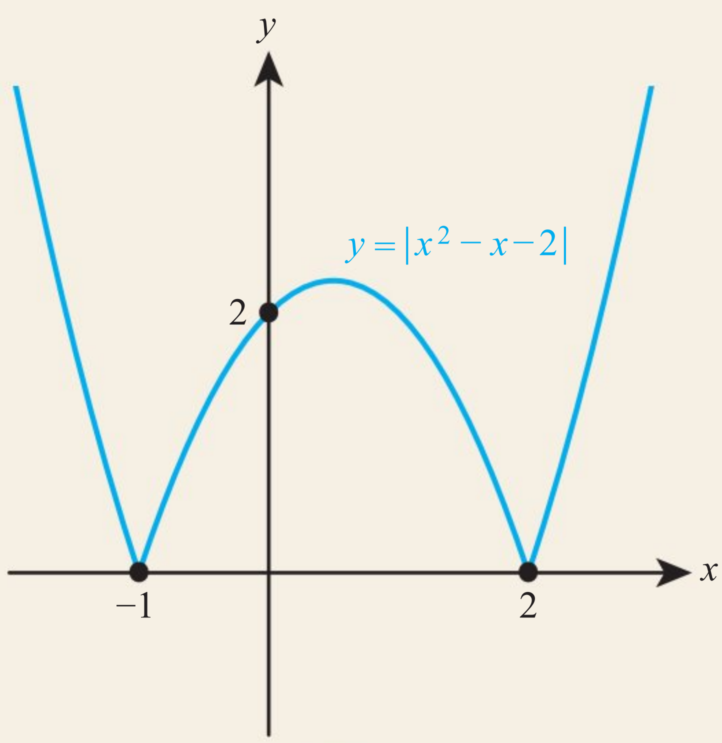


To sketch the graph of $y = f(|x|)$, note that $f(|-x|) = f(|x|)$. Therefore $y = f(|x|)$ is symmetric in the y-axis.

KEY POINT 7.5

To sketch the graph of $y = f(|x|)$, start with the graph of $y = f(x)$ for $x \geqslant 0$ and reflect that in the y-axis.

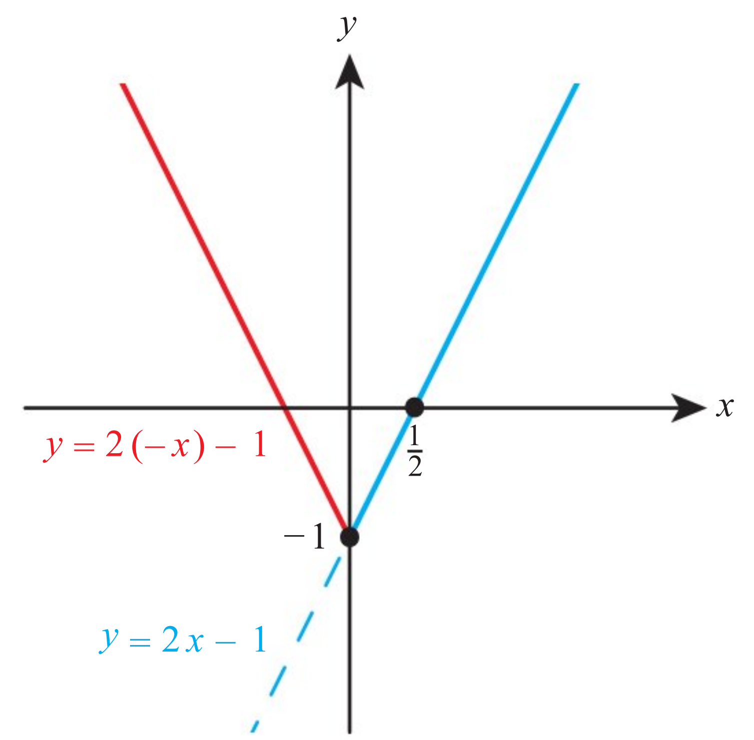
WORKED EXAMPLE 7.7

Sketch the graph of

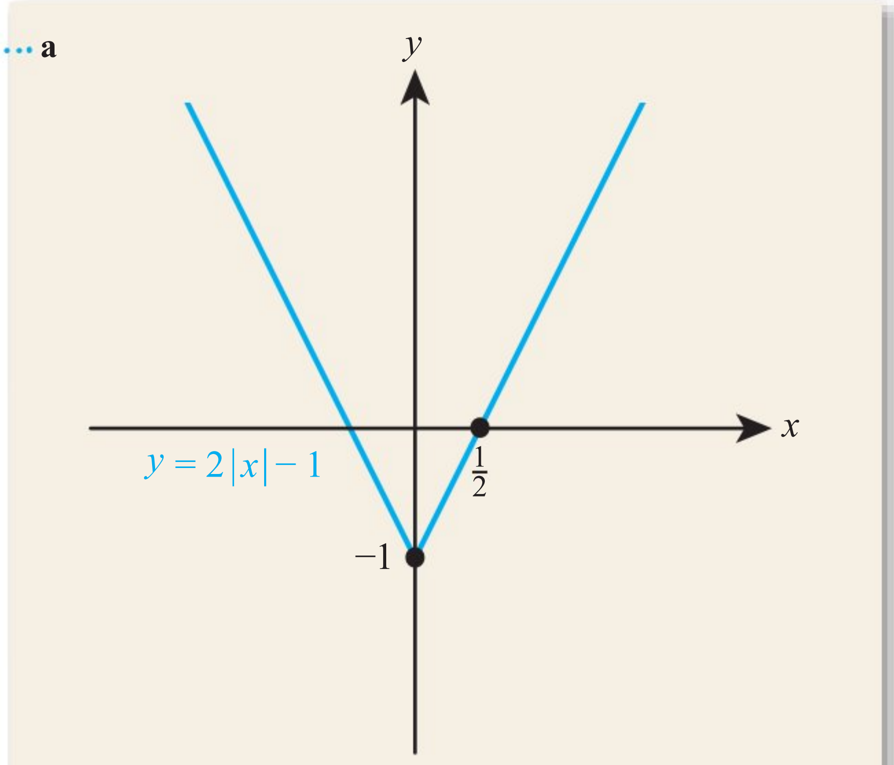
a $y = 2|x| - 1$

b $y = |x|^2 - |x| - 2$.

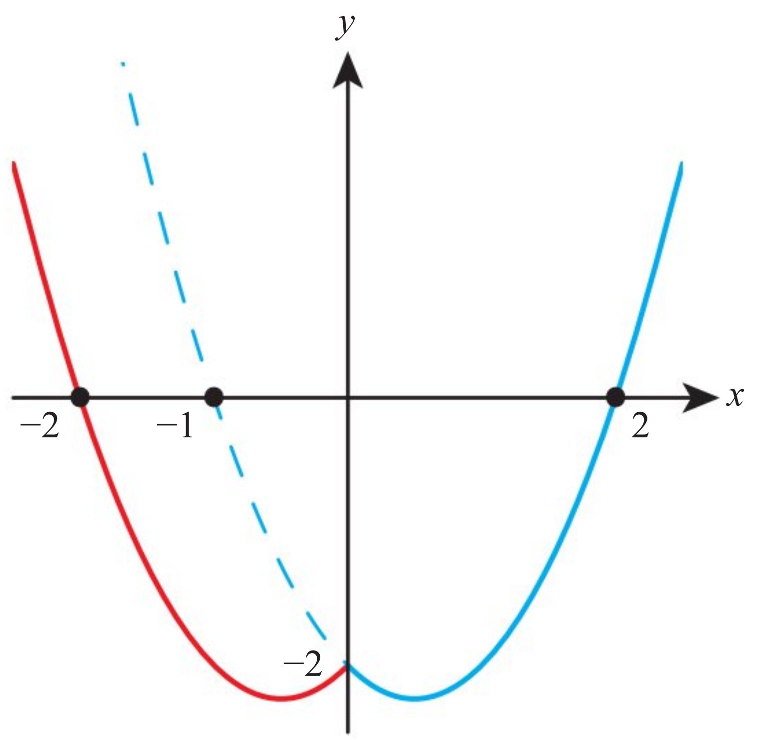
Sketch $y = 2x - 1$ for $x \geqslant 0$ and reflect it in the y-axis



a



Sketch $y = x^2 - x - 2$ for $x \geqslant 0$ and reflect it in the y-axis



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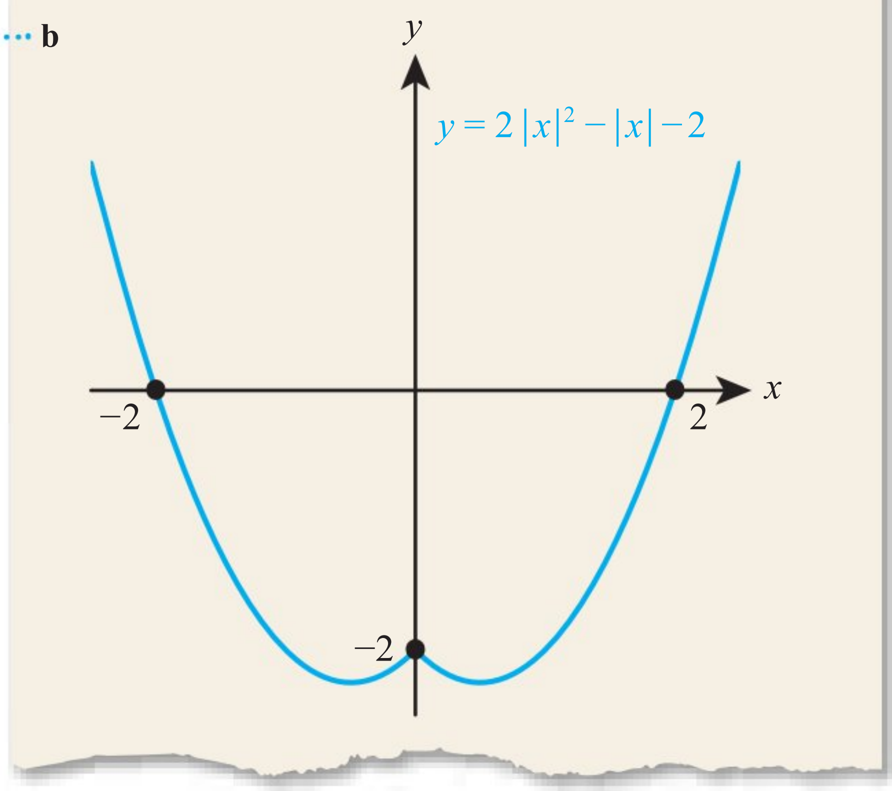


Solutions of modulus equations and inequalities

It is useful to sketch the relevant graphs before solving equations and inequalities involving the modulus function.

The graphs enable you to decide whether any intersections are on the reflected or the original part of the graph. If on the original part you can rewrite the equation without the modulus sign in; if on the reflected part you need to replace the modulus sign by a minus sign.

WORKED EXAMPLE 7.8

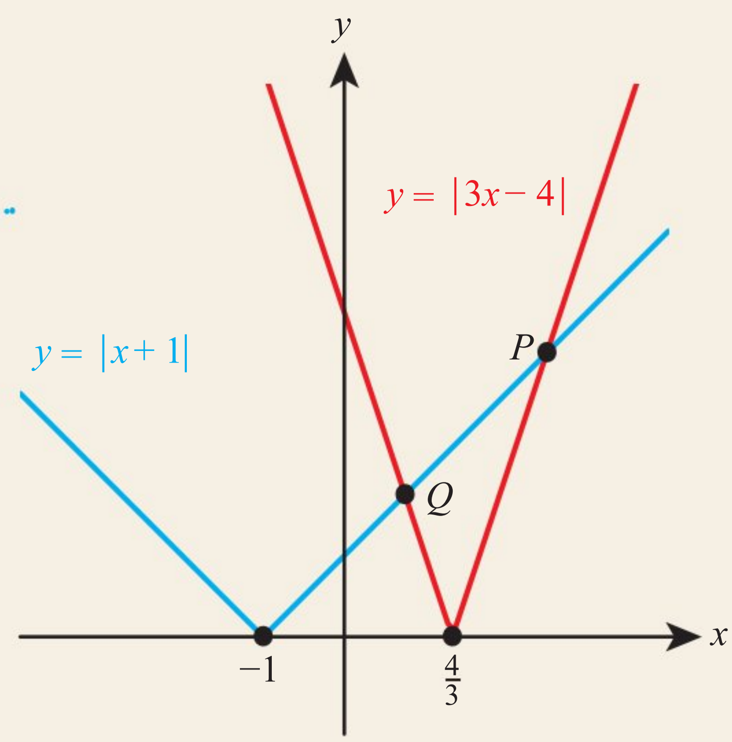
Solve the equation $|3x - 4| = |x + 1|$.

Sketch the graphs of $y = |3x - 4|$ and $y = |x + 1|$

There are two intersections:

Q is on the original part of the red graph and the original part of the blue graph

P is on the reflected part of the red graph and the original part of the blue graph



For the intersection of two original parts, just remove the modulus signs

$$3x - 4 = x + 1$$
$$2x = 5$$
$$x = \frac{5}{2}$$

For a reflected part, multiply the equation by −1

$$-(3x - 4) = x + 1$$
$$-3x + 4 = x + 1$$
$$4x = 3$$
$$x = \frac{3}{4}$$

So, solutions are $x = \frac{3}{4}$, $x = \frac{5}{2}$.

The next Worked Example illustrates how useful the graphs are in identifying the solutions of the modulus equation (and therefore the solution of the inequality).

WORKED EXAMPLE 7.9

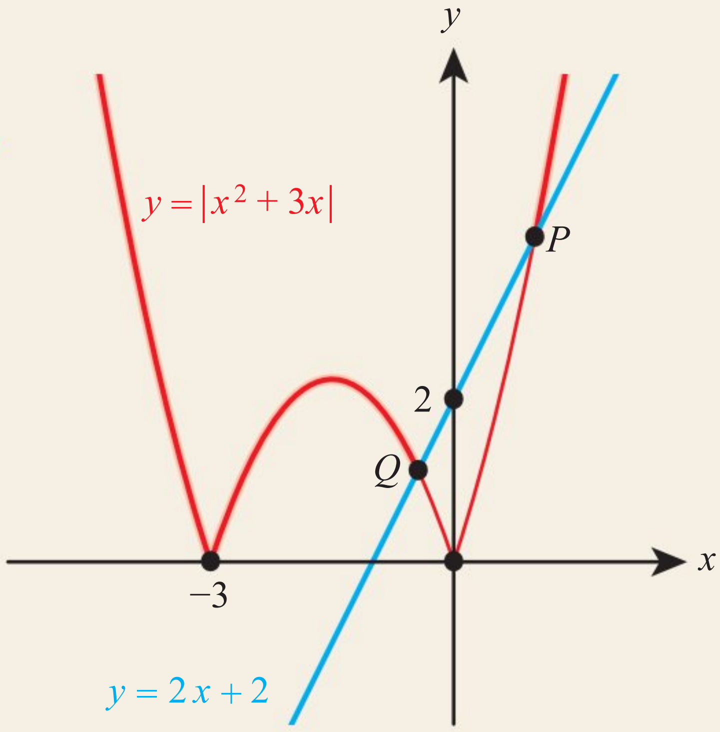
Solve the inequality $|x^2 + 3x| > 2x + 2$.

Sketch the graphs of $y = |x^2 + 3x|$ and $y = 2x + 2$ and highlight the required region

There are two intersections:

P is on the original part of the red graph

Q is on the reflected part of the red graph



For the original part, just remove the modulus sign from the equation

$$x^2 + 3x = 2x + 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, 1$$

We can see from the graph that $x = -2$ is not an intersection so disregard it

So, $x = 1$

For the reflected part, multiply by -1

$$-(x^2 + 3x) = 2x + 2$$
$$x^2 + 5x + 2 = 0$$
$$x = \frac{-5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$
$$= \frac{-5 \pm \sqrt{17}}{2}$$

Both solutions are negative but we can see from the graph that the intersection is the least negative of the two

So, $x = \dfrac{-5 + \sqrt{17}}{2}$

Refer back to the graph to describe the required region

So, $x < \dfrac{-5 + \sqrt{17}}{2}$ or $x > 1$.

Tip

You can also use your GDC to solve modulus equations and inequalities.

Be the Examiner 7.1

Solve the equation $|x-2| = 2x-3$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$x-2=2x-3$ $x=1$ $-(x-2)=2x-3$ $-x+2=2x-3$ $3x=5$ $x=\frac{5}{3}$ So, $x=1, \frac{5}{3}$	$x-2=2x-3$ $x=1$ So, $x=1$	$-(x-2)=2x-3$ $-x+2=2x-3$ $3x=5$ $x=\frac{5}{3}$ So, $x=\frac{5}{3}$

Exercise 7C

For questions 1 to 6, use the method demonstrated in Worked Example 7.6 to sketch the graph of $y = |f(x)|$, marking on axis intercepts.

1 a $y=|x+4|$
b $y=|x-1|$

2 a $y=|3x-2|$
b $y=|2x+5|$

3 a $y=|x^2-x-12|$
b $y=|x^2-4x+3|$

4 a $y=|x^3-4x|$
b $y=|x^3-6x^2+8x|$

5 a $y=|\sin x|, -2\pi \leqslant x \leqslant 2\pi$
b $y=|\tan x|, -\pi \leqslant x \leqslant \pi$

6 a $y=|\ln x|$
b $y=|\ln(x+2)|$

For questions 7 to 12, use the method demonstrated in Worked Example 7.7 to sketch the graph of $y = f(|x|)$, marking on axis intercepts.

7 a $y=|x|+4$
b $y=|x|-1$

8 a $y=3|x|-2$
b $y=2|x|+5$

9 a $y=|x|^2+|x|-12$
b $y=|x|^2-4|x|+3$

10 a $y=|x|^3-4|x|$
b $y=|x|^3-6|x|^2+8|x|$

11 a $y=\sin|x|, -2\pi \leqslant x \leqslant 2\pi$
b $y=\tan|x|, -\pi \leqslant x \leqslant \pi$

12 a $y=\ln|x|$
b $y=\ln|x+2|$

For questions 13 to 18, use the method demonstrated in Worked Example 7.8 to solve the modulus equation.

13 a $|2x-1|=5$
b $|3x+2|=8$

14 a $|3x-5|=|x+2|$
b $|4x+1|=|x-3|$

15 a $|5+2x|=3-4x$
b $|3x-4|=8-x$

16 a $|x^2+x-6|=6$
b $|x^2-5x+3|=3$

17 a $|x^2-3x-10|=x+2$
b $|x^2+5x+6|=x+3$

18 a $2|\cos x|=1, -\pi \leqslant x \leqslant \pi$
b $\sqrt{2}|\sin x|=1, -\pi \leqslant x \leqslant \pi$

For questions 19 to 24, use the method demonstrated in Worked Example 7.9 to solve the modulus inequality.

19 a $|2x-1|>5$
b $|3x+2|<8$

20 a $|3x-5| \leqslant |x+2|$
b $|4x+1| \geqslant |x-3|$

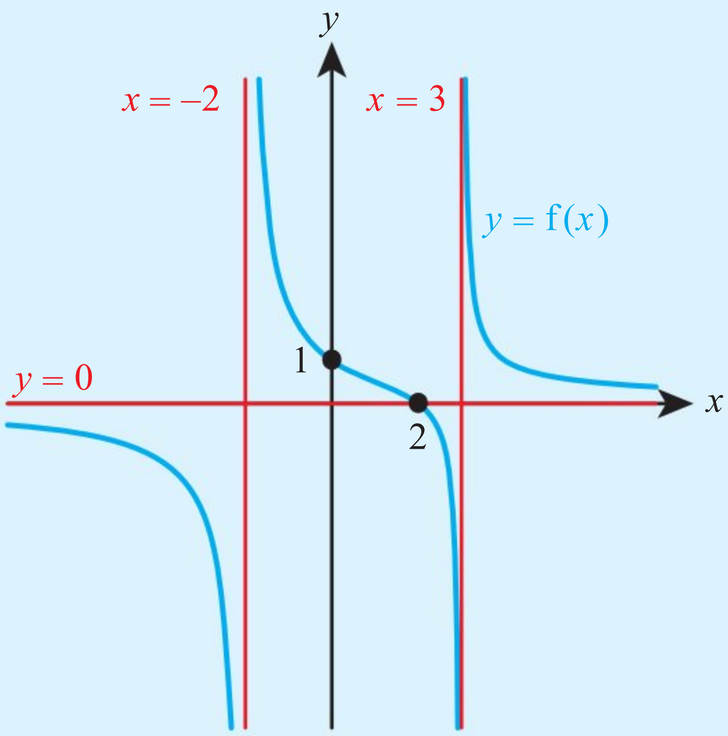
21 a $|5+2x|>3-4x$
b $|3x-4|<8-x$

22 a $|x^2+x-6| \geqslant 6$
b $|x^2-5x+3| \leqslant 3$

23 a $|x^2-3x-10|<x+2$
b $|x^2+5x+6|>x+3$

24 a $|\cos x|<\frac{1}{2}, -\pi \leqslant x \leqslant \pi$
b $|\sin x|>\frac{1}{\sqrt{2}}, -\pi \leqslant x \leqslant \pi$

25 The graph of $y = f(x)$ is shown below.

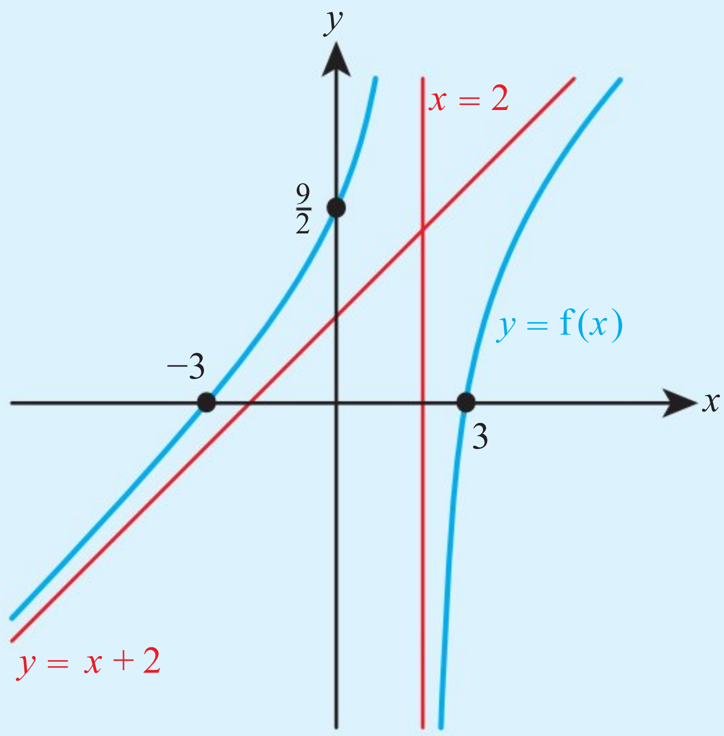


On separate axes, sketch the graph of

a $y = |f(x)|$

b $y = f(|x|)$.

26 The graph of $y = f(x)$ is shown below.



On separate axes, sketch the graph of

a $y = |f(x)|$

b $y = f(|x|)$.

27 Sketch the graph of $y = 2 - 3|x+1|$, labelling the axis intercepts.

28 Given that $a < 0 < b$, sketch the graph of $y = |(x-a)(x-b)|$.

29 Given that $a < 0 < b < c$, sketch the graph of $y = |(x-a)(x-b)(x-c)|$.

30 Solve the inequality $|xe^x| \leqslant 2 - x^2$.

31 Find the set of values of x for which $\left|\frac{3x-2}{x+4}\right| < 11 - 2x$.

32 Solve the inequality $|4x \arccos x| > 1$.

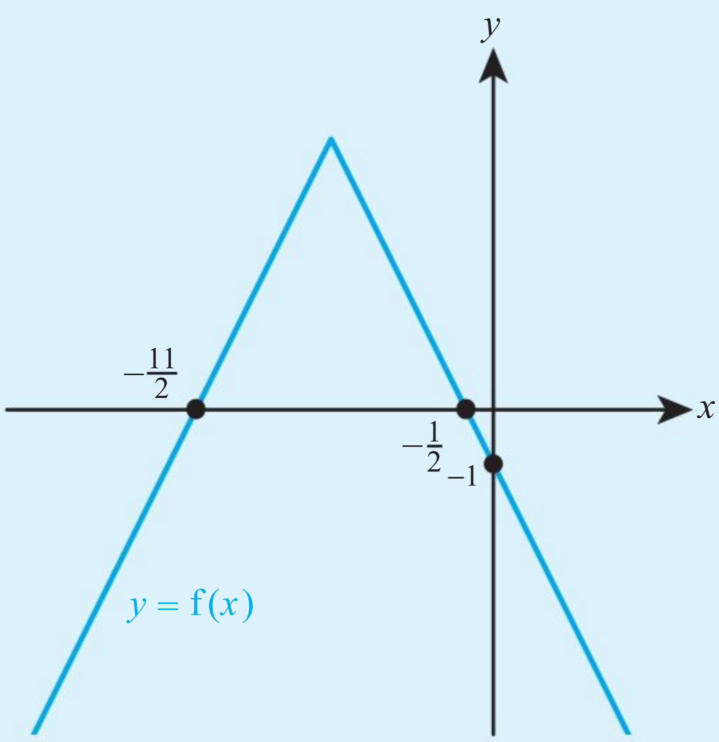
33 **a** On the same axes, sketch the graphs of $y = |2x-5|$ and $y = 3|x| + 1$.

b Hence solve the inequality $|2x-5| < 3|x| + 1$.

34 **a** Sketch the graph of $y = 3e^{|x|}$.

b Hence find the set of values of x for which $3e^{|x|} > 5$.

35 The graph of $y = a|x+b| + c$ is shown below.



Find the values of a, b and c.

36 Solve the inequality $|2x+3| > 3x+7$.

37 Find the values of x for which $|x^2-3x-5| = 3-x$.

38 Solve the inequality $|x^2-5x+4| > 2$.

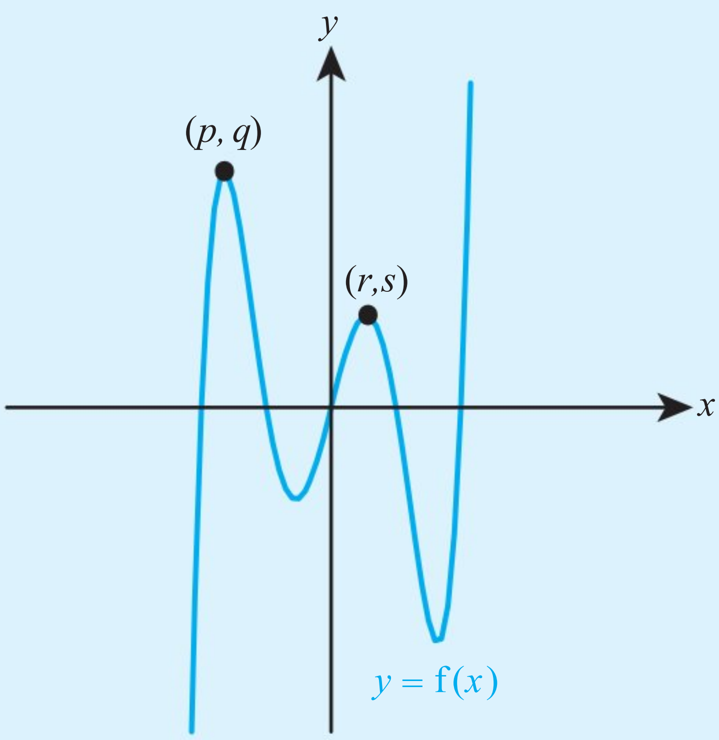
39 By sketching appropriate graphs or otherwise, solve the equation $|x+1|+|x-1| = x+4$.

40 By sketching appropriate graphs or otherwise, solve the equation $|3x-1| = x+|2-x|$.

41 **a** Sketch the graph of $y = x|x|$.

b Solve the equation $x|x| = kx$ where $k > 0$.

42 The graph of $y = \mathrm{f}(x)$ is shown below.



Sketch the graph of $y = \mathrm{f}(x) + |\mathrm{f}(x)|$.

43 Solve the equation $|x+a^2| = |x-2a^2|$, giving your answer in terms of a.

44 Find the condition on the constant k such that the equation $|x^2+4x-7| = k$ has four solutions.

45 Find the condition on the constant k such that the equation $|x^3-12x+4| = k$ has four solutions.

7D The graphs of the functions $y = \frac{1}{f(x)}$, $y = f(ax + b)$ and $y = [f(x)]^2$

The graph of $y = \frac{1}{f(x)}$

Given the graph of $y = f(x)$ we can draw the graph of $y = \frac{1}{f(x)}$ by considering a few key features.

KEY POINT 7.6

To sketch the graph of $y = \frac{1}{f(x)}$ consider the following key features:

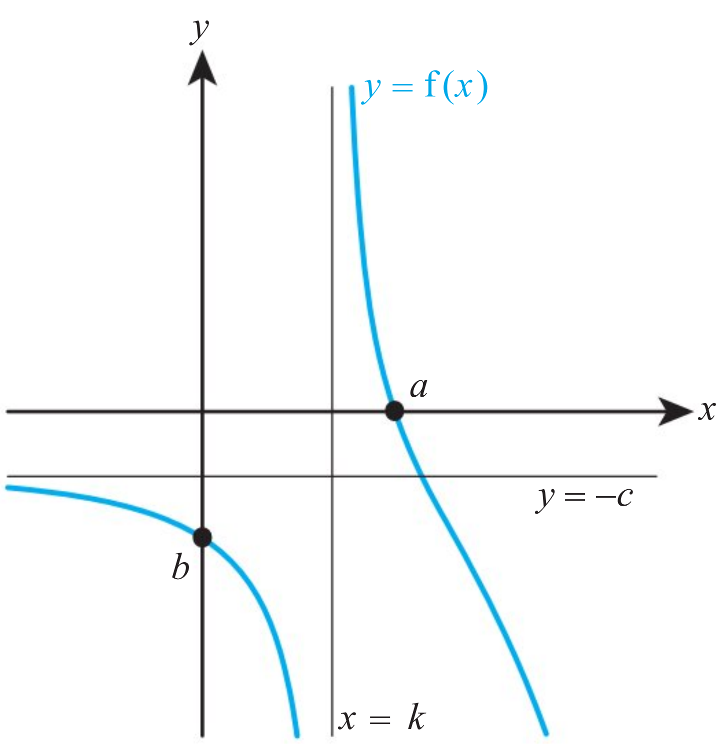
Feature of $y = f(x)$	**Feature of $y = \frac{1}{f(x)}$**
x-intercept at $(a, 0)$	$x = a$ is a vertical asymptote
y-intercept at $(0, b)$, $b \neq 0$	y-intercept at $\left(0, \frac{1}{b}\right)$
$x = a$ is a vertical asymptote	x-intercept at $(a, 0)$
$y = a$ is a horizontal asymptote, $a \neq 0$	$y = \frac{1}{a}$ is a horizontal asymptote
$y = 0$ is a horizontal asymptote	$y \to \pm\infty$
$y \to \pm\infty$	$y = 0$ is a horizontal asymptote
(a, b) is a turning point, $b \neq 0$)	$\left(a, \frac{1}{b}\right)$ is the opposite turning point

Tip

If you are unsure about which side of an asymptote the graph lies on, check a few points.

WORKED EXAMPLE 7.10

The diagram shows the graph of $y = f(x)$.



Sketch the graph of $y = \frac{1}{f(x)}$.

$y = \text{f}(x)$ has an x-intercept at $(a, 0)$ so $y = \frac{1}{\text{f}(x)}$ has a vertical asymptote at $x = a$

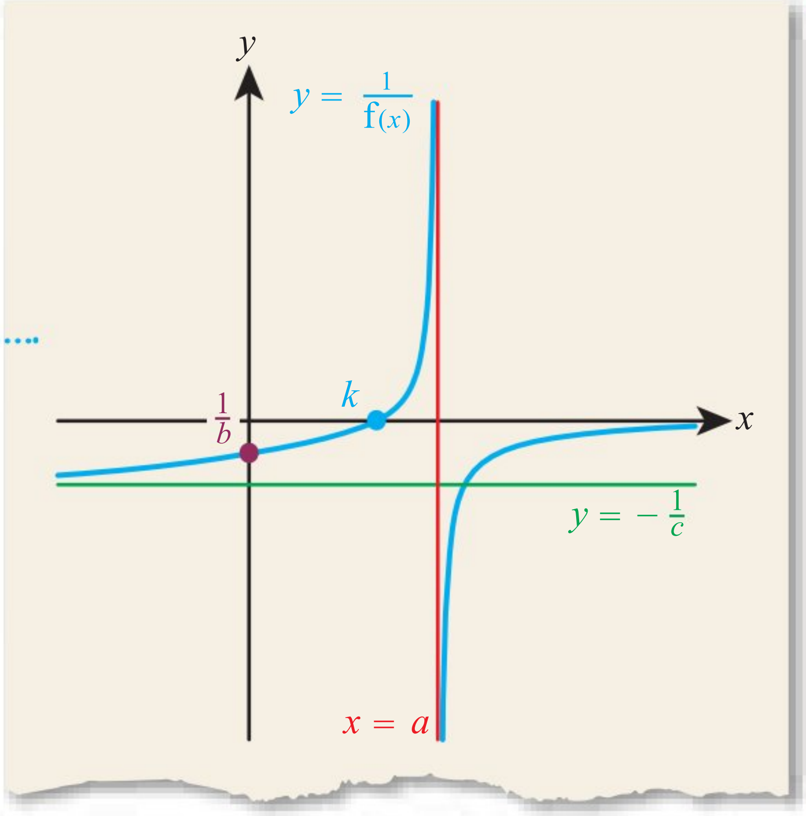
$y = \text{f}(x)$ has a y-intercept at $(0, b)$ so $y = \frac{1}{\text{f}(x)}$ has a y-intercept at $\left(0, \frac{1}{b}\right)$

$y = \text{f}(x)$ has a vertical asymptote at $x = k$ so $y = \frac{1}{\text{f}(x)}$ has an x-intercept at $(k, 0)$

$y = \text{f}(x)$ has a horizontal asymptote at $y = -c$ so $y = \frac{1}{\text{f}(x)}$ has a horizontal asymptote at $y = -\frac{1}{c}$

As $x \to \infty$, $\text{f}(x) \to -\infty$ so $y = \frac{1}{\text{f}(x)}$ has a horizontal asymptote at $y = 0$

$y = \text{f}(x)$ has no turning points so $y = \frac{1}{\text{f}(x)}$ has no turning points



The graph of $y = \text{f}(ax + b)$

In Section 16A of Mathematics: analysis and approaches SL you saw how to apply two vertical transformations, or one vertical and one horizontal transformation.

We now need to be able to apply two horizontal transformations.

Tip

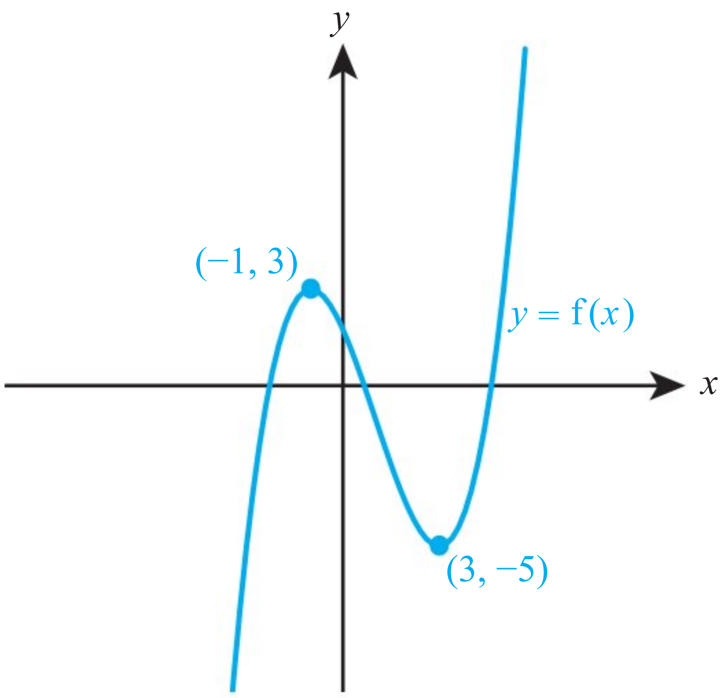
Notice that the transformations are in the 'wrong' order: the addition is done before the multiplication.

KEY POINT 7.7

When two horizontal transformations are applied, the order matters: $y = \text{f}(ax + b)$ is a horizontal translation by $-b$ followed by a horizontal stretch with scale factor $\frac{1}{a}$.

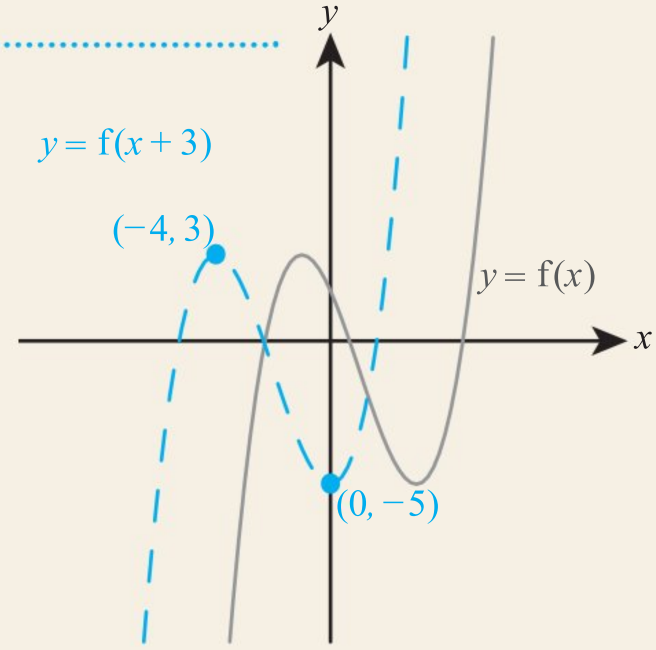
WORKED EXAMPLE 7.11

Below is the graph of $y = \text{f}(x)$. Sketch the graph of $y = \text{f}(2x + 3)$.

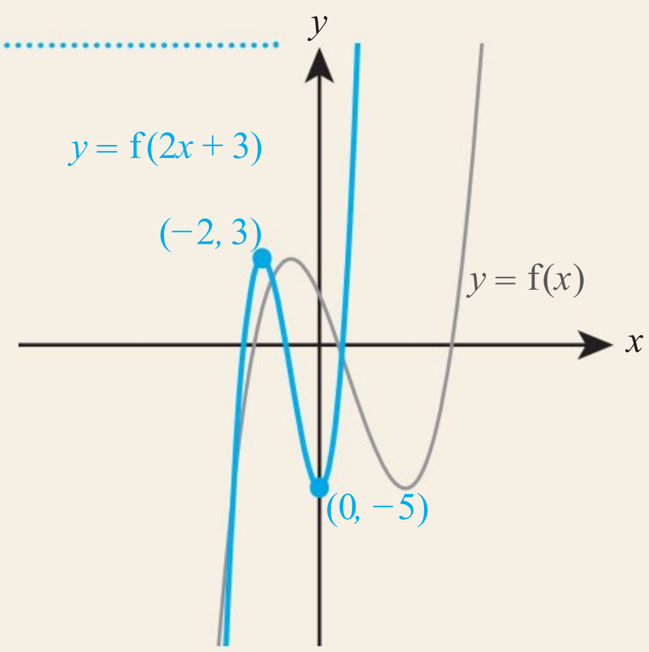


There are two horizontal transformations so the order is important.

First translate horizontally by −3...



... and then stretch horizontally with scale factor $\frac{1}{2}$



WORKED EXAMPLE 7.12

Describe a sequence of two horizontal transformations that maps the graph of $y = 9x^2$ to the graph of $y = x^2 - 6x + 9$.

Let $f(x) = 9x^2$

Express the second equation in function notation, related to the first

$$\text{Then, } y = x^2 - 6x + 9$$
$$= (x-3)^2$$
$$= 9\left[\frac{1}{9}(x-3)^2\right]$$

The factor of $\frac{1}{9}$ goes inside the squared bracket as a factor of $\frac{1}{3}$

$$= 9\left(\frac{1}{3}x - 1\right)^2$$
$$= f\left(\frac{1}{3}x - 1\right)$$

State the transformation, making sure the translation comes before the stretch

Horizontal translation by 1 followed by horizontal stretch with scale factor 3.

Be the Examiner 7.2

The graph of $y = f(x)$ is stretched horizontally with scale factor 2 and translated horizontally by 3. Find the equation of the transformed graph.

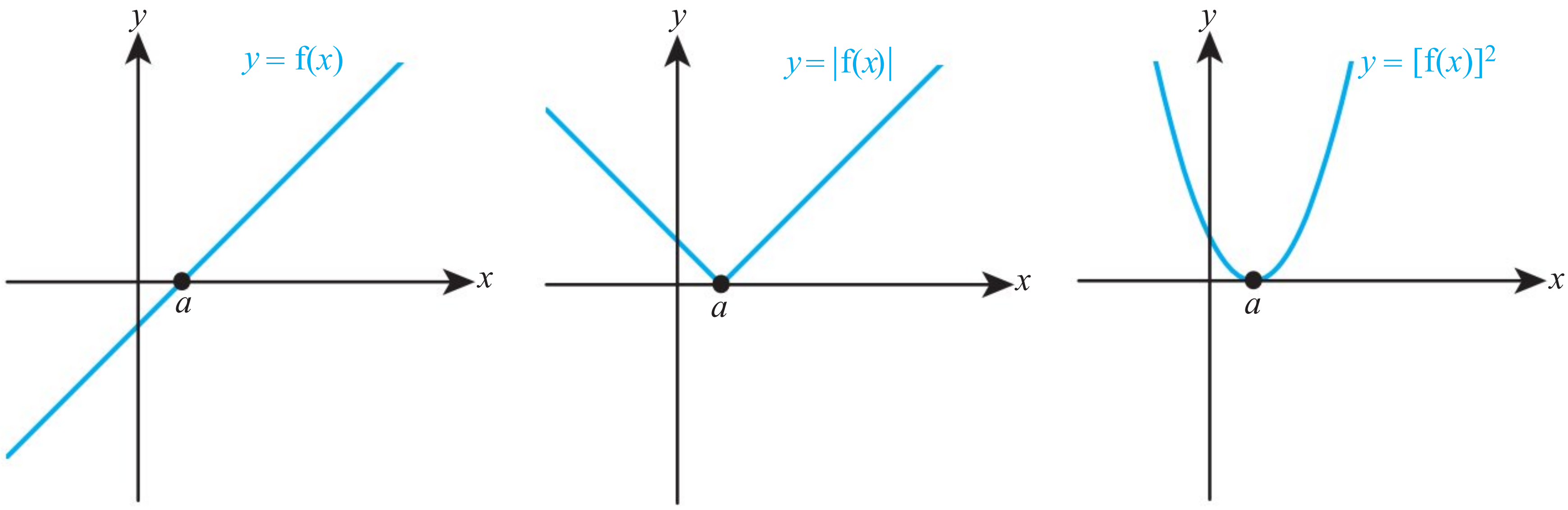
Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$y = f\left(\frac{1}{2}x - 3\right)$	$y = f\left(\frac{x-3}{2}\right)$	$y = f(2x - 6)$

The graph of $y = [f(x)]^2$

Sketching the graph of $y = [f(x)]^2$ is rather like sketching the graph of $y = |f(x)|$ in the sense that any parts of the graph of $y = f(x)$ that are below the x-axis will now be above the x-axis.

The difference is that all the y-values will change magnitude since they are being squared, except for any points where $f(x) = \pm 1$. This has the effect of smoothing the function where it touches the x-axis so that such points become turning points.



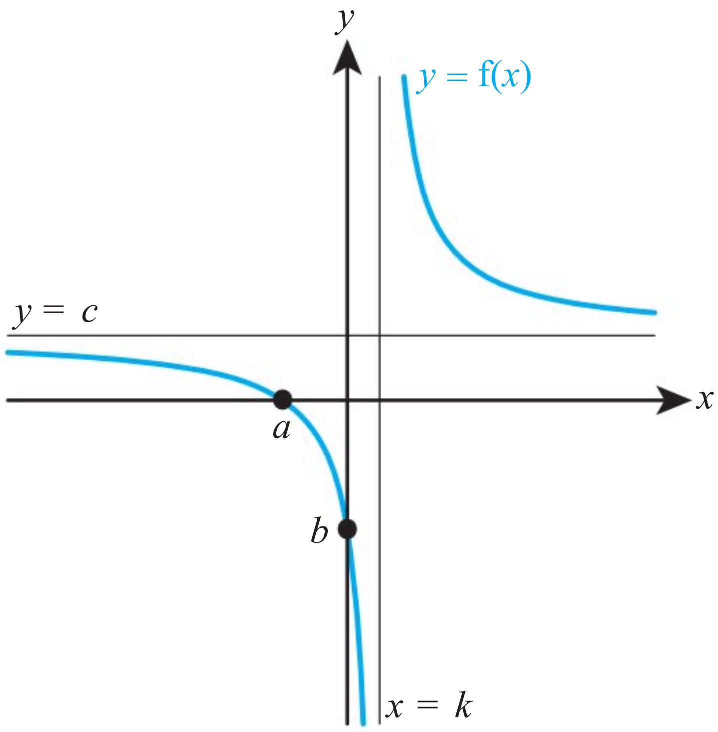
KEY POINT 7.8

To sketch the graph of $y = [f(x)]^2$ consider the following key features:

Feature of $y = f(x)$	Feature of $y = [f(x)]^2$
$y < 0$	$y > 0$
x-intercept at $(a, 0)$	Local minimum at $(a, 0)$
y-intercept at $(0, b)$	y-intercept at $(0, b^2)$
$x = a$ is a vertical asymptote	$x = a$ is a vertical asymptote
$y = a$ is a horizontal asymptote	$y = a^2$ is a horizontal asymptote
$y \to \pm\infty$	$y \to \infty$

WORKED EXAMPLE 7.13

Below is the graph of $y = f(x)$. Sketch the graph of $y = [f(x)]^2$.



All negative values on $y = f(x)$ now become positive on $y = [f(x)]^2$

$y = f(x)$ has an x-intercept at $(a, 0)$ so $y = [f(x)]^2$ has a local minimum at $(a, 0)$

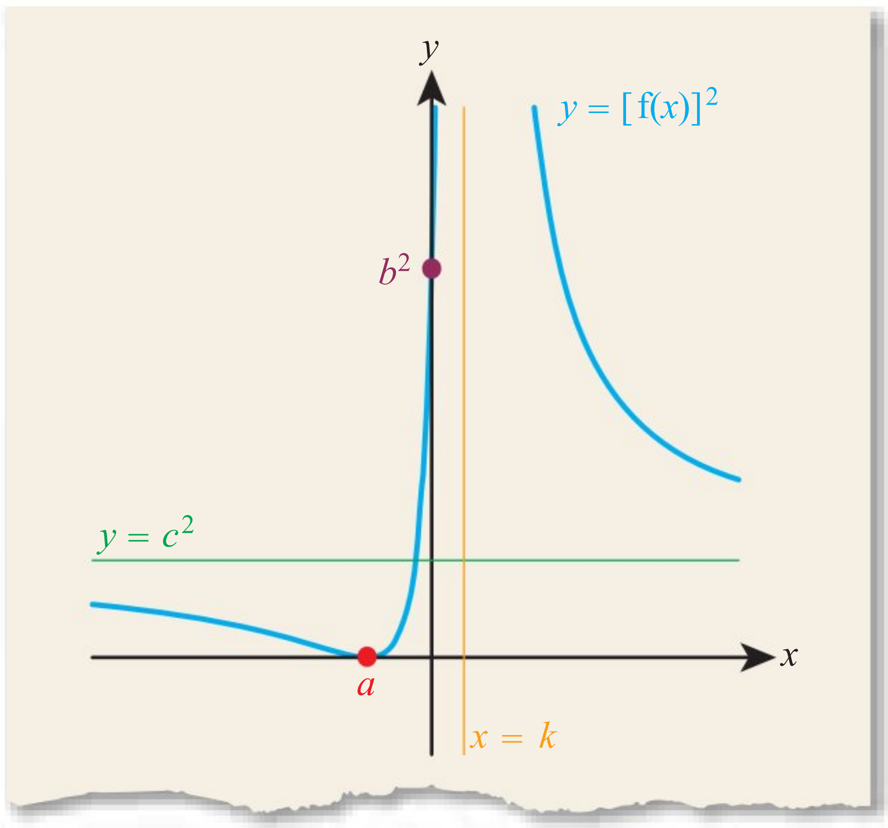
$y = f(x)$ has a y-intercept at $(0, b)$ so $y = [f(x)]^2$ has a y-intercept at $(0, b^2)$

The vertical asymptote is unaffected at $x = k$

$y = f(x)$ has a horizontal asymptote at $y = c$ so $y = [f(x)]^2$ has a horizontal asymptote at $y = c^2$

Where $f(x) > 1$, the graph of $y = [f(x)]^2$ will be above the graph of $y = f(x)$

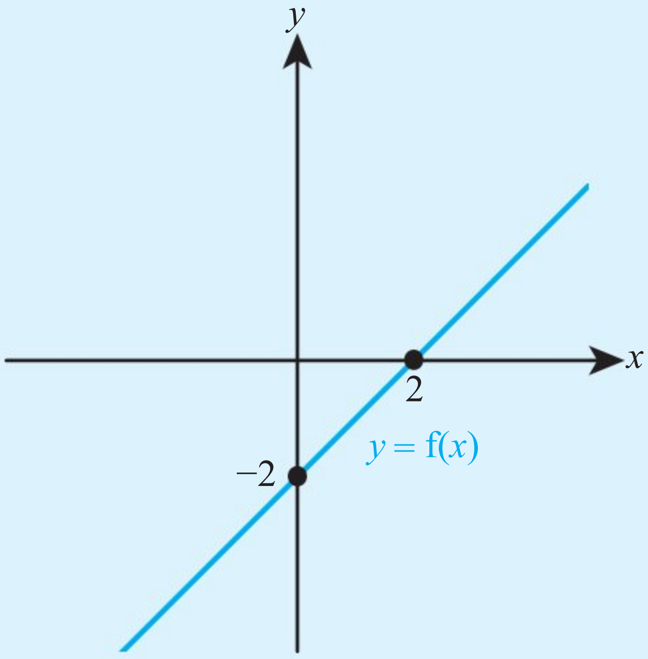
Where $0 < f(x) < 1$, the graph of $y = [f(x)]^2$ will be below the graph of $y = f(x)$



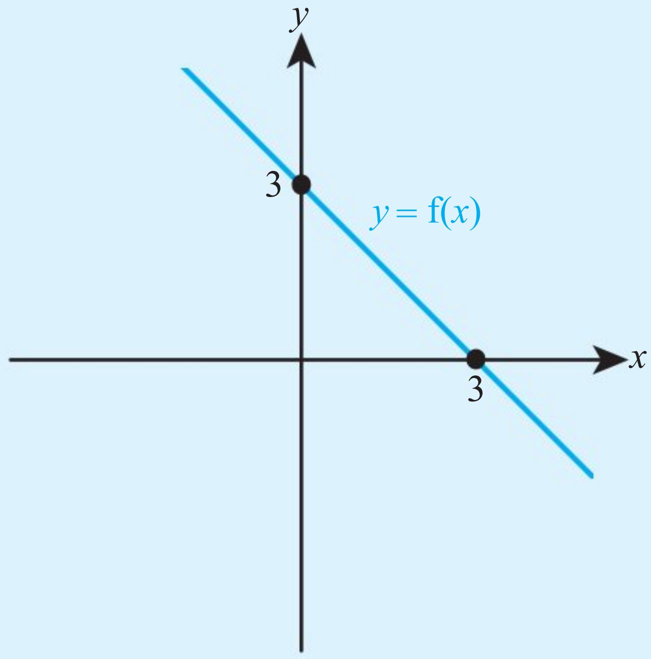
Exercise 7D

For questions 1 to 5, use the method demonstrated in Worked Example 7.10 to sketch the graph of $y = \frac{1}{f(x)}$. Label any axis intercepts, turning points and asymptotes.

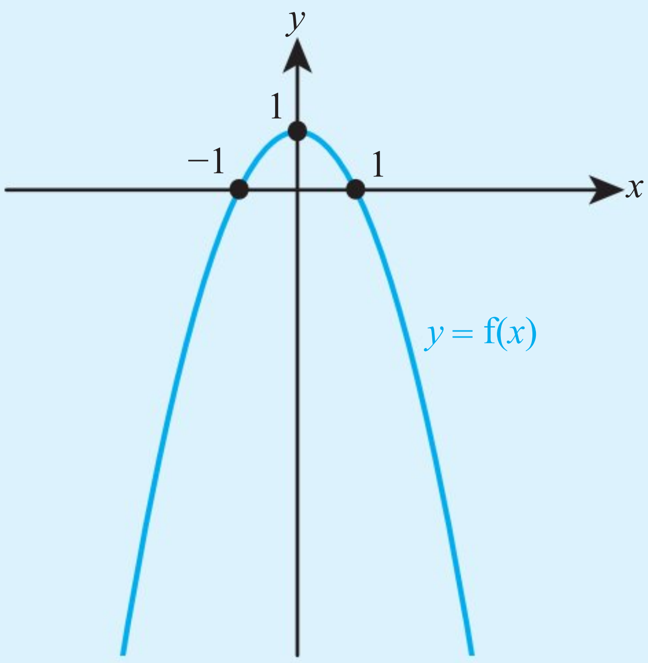
1 a



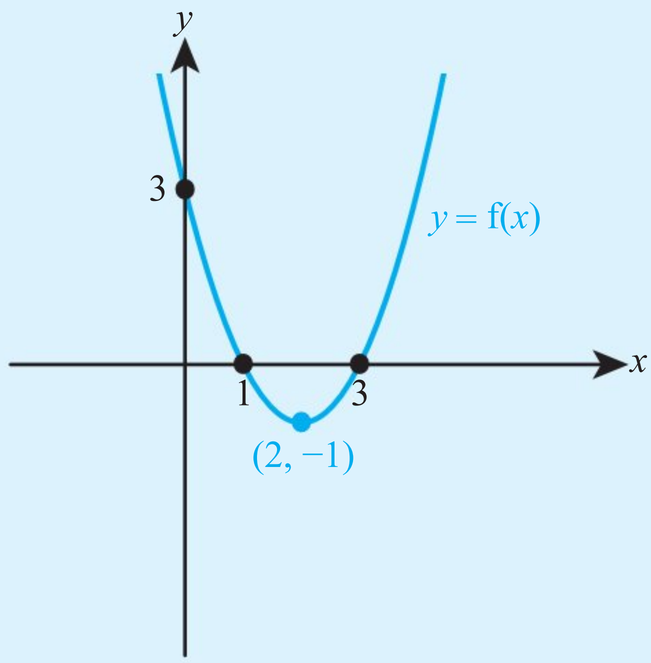
b



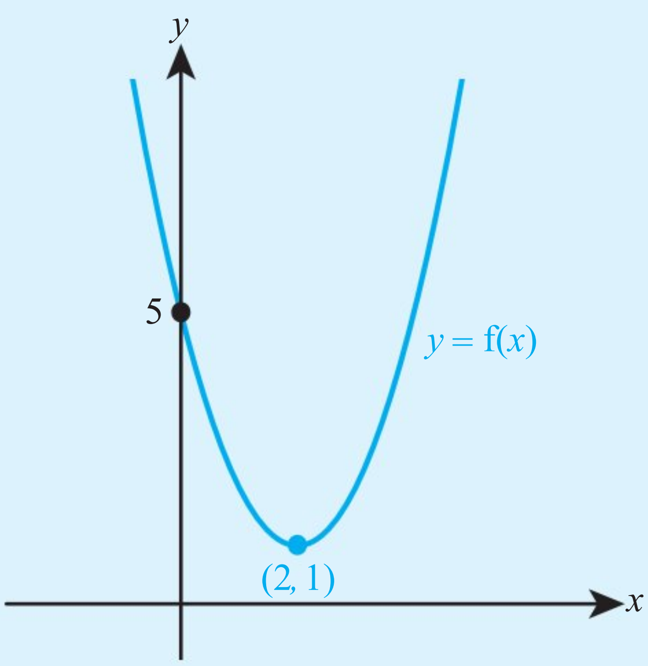
2 a



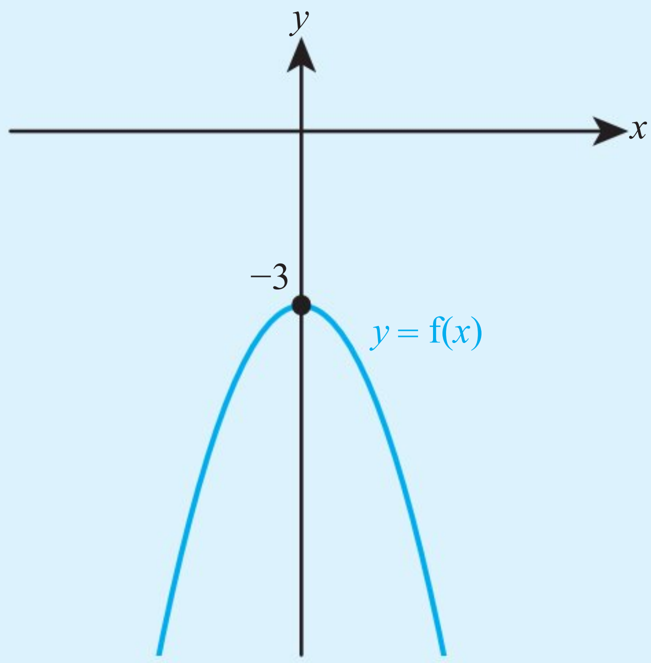
b



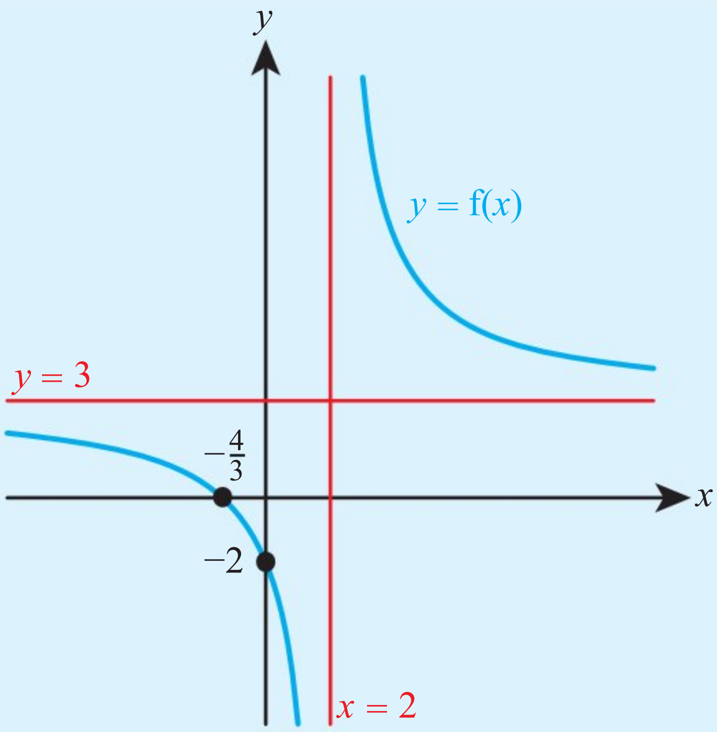
3 a



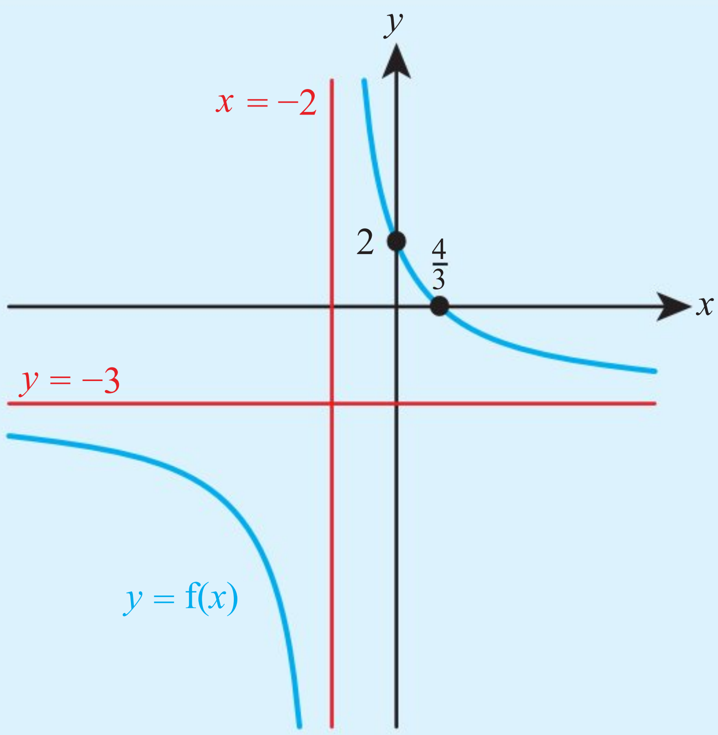
b



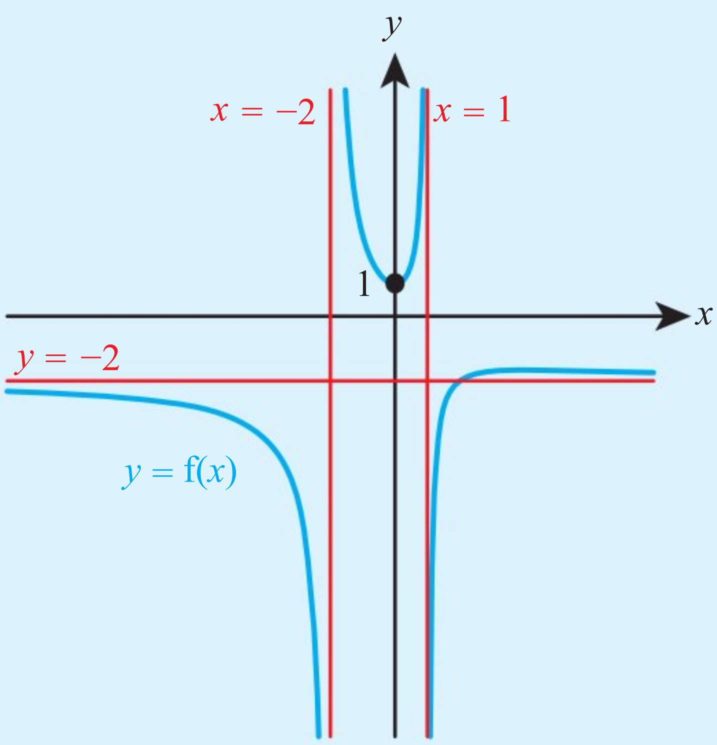
4 a



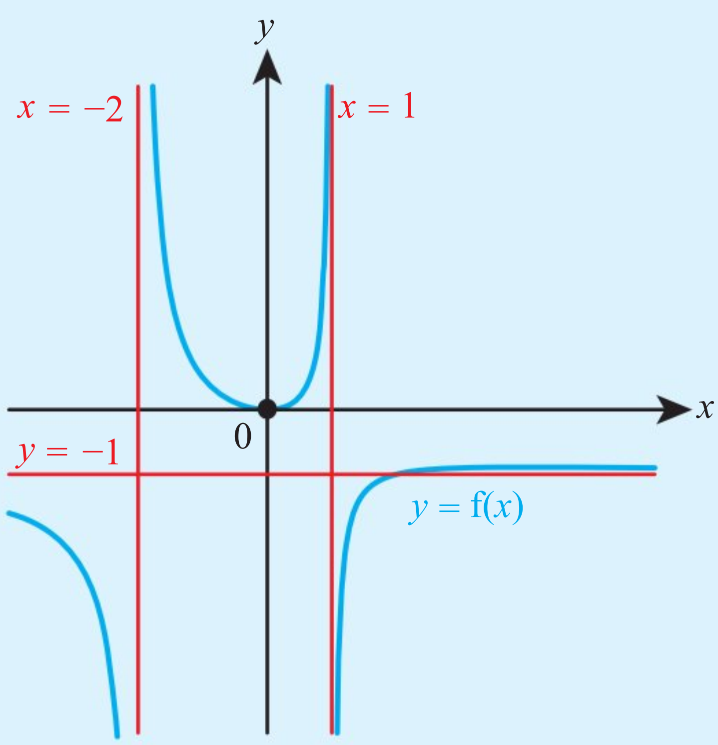
b



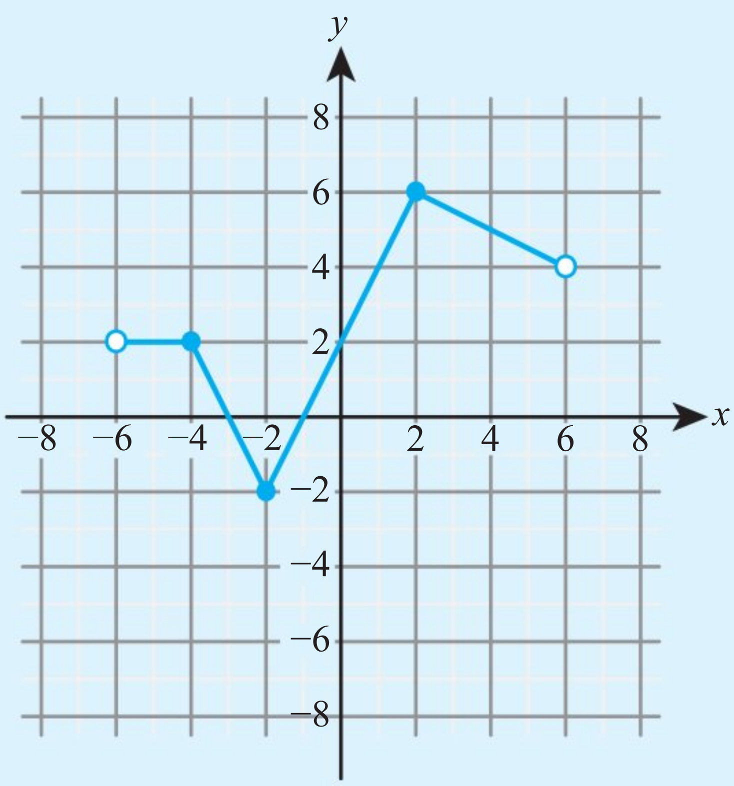
5 a



b



For questions 6 to 8, use the method demonstrated in Worked Example 7.11 to sketch the required graph.
The graph of $y = f(x)$ is given below.



6 **a** $y = f(3x - 2)$

b $y = f(4x + 1)$

7 **a** $y = f\left(\frac{1}{2}x + 3\right)$

b $y = f\left(\frac{1}{3}x - 1\right)$

8 **a** $y = f\left(\frac{x+1}{4}\right)$

b $y = f\left(\frac{x-4}{3}\right)$

For questions 9 to 11, use the method demonstrated in Worked Example 7.12 to describe a sequence of two horizontal transformations that maps the graph of $y = x^2$ to the given graph.

9 **a** $\frac{1}{16}x^2 - x + 4$

b $\frac{1}{9}x^2 + \frac{2}{3}x + 1$

10 **a** $9x^2 - 6x + 1$

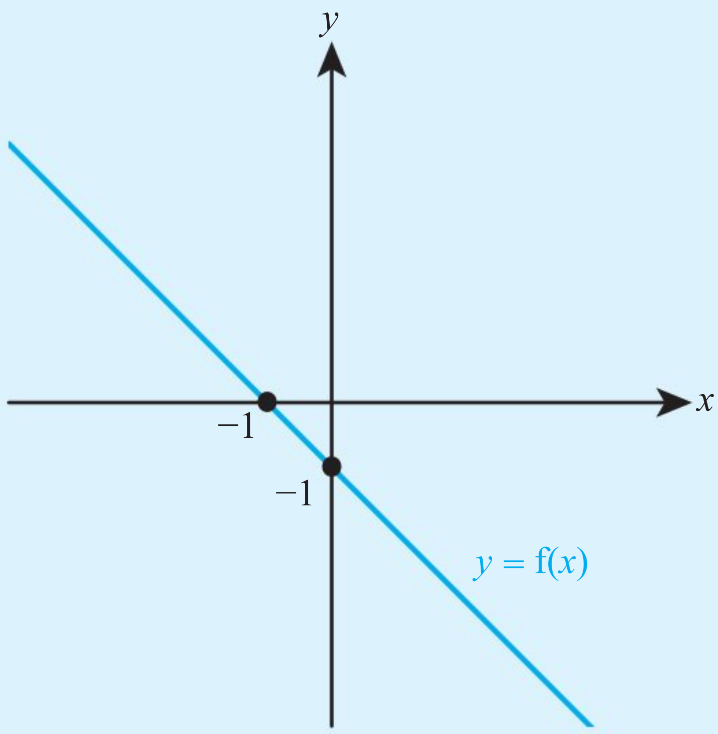
b $4x^2 + 12x + 9$

11 **a** $x^2 + 6x + 9$

b $x^2 - 4x + 4$

For questions 12 to 16, use the method demonstrated in Worked Example 7.10 to sketch the graph of $y = [f(x)]^2$. Label any axis intercepts, turning points and asymptotes.

12 **a**



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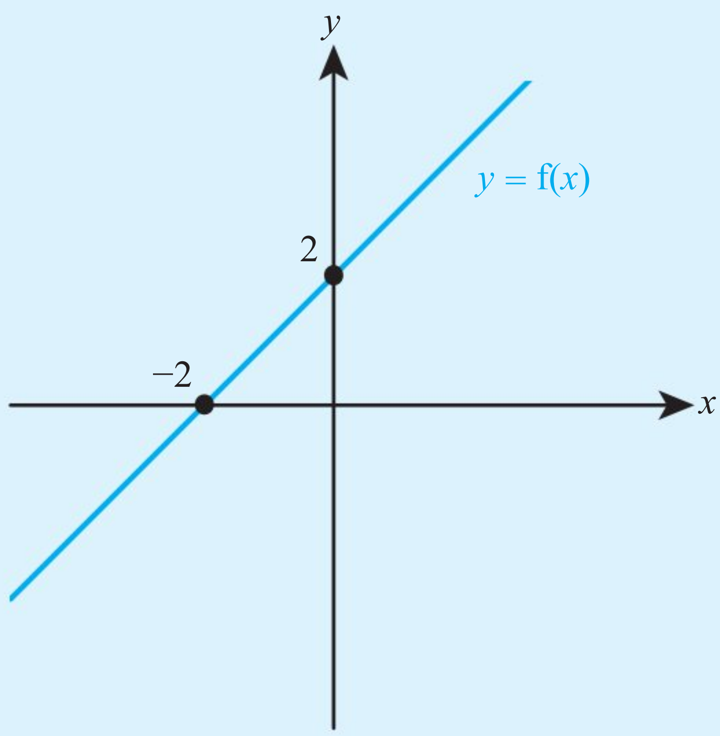
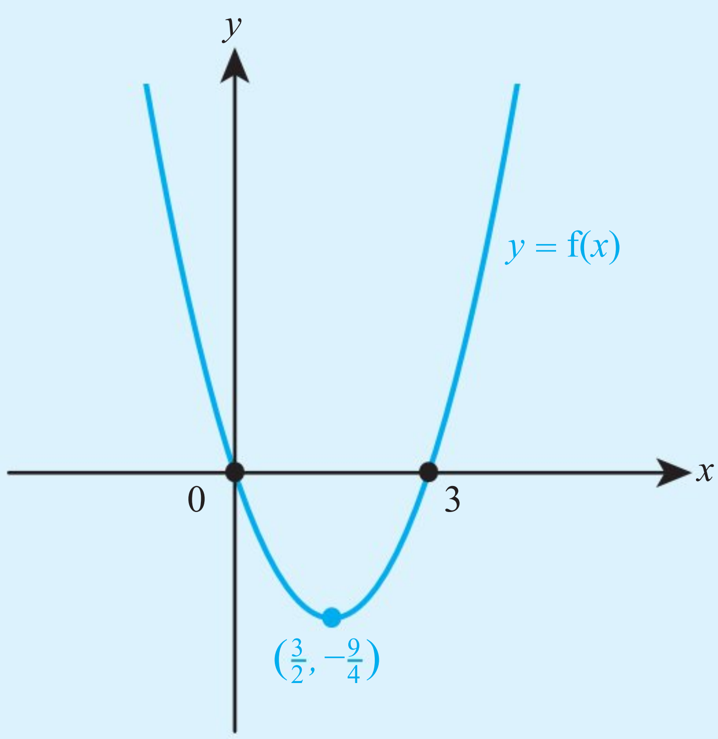


13 **a**



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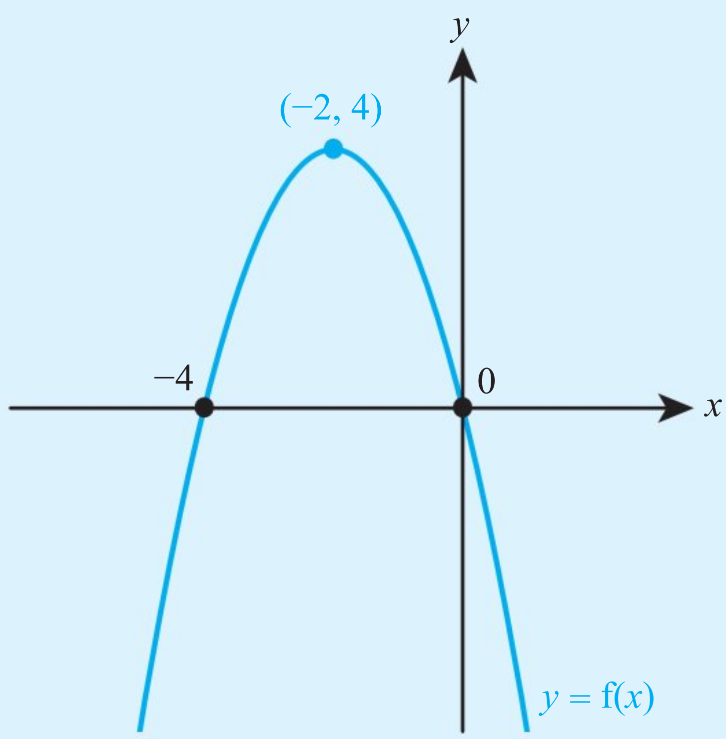
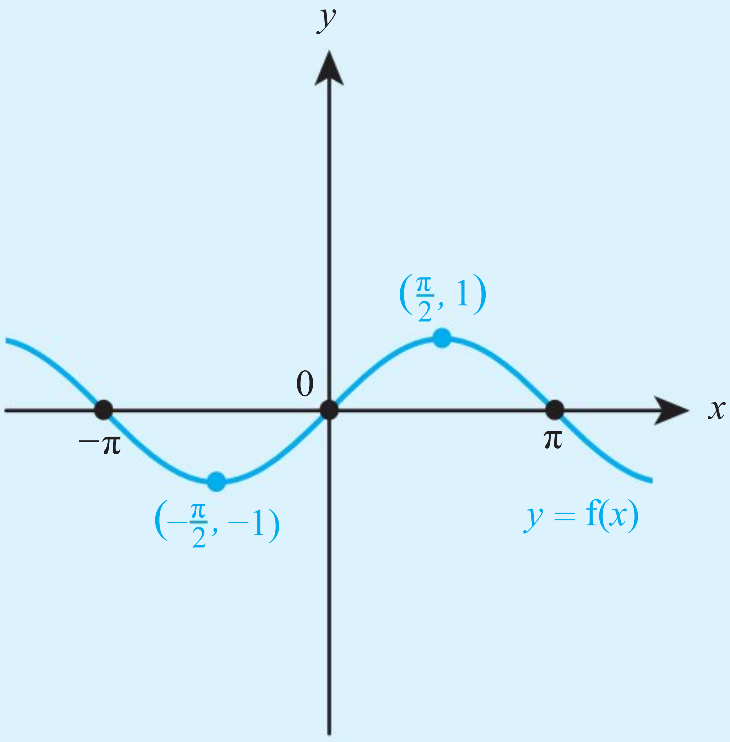


14 a



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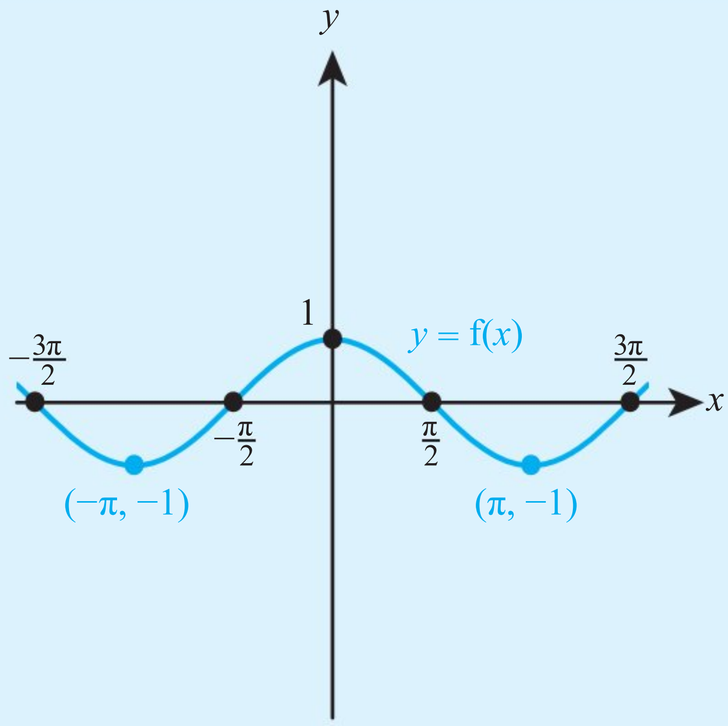
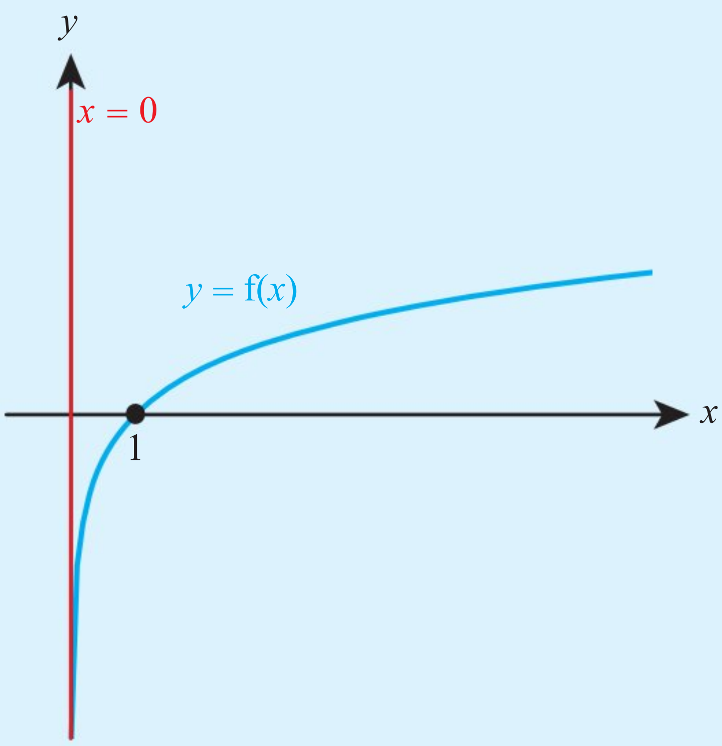
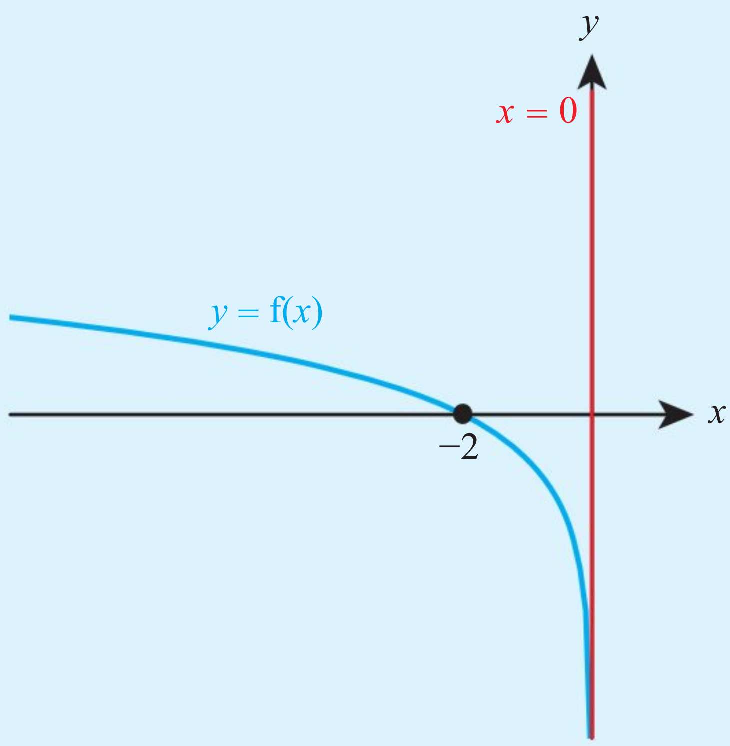


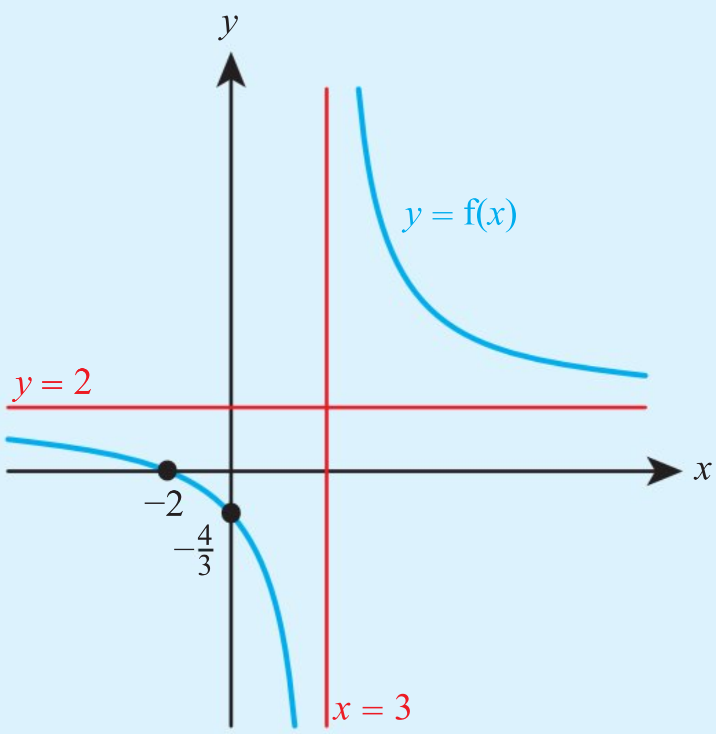
15 a



b



16 a



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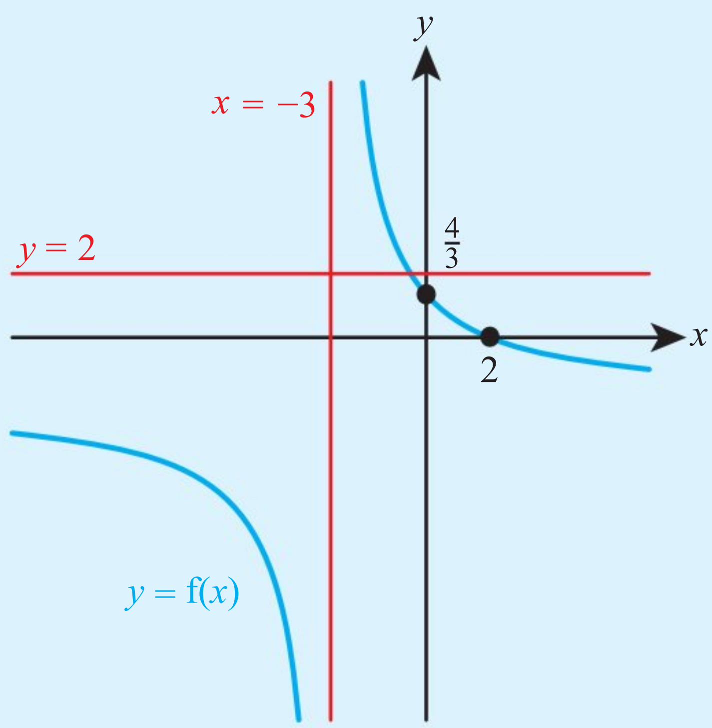
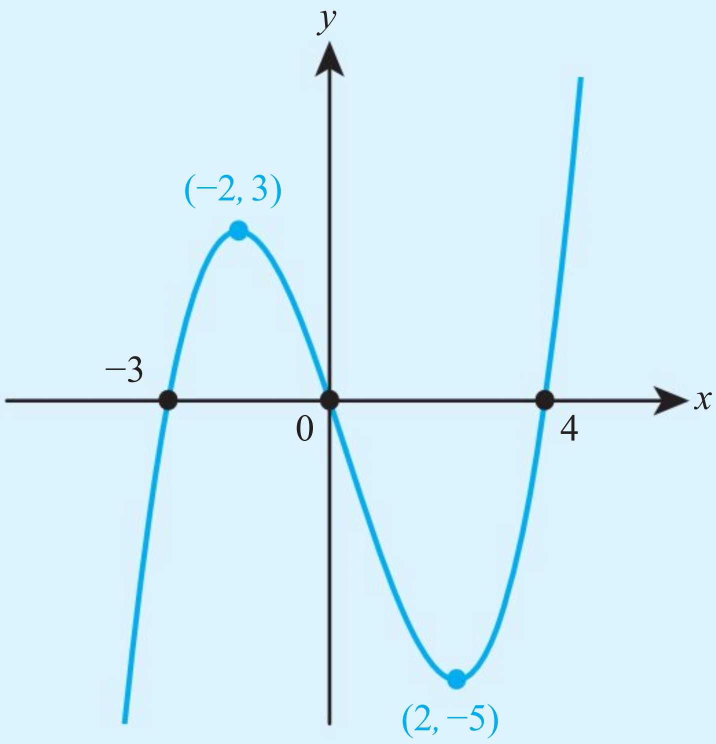


17 The graph of $y = f(x)$ is shown.

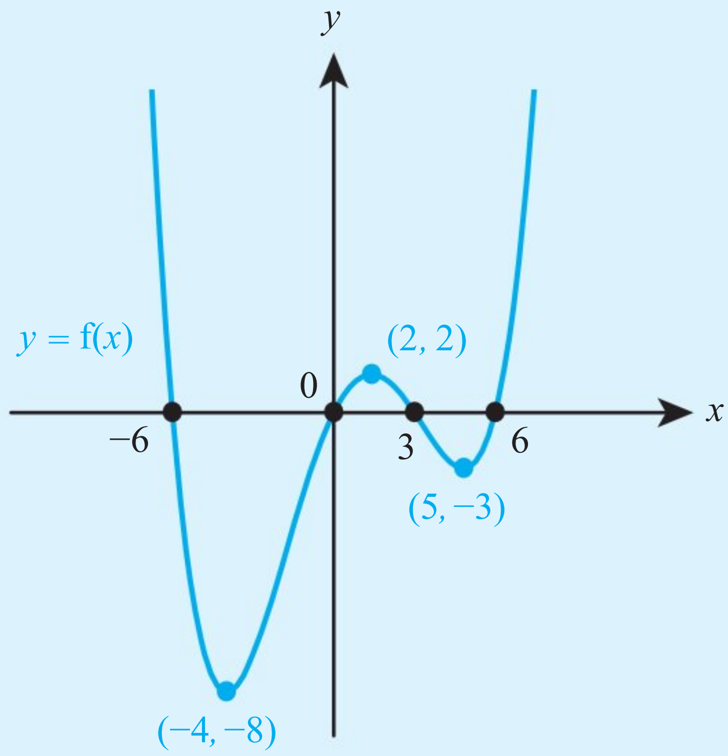


Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \dfrac{1}{f(x)}$

b $y = [f(x)]^2$

18 The graph of $y = f(x)$ is shown.

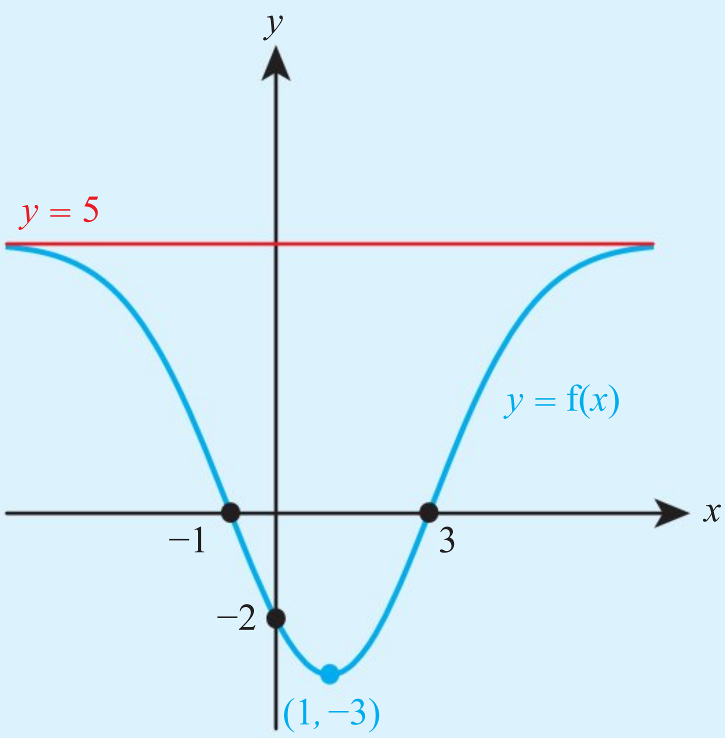


Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \dfrac{1}{f(x)}$

b $y = [f(x)]^2$

19 The graph of $y = f(x)$ is shown.



Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \dfrac{1}{f(x)}$

b $y = [f(x)]^2$

20 The point P with coordinates $(3, -4)$ lies on the graph $y = f(x)$. Find the coordinates of the transformed point P' on the graph of:

a $y = \dfrac{1}{f(x)}$

b $y = [f(x)]^2$

c $y = f\left(\dfrac{1}{2}x + 2\right)$

21 State a sequence of two transformations that maps the graph of $y = \cos x$ onto the graph of $y = \cos\left(3x - \frac{\pi}{4}\right)$.

22 State a sequence of two transformations that maps the graph of $y = \ln x$ onto the graph of $y = \ln\left(\frac{2}{5}x + 3\right)$.

23 The graph of $y = 3x^2 + 4x$ is stretched horizontally with scale factor $\frac{1}{2}$ and then translated in the positive x-direction by 1 unit.

Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

24 The following sequence of transformations is applied to the graph of $y = f(x)$:

- Horizontal translation by 4 units in the positive x direction
- Horizontal stretch by scale factor $\frac{1}{2}$
- Vertical translation by 3 units in the positive y direction
- Vertical stretch by scale factor 2

Find the equation of the resulting graph.

25 The following sequence of transformations is applied to the graph of $y = f(x)$:

- Vertical stretch by scale factor $\frac{1}{3}$
- Vertical translation by 4 units in the negative y direction
- Horizontal stretch by scale factor 2
- Horizontal translation by 1 unit in the negative x direction

Find the equation of the resulting graph.

26 Transformation A is reflection in the y-axis.

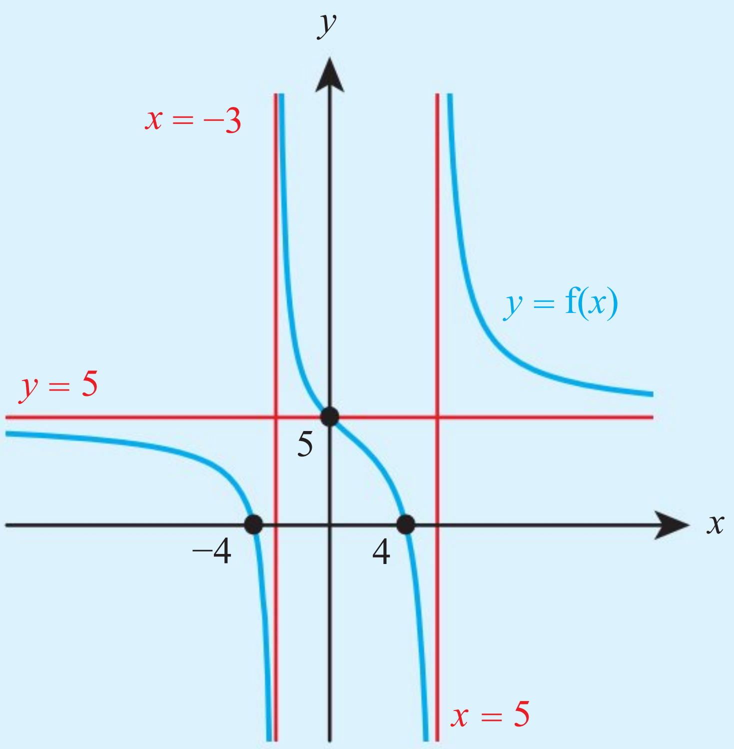
Transformation B is translation right by 5 units.

Find the equation of the resulting graph if $y = f(x)$ is transformed by:

a A and then B

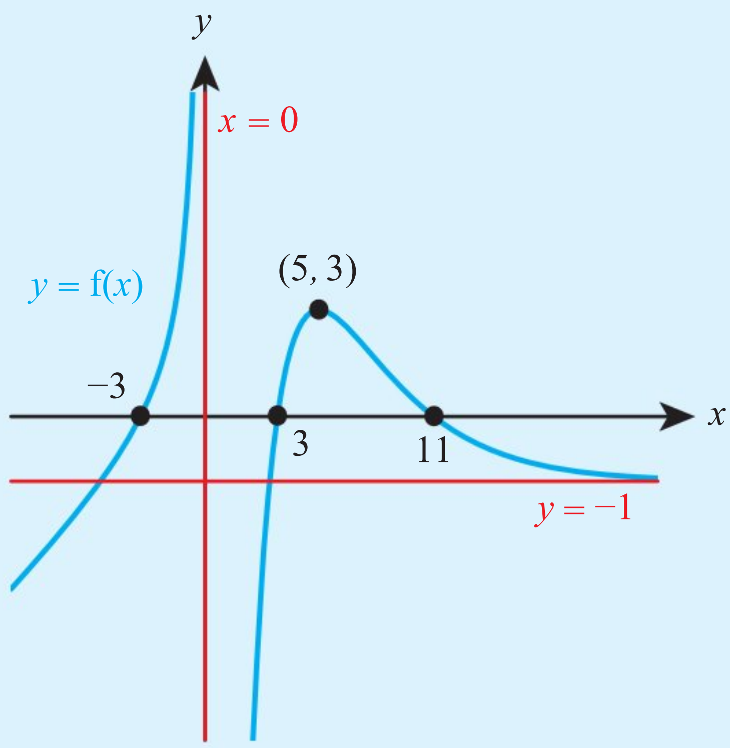
b B and then A

27 The graph of $y = \frac{1}{f(x)}$ is shown.



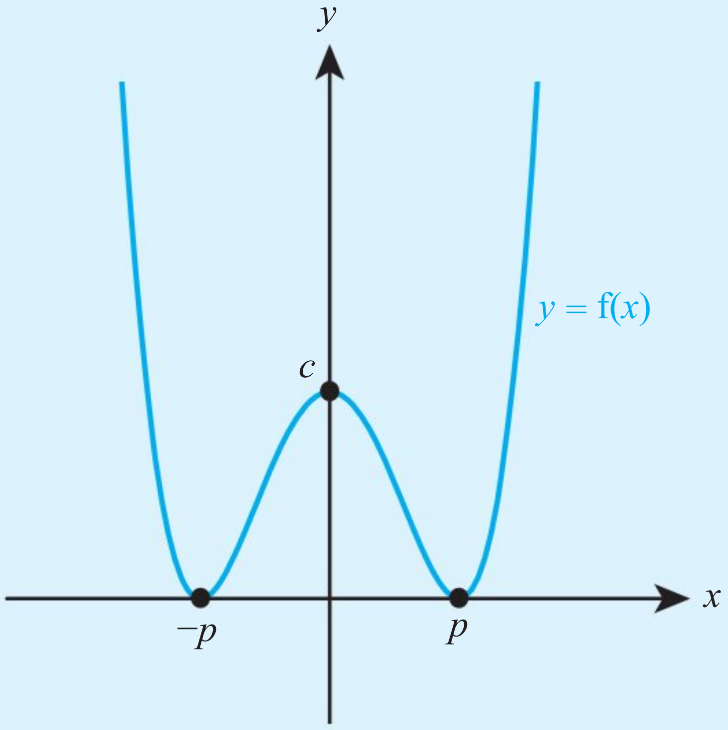
Sketch the graph of $y = f(x)$, labelling all axis intercepts and asymptotes.

28 The graph of $y = f(x)$ is shown.



Sketch the graph of $y = [f(x)]^2$, labelling all axis intercepts, turning points and asymptotes.

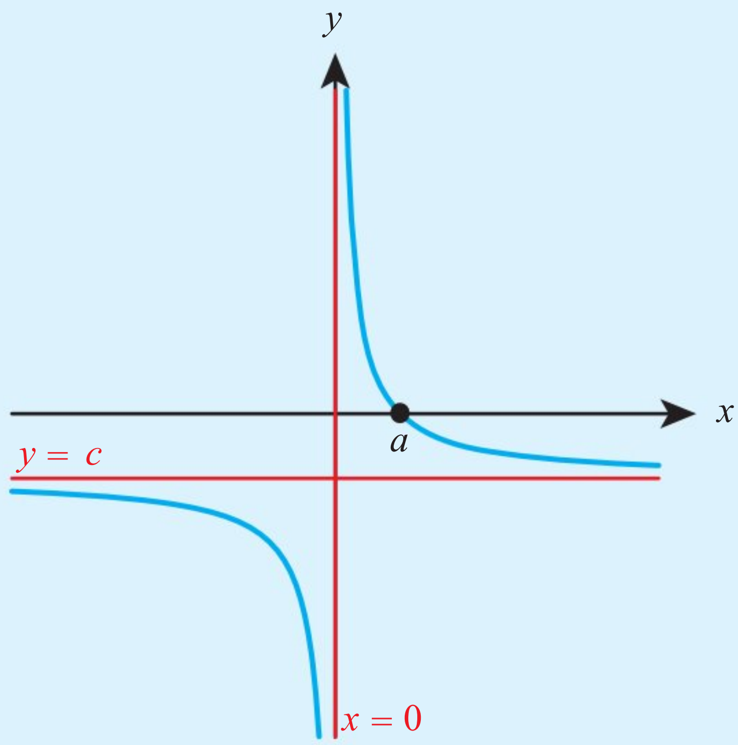
29 The graph of $y = f(x)$ is shown.



Sketch the graph of $y = \frac{1}{f(ax - b)}$, where $a, b > 0$.

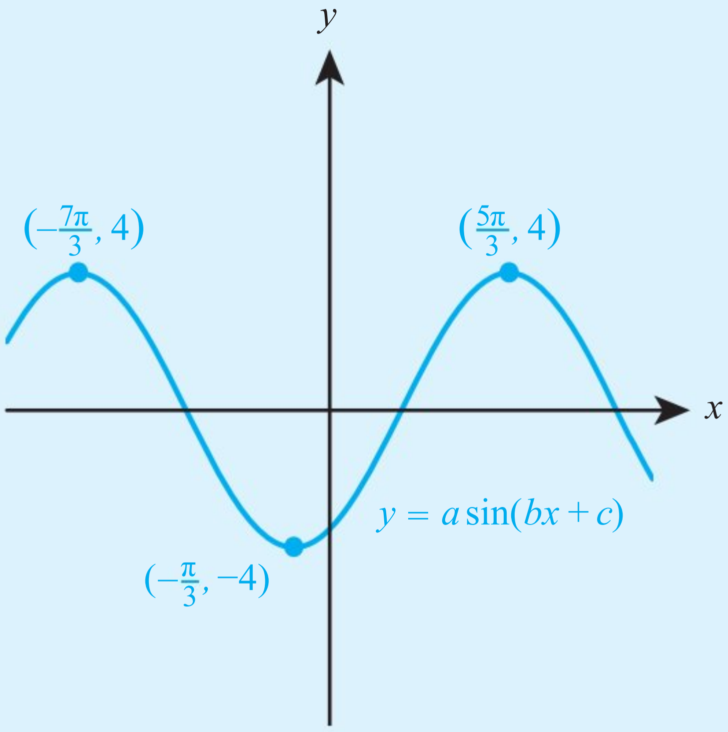
Label any axis intercepts, turning points and asymptotes.

30 The graph of $y = f(x)$ is shown.



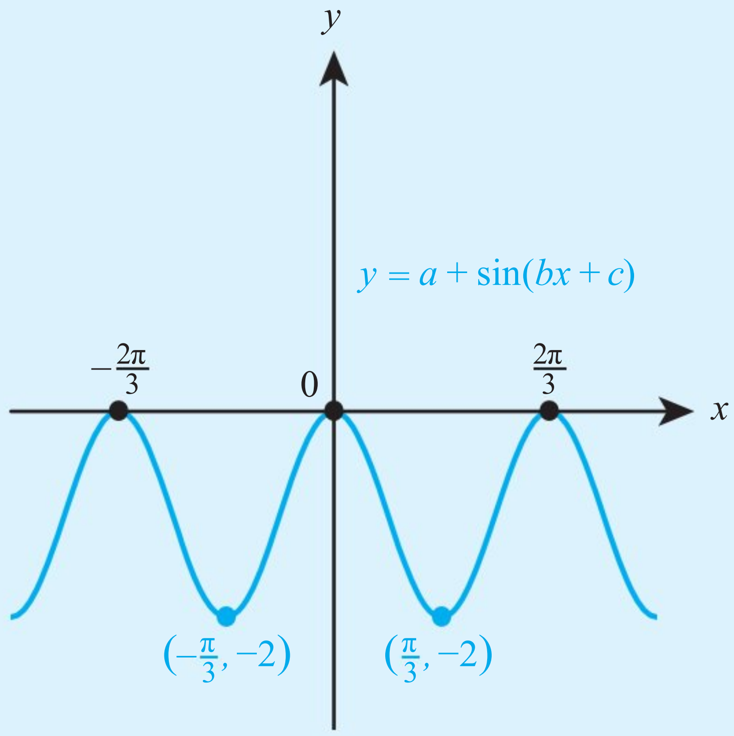
Sketch the graph of $y = f(ax + b)^2$, where $a, b > 0$.
Label any axis intercepts, turning points and asymptotes.

31 The graph of $y = a\sin(bx + c)$ is shown.



Find the values of the constants a, b and c.

32 The graph of $y = a + \sin(bx + c)$ is shown.



Find the values of the constants a, b and c.

33 The graph of $y = f(x)$ is translated right by 2 and then stretched horizontally with a scale factor $\frac{1}{5}$ to give the graph $y = g(x)$.

Find a different sequence of two transformations that maps $y = f(x)$ onto $y = g(x)$.

34 The graph of $y = ax^2 + bx + c$ is transformed by the following sequence:

- Translation by 2 units in the positive x direction
- Horizontal stretch with scale factor 3

The resulting graph is $y = x^2 + cx + 14$. Find the values of a, b and c.

35 The graph of $y = ax^3 + bx + c$ is transformed by the following sequence:

- Horizontal stretch with scale factor $\frac{1}{2}$
- Translation by 1 unit in the negative x direction

The resulting graph is $y = 2x^3 + 6x^2 - bx - 2$. Find the values of a, b and c.

36 State a sequence of two transformations that map the graph of $y = f(2x + 1)$ onto the graph of $y = f(3x)$.

37 The graph of $y = \tan\left(3x - \frac{\pi}{2}\right)$ is translated by $\frac{\pi}{6}$ in the negative x direction and then reflected in the y-axis.
Find the equation of the transformed graph.

38 The graph of $y = 8^x$ is stretched vertically with scale factor 5.
The resulting graph is the same as that found when the graph of $y = 2^x$ is translated right by c units and then stretched horizontally with scale factor $\frac{1}{3}$. Find the value of c.

7E Properties of functions

Odd and even functions

Among the many features of trigonometric functions that you met in Mathematics: analysis and approaches SL Chapter 18 were the following:

- $\cos(-x) = \cos x$
- $\sin(-x) = -\sin x$
- $\tan(-x) = -\tan x$.

We say that sin and tan are **odd functions**, while cos is an **even function**.

Tip

Note that a function can be neither odd nor even if it does not satisfy either of the conditions in Key Point 7.9.

KEY POINT 7.9

A function is
- odd if $f(-x) = -f(x)$ for all x in the domain of f
- even if $f(-x) = f(x)$ for all x in the domain of f.

WORKED EXAMPLE 7.14

Determine algebraically whether $f(x) = x \sin x$ is an odd function, an even function or neither.

Find an expression for $f(-x)$: $f(-x) = (-x)\sin(-x)$

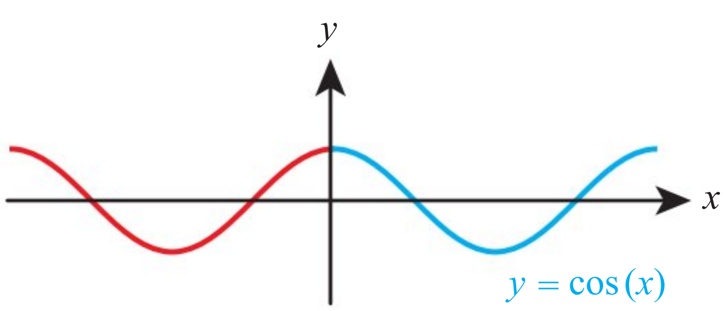
Use $\sin(-x) = -\sin x$: $= (-x)(-\sin x)$

$= x \sin x$

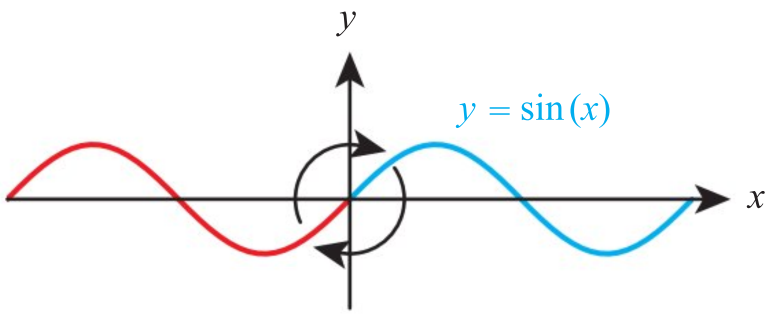
$f(-x) = f(x)$ so the function is even: $= f(x)$

So, f is an even function.

When you met the properties above for sin, cos and tan you related them to the graphs of these functions.



The cos graph is symmetric with respect to the y-axis, i.e. its graph remains unchanged after reflection in the y-axis.



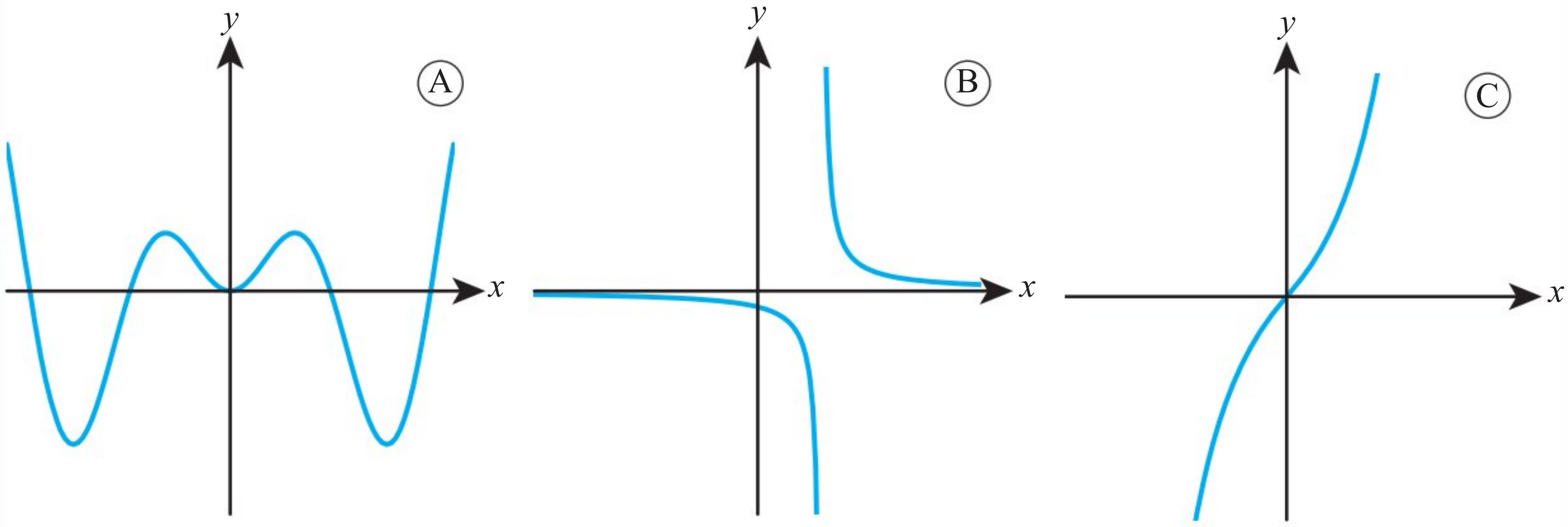
The sin graph is symmetric with respect to the origin, that is, its graph remains unchanged after rotation of 180° about the origin.

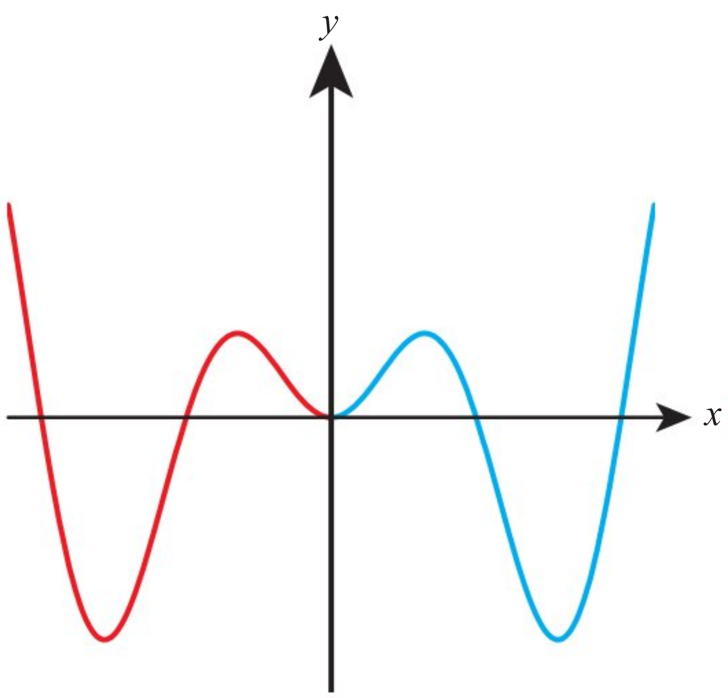
KEY POINT 7.10

The graph of
- an odd function is symmetric with respect to the origin
- an even function is symmetric with respect to the y-axis.

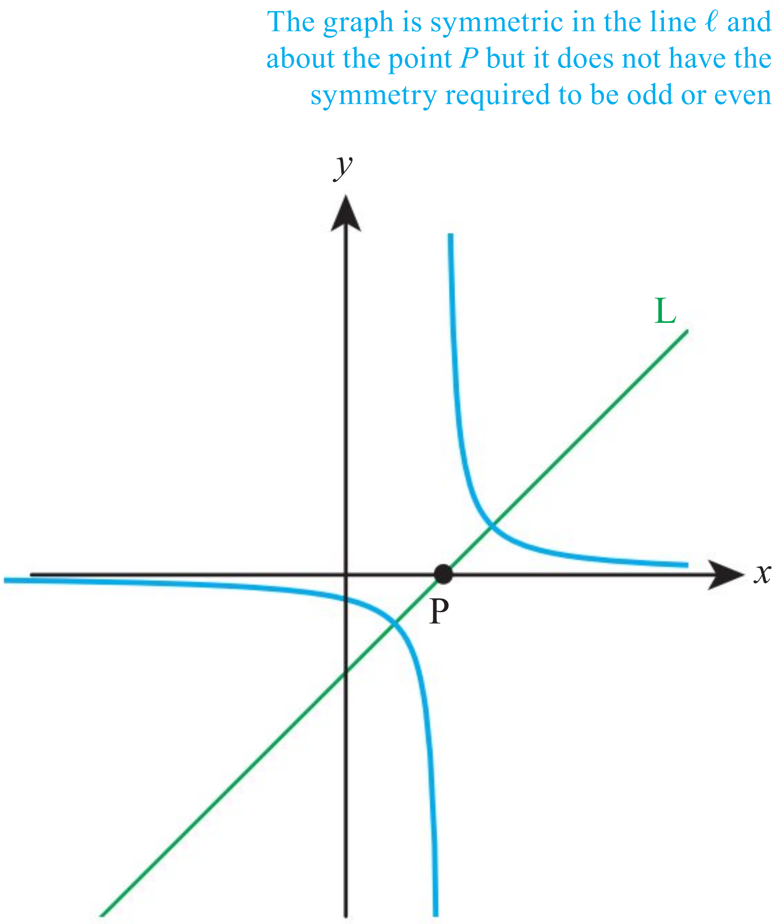
WORKED EXAMPLE 7.15

From their graphs, identify whether the functions are odd, even or neither.

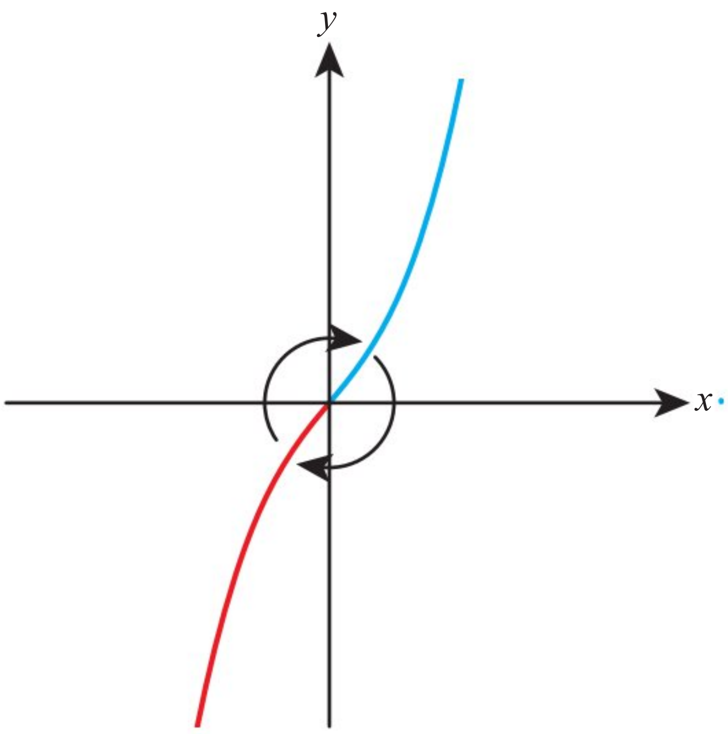




A is symmetric in the y-axis so is an even function.



B is not symmetric in the origin and not symmetric in the y-axis so it is neither odd nor even.



C is symmetric in the origin so is an odd function.

Tip

Since a function with maximum or minimum points cannot be one-to-one, when restricting the domain you want to start by looking for turning points.

Finding the inverse function $f^{-1}(x)$, including domain restriction

You know from Mathematics: analysis and approaches SL Chapter 14 that for an inverse function, f^{-1}, to exist, the original function, f, must be one-to-one.

We can make a function one-to-one by restricting its domain.

WORKED EXAMPLE 7.16

The function f is defined by $f(x) = x^2 + 6x + 4$, $x \geqslant k$.

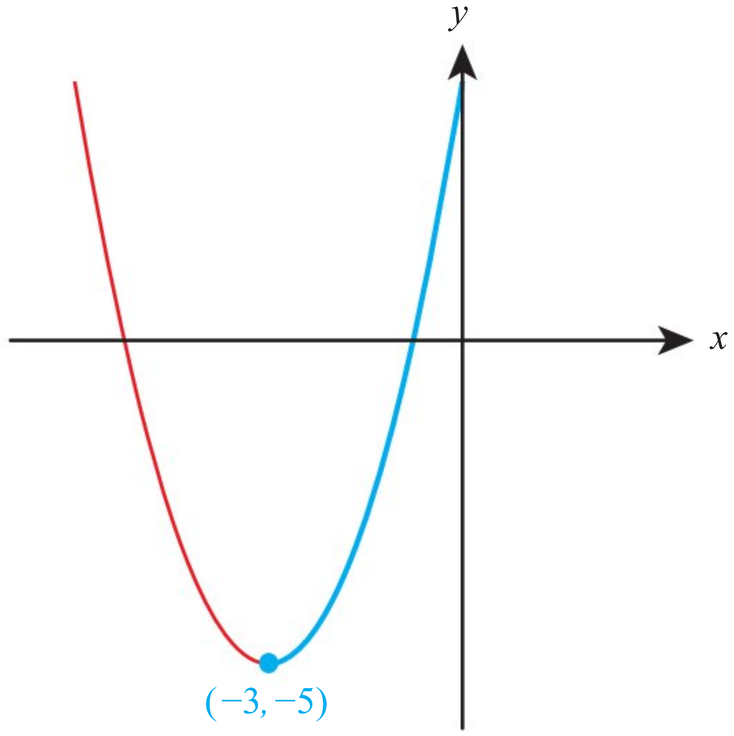
Find the smallest value of k such that the inverse function f^{-1} exists.

For f^{-1} to exist, f must be one-to-one, so find the turning point of f

$$f(x) = (x+3)^2 - 9 + 4$$
$$= (x+3)^2 - 5$$

The turning point of f is (–3, –5) so f is one-to-one for $x \geqslant -3$

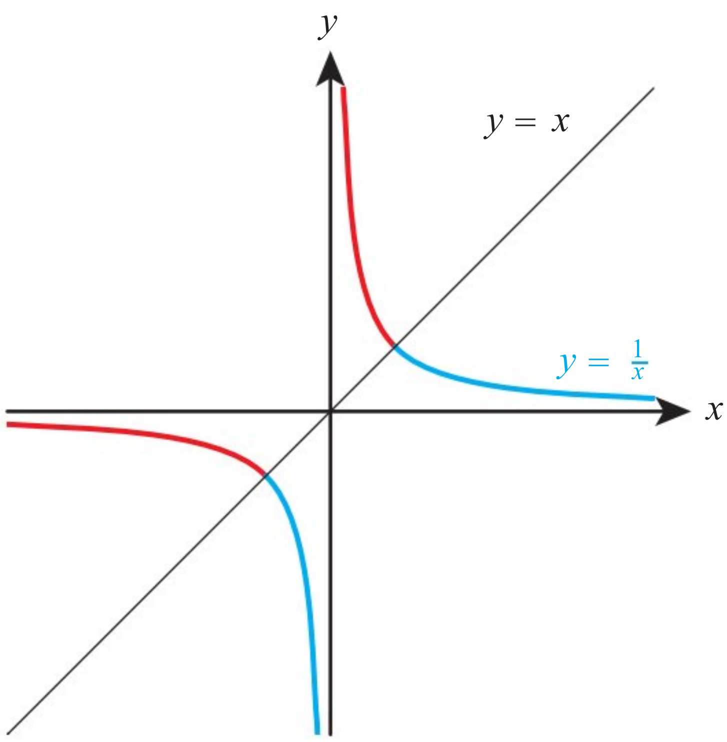
So, the smallest possible value of k is –3.



Self-inverse functions

We commented in Mathematics: analysis and approaches SL Chapter 16 that the function $f(x) = \frac{1}{x}$ is the same as its inverse, $f^{-1}(x) = \frac{1}{x}$. Functions such as this are said to **self-inverse**.

Since the inverse function is a reflection in the line $y = x$ of the original function, this means that self-inverse functions must be symmetric with respect to the line $y = x$.



KEY POINT 7.11

- A function f is said to be self-inverse if $f^{-1}(x) = f(x)$ for all x in the domain of f.
- The graph of a self-inverse function is symmetric in the line $y = x$.

WORKED EXAMPLE 7.17

Show that the function $f(x) = \frac{x}{x-1}$ is self-inverse.

Use the standard procedure for finding f^{-1}

Let $y = (x)$

$$y = \frac{x}{x-1}$$

$$xy - y = x$$

$$xy - x = y$$

$$x(y-1) = y$$

$$x = \frac{y}{y-1}$$

So,

$$f^{-1}(x) = \frac{x}{x-1}$$

Conclude by stating that f and its inverse are the same

$f(x) = f^{-1}(x)$ so f is self-inverse.

Exercise 7E

For questions 1 to 4, use the method demonstrated in Worked Example 7.14 to determine whether the given function is odd, even or neither.

1 **a** $f(x) = x^3 - 4x + 1$
b $f(x) = x^4 - 3x^2 + 2$

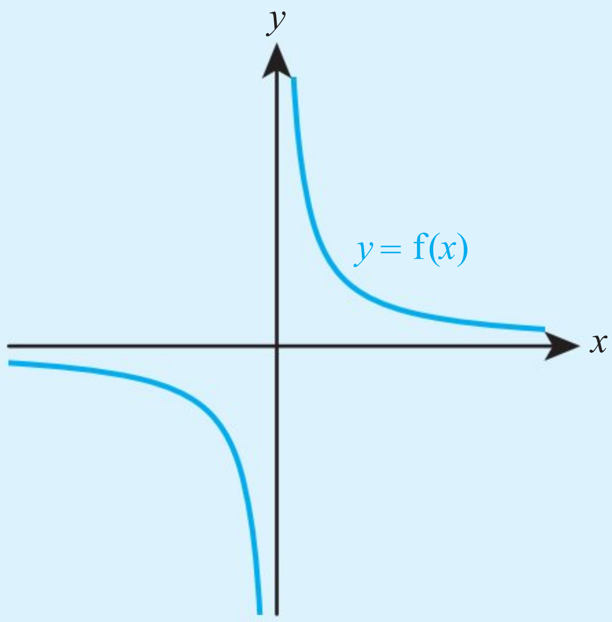
2 **a** $f(x) = 2x + \cos x$
b $f(x) = 2x + \tan x$

3 **a** $f(x) = e^{x^3}$
b $f(x) = e^{x^2}$

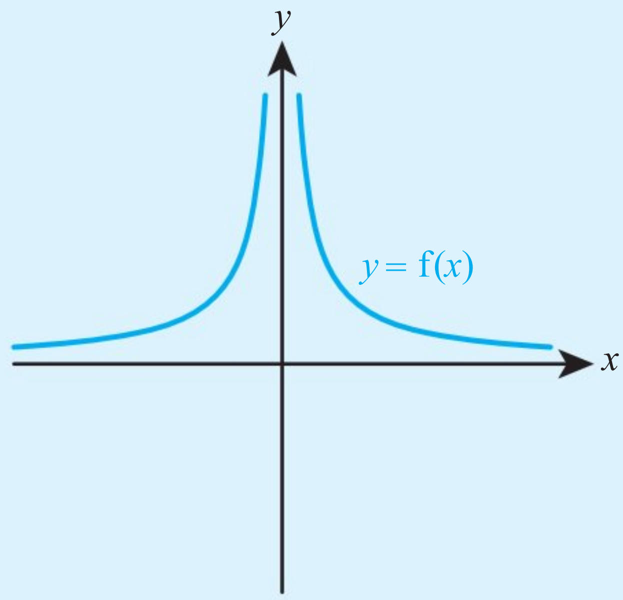
4 **a** $f(x) = |x| - 3$
b $f(x) = |x - 3|$

For questions 5 to 8, use the method demonstrated in Worked Example 7.15 to determine from its graph whether the function is odd, even or neither.

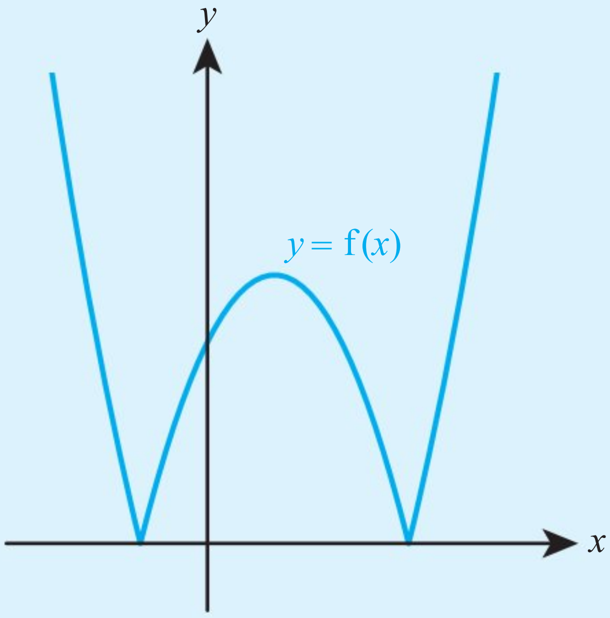
5 **a**



b



6 a



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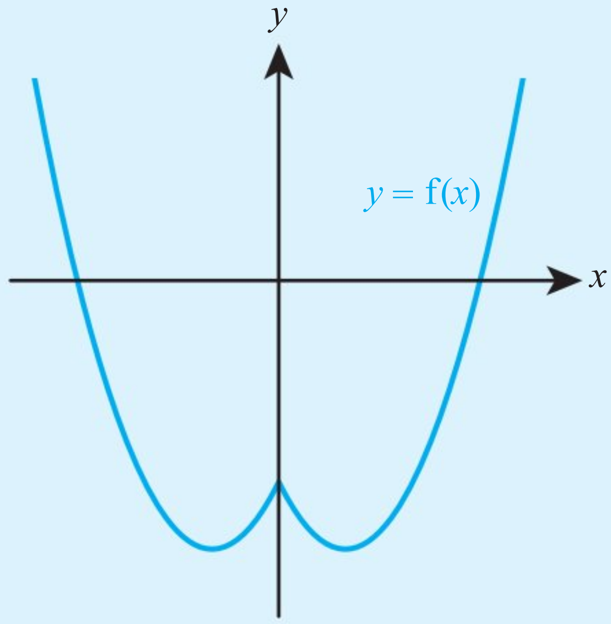
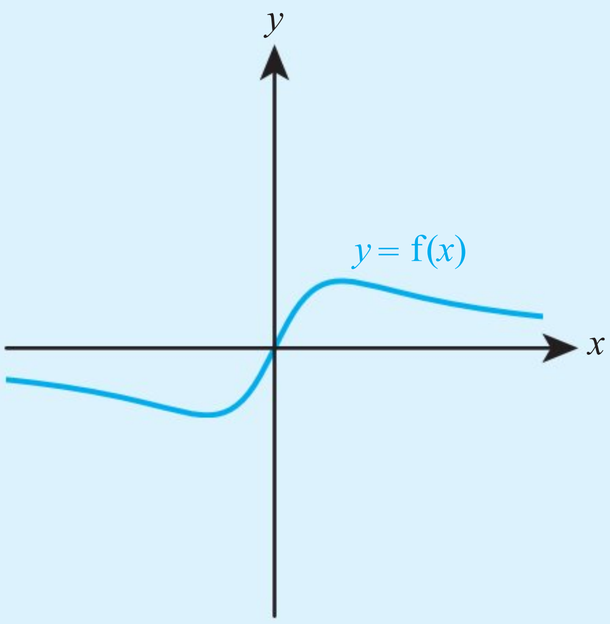


7 a



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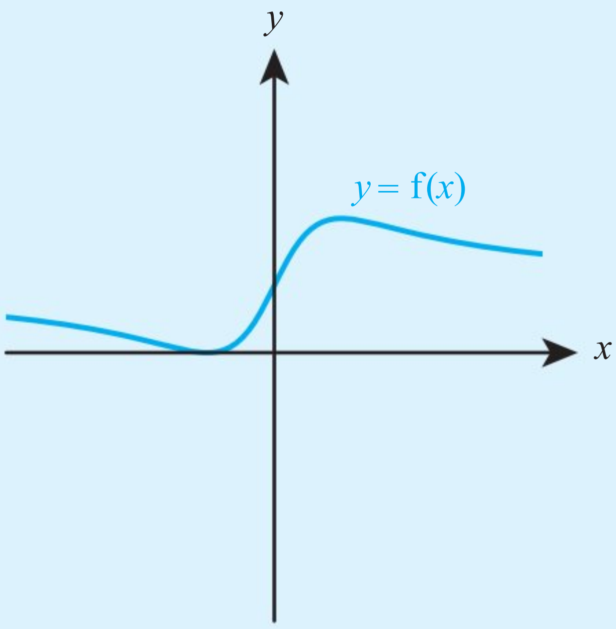
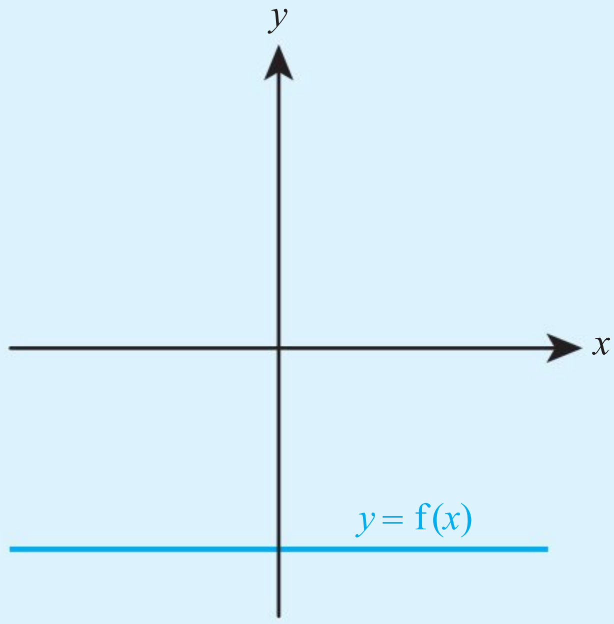


8 a



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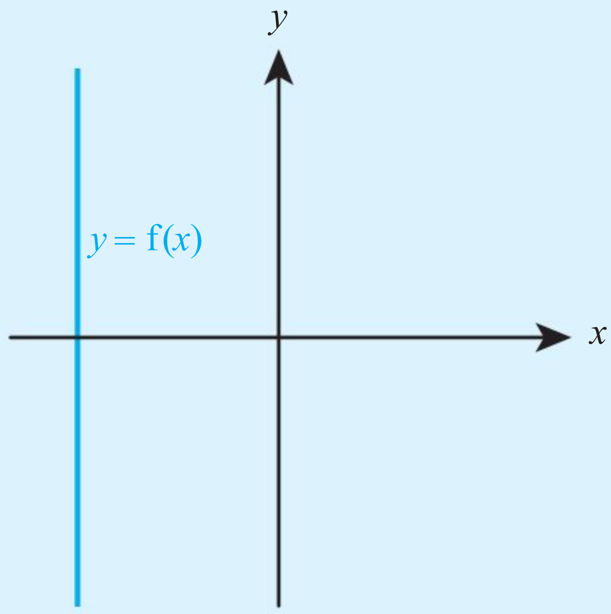
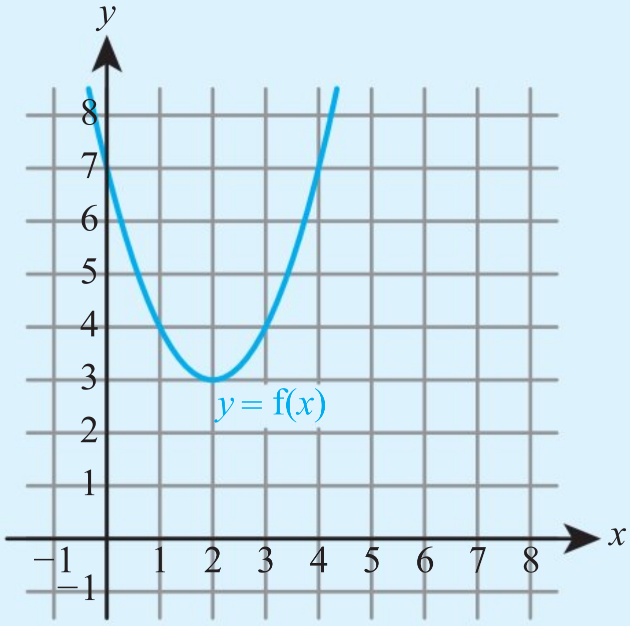


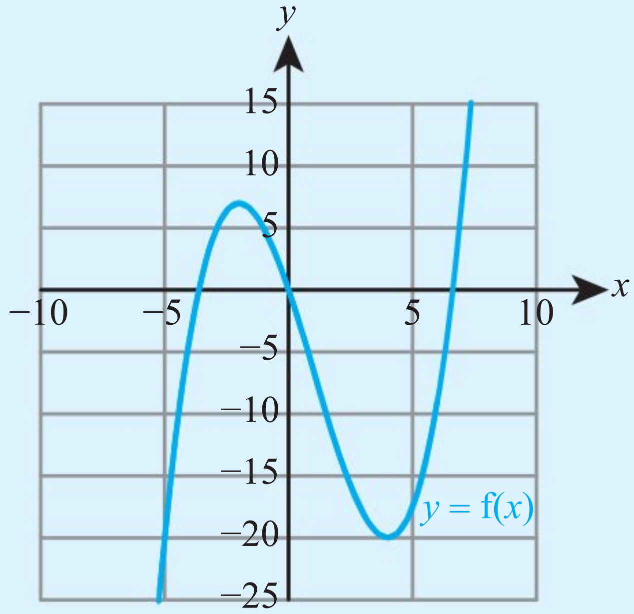
For questions 9 to 11, use the method demonstrated in Worked Example 7.16 to determine from the graph the largest possible domain of the given form for which the inverse function exists.

9



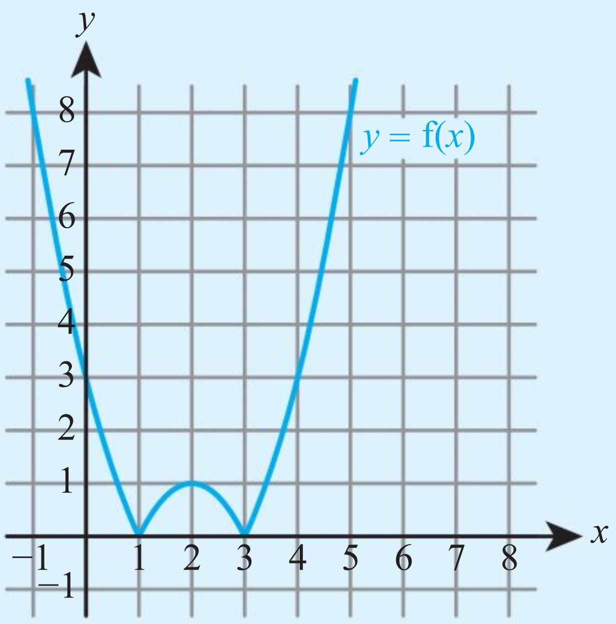
a $x \leqslant k$ **b** $x \geqslant k$

10



a $x \geqslant k$ **b** $x \leqslant k$

11



$|x^2 - 4x + 3|$

a $x \leqslant k$ **b** $x \geqslant k$

For questions 12 to 14, use the method demonstrated in Worked Example 7.15 to determine whether or not the function is self-inverse.

12 **a** $f(x) = \frac{1}{2x}$ **b** $f(x) = -\frac{5}{x}$

13 **a** $f(x) = 3 - 2x$ **b** $f(x) = 4 - x$

14 **a** $f(x) = \frac{x}{x+1}$ **b** $f(x) = \frac{3x+1}{2x-3}$

15 Determine algebraically whether the function $f(x) = \frac{x^3}{x^2 - 6}$ is odd, even or neither.

16 Determine algebraically whether the function $f(x) = \tan x + 3x^2$ is odd, even or neither.

17 Determine algebraically whether the function $f(x) = x \cos x - \sin x$ is odd, even or neither.

18 Let $f(x) = x^2 + 8x + 19$, $x \geqslant k$.

a Find the smallest value of k such that f^{-1} exists.

b For this value of k, find f^{-1} and its domain.

19 Let $f(x) = x^2 - 3x + 1$.

a Given that f^{-1} exists, find the largest possible domain of f of the form $x \leqslant k$.

b For the domain in part **a**, find f^{-1} and its domain.

20 **a** Show that the function $\mathrm{f}(v) = \frac{3-2x}{2}$, $b \in \mathbb{R}$ is self-inverse.

b State the domain of f^{-1}.

21 The function f is even and the function g is odd.

a Prove that the function h defined by $\mathrm{h}(x) = \frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is odd.

b Determine, with proof, whether the sum of an even and an odd function is odd, even or neither.

22 Prove that the only function that is both odd and even is $\mathrm{f}(x) = 0$.

23 Determine whether the function $\mathrm{f}(x) = |x-1| + |x+1|$ is even, odd or neither, fully justifying your answer.

24 The function f is defined by $\mathrm{f}(x) = x^3 + 6x^2 + 9x - 2$, $-5 \leqslant x \leqslant 1$.

a Find the largest possible domain of the form $a \leqslant x \leqslant b$ for which f has an inverse function.

b For the domain in part **a**, find the domain of f^{-1}.

25 The function f is defined by $\mathrm{f}(x) = x^4 - 8x^2 + 5$, $x \geqslant k$

a Find the smallest value of k such that f has an inverse function.

b For this value of k, find the domain of f^{-1}.

26 The function f is defined by $\mathrm{f}(x) = \mathrm{e}^x - 4x$, $x \leqslant k$.

a Find the largest value of k such that f^{-1} exists.

b For this value of k, find the domain of f^{-1}.

27 **a** Show that the function $\mathrm{f}(x) = \frac{2x+1}{3x-2}$, $x \neq 2$ is self-inverse.

b Find the domain of f^{-1}.

28 Determine, with proof, the condition on $a, b \in \mathbb{R}$ such that $\mathrm{f}(x) = ax + b$ is self-inverse.

29 **a** For any function f, show that $\mathrm{f}(x) + \mathrm{f}(-x)$ is an even function.

b For any function f, show that $\mathrm{f}(x) - \mathrm{f}(-x)$ is an odd function.

c Hence show that any function can be expressed as the sum of an even and an odd function.

30 Find the value of c for which the function $\mathrm{f}(x) = \frac{3-2x}{x+c}$ is self-inverse.

Checklist

- You should be able to sketch the graphs of functions of the form $\mathrm{f}(x) = \frac{ax+b}{cx^2+dx+e}$ and $\mathrm{f}(x) = \frac{ax^2+bx+c}{dx+e}$.
  - If $y = \frac{ax+b}{cx^2+dx+e}$, then
    - the y-intercept is $\left(\frac{b}{e}, 0\right)$
    - the x-intercept is $\left(0, -\frac{b}{a}\right)$
    - the horizontal asymptote is at $y = 0$
    - any vertical asymptotes occur at solutions of $cx^2 + dx + e = 0$.
  - If $y = \frac{ax^2+bx+c}{dx+e}$, then
    - the y-intercept is $\left(0, \frac{c}{e}\right)$
    - any x-intercepts occur at solutions of $ax^2 + dx + e = 0$
    - the vertical asymptote is at $x = -\frac{e}{d}$
    - there will be an oblique asymptote of the form $y = px + q$.
- You should be able to solve cubic inequalities.
- You should be able to solve other inequalities graphically using your GDC.

- You should be able to sketch graphs of the functions $y = |f(x)|$ and $y = f(|x|)$.
  - $|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$
  - To sketch the graph of $y = |f(x)|$, start with the graph of $y = f(x)$ and reflect in the x-axis any parts that are below the x-axis.
  - To sketch the graph of $y = f(|x|)$, start with the graph of $y = f(x)$ for $x \geqslant 0$ and reflect that in the y-axis.
- You should be able to solve modulus equations and inequalities.
- You should be able to sketch graphs of the form $y = \dfrac{1}{f(x)}$.

  To sketch the graph of $y = \dfrac{1}{f(x)}$ consider the following key features:

Feature of $y = f(x)$	**Feature of $y = \dfrac{1}{f(x)}$**
x-intercept at $(a, 0)$	$x = a$ is a vertical asymptote
y-intercept at $(0, b)$, $b \neq 0$	y-intercept at $\left(0, \dfrac{1}{b}\right)$
$x = a$ is a vertical asymptote	x-intercept at $(a, 0)$
$y = a$ is a horizontal asymptote, $a \neq 0$	$y = \dfrac{1}{a}$ is a horizontal asymptote
$y = 0$ is a horizontal asymptote	$y \to \infty$
$y \to \pm\infty$	$y = 0$ is a horizontal asymptote
(a, b) is a turning point, $b \neq 0$	$\left(a, \dfrac{1}{b}\right)$ is the opposite turning point

- You should be able to sketch graphs of the form $y = f(ax + b)$.

  When two horizontal transformations are applied, the order matters: $y = f(ax + b)$ is a horizontal translation by $-b$ followed by a horizontal stretch with scale factor $\dfrac{1}{a}$.
- You should be able to sketch graphs of the form $y = [f(x)]^2$.

  To sketch the graph of $y = [f(x)]^2$ consider the following key features:

Feature of $y = f(x)$	**Feature of $y = [f(x)]^2$**
$y < 0$	$y > 0$
x-intercept at $(a, 0)$	Local minimum at $(a, 0)$
y-intercept at $(0, b)$	y-intercept at $(0, b^2)$
$x = a$ is a vertical asymptote	$x = a$ is a vertical asymptote
$y = a$ is a horizontal asymptote	$y = a^2$ is a horizontal asymptote
$y \to \pm\infty$	$y \to \infty$

- You should be able to determine whether a function is odd, even or neither.
  - A function is
    — odd if $f(-x) = -f(x)$ for all x in the domain of f
    — even if $f(-x) = f(x)$ for all x in the domain of f.
  - The graph of
    — an odd function is symmetric with respect to the origin
    — an even function is symmetric with respect to the y-axis.
- You should be able to restrict the domain of a many-to-one function so that the inverse function exists.
- You should be able to determine whether a function is self-inverse.
  - A function is self-inverse if $f^{-1}(x) = f(x)$ for all x in the domain of f.
  - The graph of a self-inverse function is symmetric in the line $y = x$.

Mixed Practice

1 Let $f(x) = \dfrac{2x+1}{(3x-2)(x+2)}$.

a State the equation of the vertical asymptotes.
b Find the coordinates of the axis intercepts.
c Sketch the graph of $y = f(x)$.

2 Let $f(x) = x - 2 - \dfrac{8}{x-4}$.

a State the equation of
  i the vertical asymptote
  ii the oblique asymptote.
b Find the coordinates of the axis intercepts.
c Sketch the graph of $y = f(x)$.

3 Find the set of values of x for which $6x + x^2 - 2x^3 < 0$.

4 **a** Show that $(x+2)$ is a factor of $x^3 - 3x^2 - 6x + 8$.
b Hence solve the inequality $x^3 - 1 \geqslant 3(x^2 + 2x - 3)$.

5 Solve the inequality $2x^4 - 5x^2 + x + 1 < 0$.

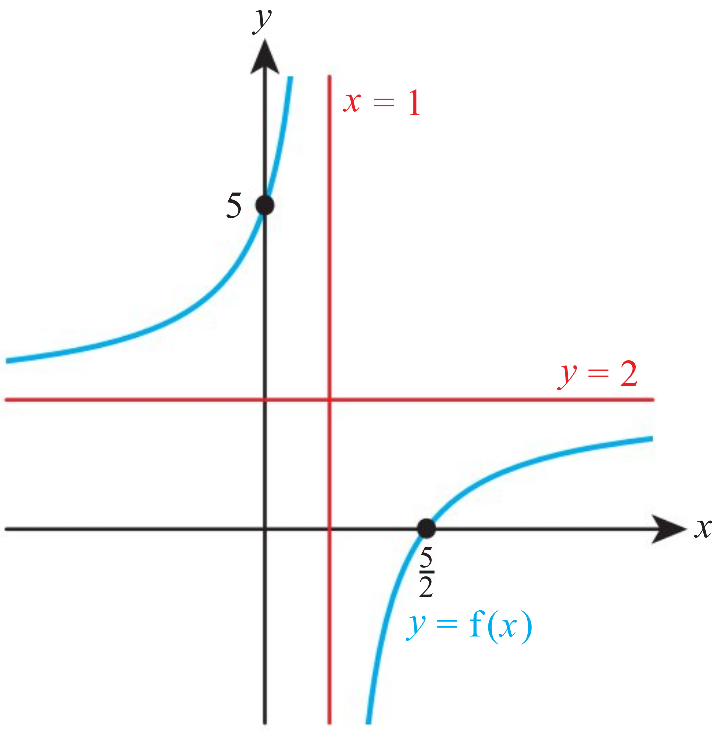
6 Solve the inequality $\ln x \leqslant e^{\sin x}$ for $0 < x \leqslant 10$.

7 **a** Sketch the graph of $y = |\cos 3x|$ for $0 \leqslant x \leqslant \pi$.
b Solve $|\cos 3x| = \dfrac{1}{2}$ for $0 \leqslant x \leqslant \pi$.

8 **a** On the same axes, sketch the graphs of $y = |4 + x|$ and $y = |5 - 3x|$, labelling any axis intercepts.
b Hence solve the inequality $|4 + x| \leqslant |5 - 3x|$.

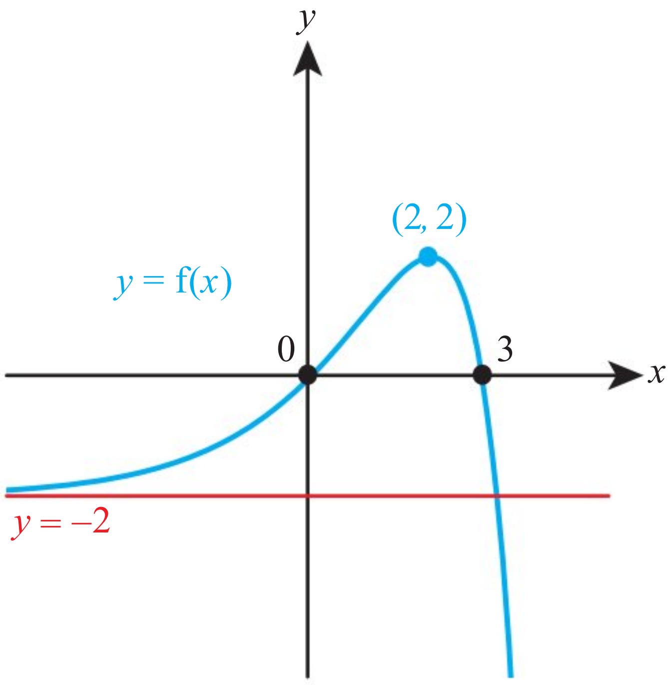
9 **a** On the same axes, sketch the graphs of $y = |5x + 1|$ and $y = 3 - x$, labelling any axis intercepts.
b Hence solve the inequality $3 - x > |5x + 1|$.

10 The graph of $y = f(x)$ is shown below.



Labelling any axis intercepts and asymptotes, on separate axes sketch the graph of
a $y = |f(x)|$
b $y = f(|x|)$.

11 The graph of $y = \text{f}(x)$ is shown below.



Labelling any x-axis intercepts, turning points and asymptotes, on separate axes sketch the graph of

a $y = \dfrac{1}{\text{f}(x)}$

b $y = [\text{f}(x)]^2$

c $y = \text{f}(2x - 1)$.

12 The function f is defined by $\text{f}(x) = 3^x + 3^{-x}$. Deterine algebraically whether f is even, odd or neither.

13 The function f is defined by $\text{f}(x) = -x^2 + 6x - 4,\ x \geqslant k$.

a Find the smallest value of k such that f has an inverse function.

b For this value of k, find $\text{f}^{-1}(x)$ and state its domain.

14 The functions f and g are defined by $\text{f}(x) = ax^2 + bx + c,\ x \in \mathbb{R}$ and $\text{g}(x) = p \sin x + qx + r,\ x \in \mathbb{R}$ where a, b, c, p, q, r are real constants.

a Given that f is an even function, show that $b = 0$.

b Given that g is an odd function, find the value of r.

The functions h is both odd and even, with domain $\mathbb{R}$.

c Find $\text{h}(x)$.

Mathematics HL May 2015 Paper 1 TZ1 Q5

15 **a** **i** Find the set of values of k for which the equation $kx^2 - 2(k+1)x + 7 - 3k = 0$ has real roots.

ii Hence determine the range of the function $\text{f}(x) = \dfrac{2x - 7}{x^2 - 2x - 3}$.

b Sketch the graph of $y = \text{f}(x)$ labelling any vertical asymptotes.

16 **a** Sketch the graph of $y = |2|x| - 3|$. State the coordinates of any axis intercepts.

b Solve the equation $|2|x| - 3| = 2$.

17 The function f is defined by $\text{f}(x) = (x - a)(x - b)$. On separate axes, sketch the graph of $y = \text{f}(|x|)$ in the case where

a $0 < b < a$ **b** $b < 0 < a$ **c** $b < a < 0$.

18 **a** Describe a sequence of two transformations that map the graph of $y = \text{f}(x)$ onto the graph of $y = \text{f}\left(\dfrac{x - 6}{3}\right)$.

b Describe a different sequence of two transformations that has the same effect as in part **a**.

19 Let $f(x) = x^2 - 3$.

a On the same axes, sketch the graphs of $y = |f(x)|$ and $y = \dfrac{1}{f(x)}$.

b Hence solve the inequality $|f(x)| \leqslant \dfrac{1}{f(x)}$.

20 Given $f(x) = |x + a| + |x + b|$, where $a, b \neq 0$, find the condition on a and b such that f is an even function.

21 The function f is defined by $f(x) = e^{2x} - 8e^x + 7$, $x \leqslant k$.

a Find the largest value of k such that f has an inverse function.

b For this value of k, find $f^{-1}(x)$ and state its domain.

22 The function f is defined by $f(x) = xe^{\frac{x}{2}}$, $x \geqslant k$.

a Find $f'(x)$ and $f''(x)$.

b Find the smallest value of k such that f has an inverse function.

c For this value of k, find the domain of f^{-1}.

23 Let $f(x) = \dfrac{3x}{x^2 + 1}$.

a **i** Show algebraically that f is an odd function.

ii What type of symmetry does this mean the graph of $y = f(x)$ must have?

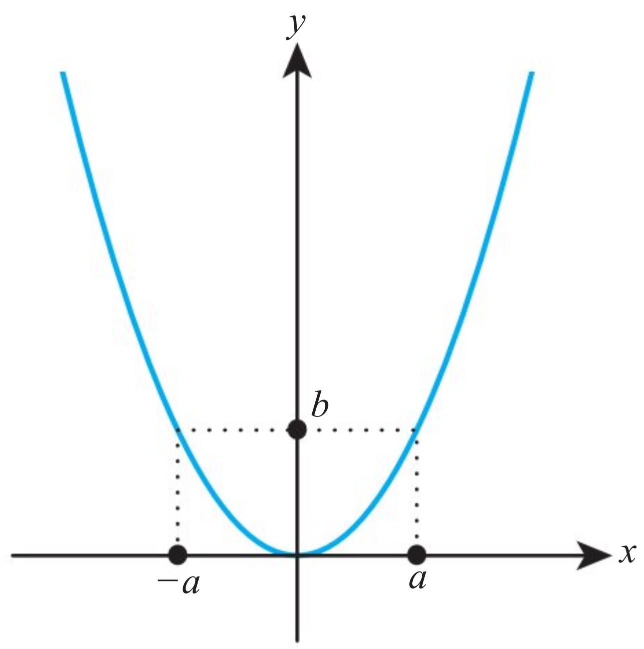
b **i** If the line $y = k$ intersects the curve, show that $4k^2 - 9 \leqslant 0$.

ii Hence find the coordinates of the turning points of the curve.

c Sketch the graph of $y = |f(x)|$.

d Solve the inequality $|f(x)| \geqslant |x|$.

24 The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$, where $b > a > 0$.



Consider the function $g(x) = \dfrac{1}{f(x - a) - b}$.

a Find the largest possible domain of the function g.

b Sketch the graph of $y = g(x)$. Indicate any asymptotes and local maxima or minima, and write down their equations and coordinates.

Mathematics HL May 2011 Paper 1 TZ1 Q10

25 The function f is defined by $f(x) = \frac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

a Express f(x) in the form $A + \frac{B}{x+2}$, where A and $B \in \mathbb{Z}$.

b Hence show that $f'(x) > 0$ on D.

c State the range of f.

d i Find an expression for $f^{-1}(x)$.

ii Sketch the graph of f(x), showing the points of intersection with both axes.

iii On the same diagram, sketch the graph of $y = f^{-1}(x)$.

e i On a different diagram, sketch the graph of $y = f(|x|)$ where $x \in D$.

ii Find all the solutions of the equation $f(|x|) = -\frac{1}{4}$.

Mathematics HL May 2013 Paper 1 TZ2 Q12

26 Let $f(x) = \frac{x^2 + 7x + 10}{x+1}$.

a Find the equation of the oblique asymptote.

b By finding a condition on k such that $f(x) = k$ has real solutions, or otherwise, find the coordiantes of the turning points of f.

c On the same axes, sketch the grap of $y = f(x)$ and $y = 2x + 7$.

d Hence solve the inequality $\frac{x^2 + 7x + 10}{x+1} < 2x + 7$.

e On a separate set of axes, sketch the graph of $y = |f(x)|$, labelling the coordinates of all axis intercepts.

f State the complete set of values of c for which $|f(x)| = c$ has two solutions.

27 The function f is defined by $f(x) = \frac{ax^2 + bx + c}{dx + e}$ and the function g is defined by $g(x) = \frac{1}{f(x)}$. f(x) has an oblique asymptote $y = x + 1$ and g(x) has vertical asymptotes $x = \frac{3}{2}$ and $x = -4$. Solve the equation $f(x) = g(x)$.

28 Find the value of c for which the function $f(x) = \frac{3x-5}{x+c}$ is self-inverse.

8 Vectors

ESSENTIAL UNDERSTANDINGS

- Geometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

- about the concept of a vector and its use in describing positions and displacements
- about different ways of representing vectors and how to perform various operations with vectors
- how to use scalar product to find the angle between two vectors
- about various forms of an equation of a straight line in three dimensions
- how to determine whether two lines intersect and find the point of intersection
- how to use vector product to find perpendicular directions and areas
- about various forms of an equation of a plane
- how to find intersections and angles between lines and planes, and use them to solve geometrical problems in three dimensions.

CONCEPTS

The following concepts will be addressed in this chapter:

- The properties of shapes depend on the dimension they occupy in **space**.
- Position and movement can be **modelled** in three-dimensional **space** using vectors.
- The **relationships** between algebraic, geometric and vector methods can help us to solve problems and **quantify** those positions and movements.

Figure 8.1 What information do you need to get from one place to another?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Find the equation of the straight line through the points (4, 3) and (−1, 5).
2. Four points have coordinates $A(3, 2)$, $B(-1, 5)$, $C(1, 6)$ and $D(9, k)$. Find the value of k for which AB and CD are parallel.
3. Solve the simultaneous equations

$$\begin{cases} 2x + 3y - z = 4 \\ 3x - 5y + 2z = 5 \\ 5x - 21y + 8z = 7. \end{cases}$$

You have probably met the distinction between scalar and vector quantities in physics. Scalar quantities, such as mass or time, can be described using a single number. Vector quantities need more than one piece of information to describe them. For example, velocity is described by its direction and magnitude (speed).

In pure mathematics we use vectors to describe positions of points and displacements between them. Although most of this chapter is concerned with using vectors to solve geometrical problems, the operations with vectors and their properties are equally applicable when vectors represent other physical quantities.

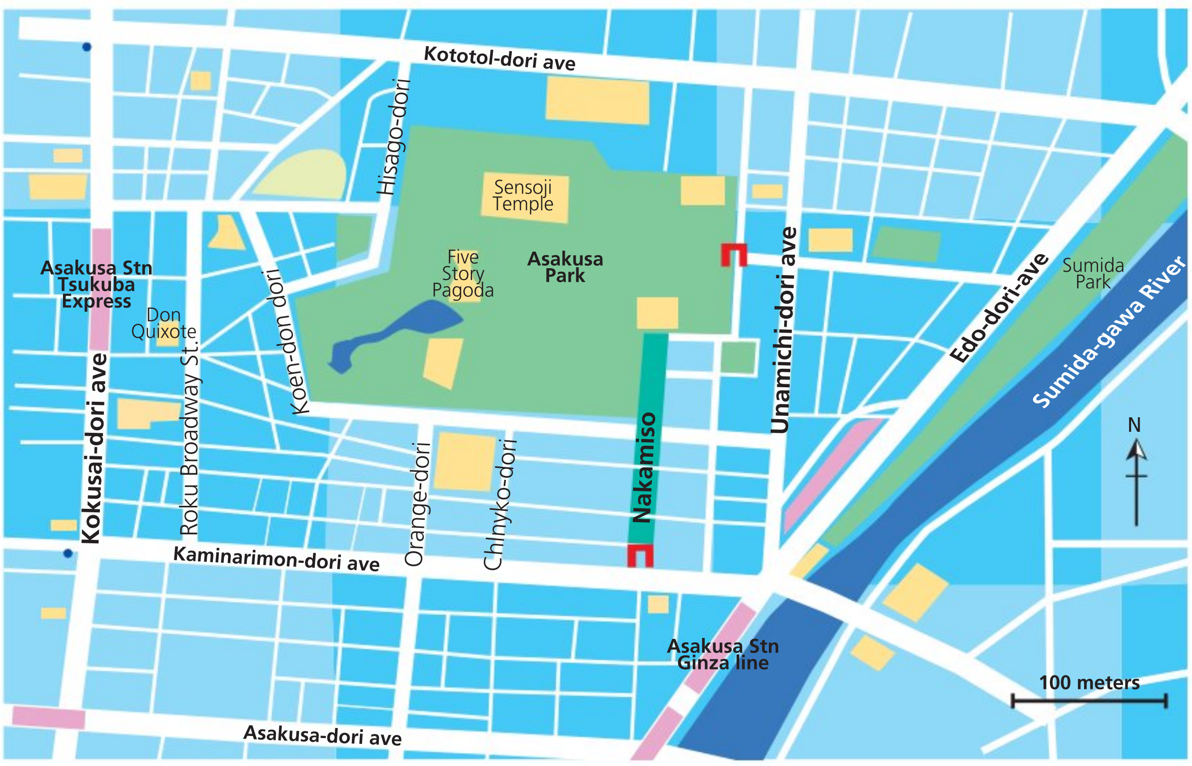
Vector equations describe geometrical objects, such as lines and planes, in three-dimensional space. Vector methods enable us to use calculations to determine properties of shapes, such as angles and lengths, in situations which may be difficult to visualize and solve geometrically.

Starter Activity

Look at the pictures in Figure 8.1. In small groups discuss how you would best give directions to get from one marked location to another (for example, from the Metropolitan Museum of Art to the 92nd Street Y).

Now look at this problem:

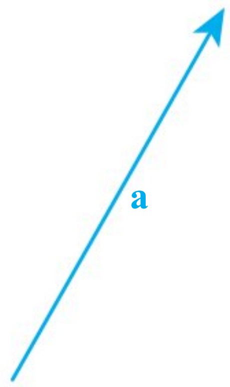
Find the size of the angle between two diagonals of a cube.



8A Introduction to vectors

A **vector** is a quantity that has both magnitude and direction. This can be represented in several different ways, either graphically or using numbers. The most useful representation depends on the precise application, but you will often need to switch between different representations within the same problem.

Representing vectors



A vector is labelled using either a bold lower case letter, for example **a**, or an underlined lower case letter $\underline{a}$.

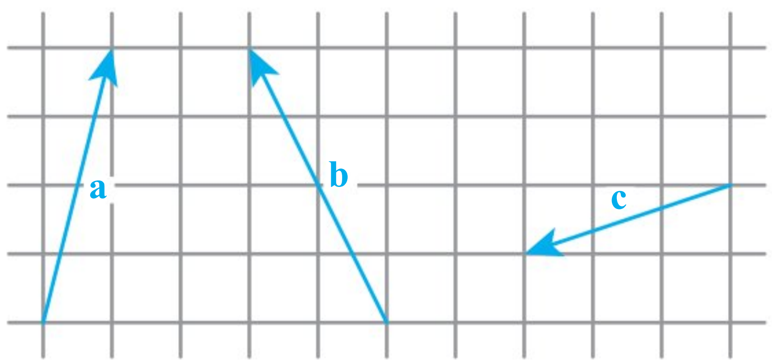
The simplest way to represent a vector is as a directed line segment, with the arrow showing the direction and the length representing the magnitude, as shown in the margin.

You will see that some problems can be solved using this diagrammatic representation, but sometimes you will also want to do numerical calculations. It that case, it may be useful to represent a vector using its **components**.

In two dimensions you can represent any vector by two numbers. We select two directions, which we will call 'horizontal' and 'vertical'. Then the components of a vector are given by the number of units in the two directions required to get from the 'tail' to the 'head' of the arrow. The components are written as a **column vector**; for example, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ means 3 units to the right and 2 units up.

WORKED EXAMPLE 8.1

Write the following as column vectors (each grid space represents one unit).



The line labelled **a** goes 1 unit to the right and 4 units up ⋯ $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

The line labelled **b** goes two units to the left and four units up ⋯ $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

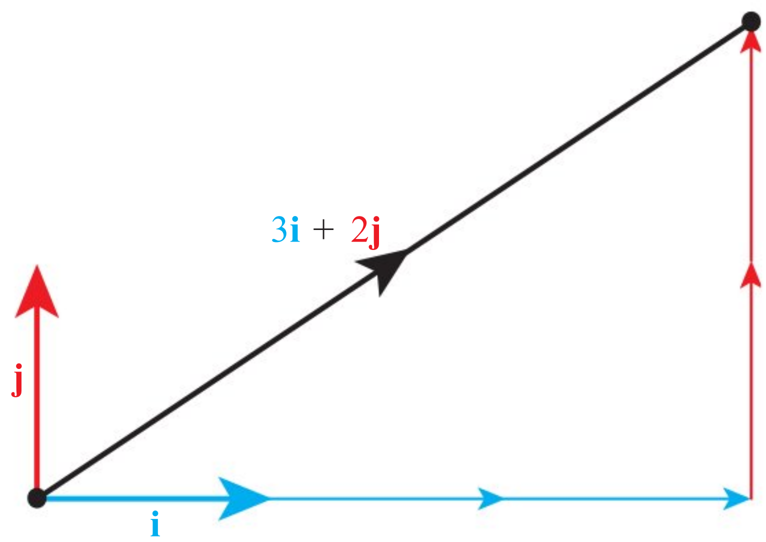
The line labelled **c** goes three units to the left and one unit down ⋯ $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Tip

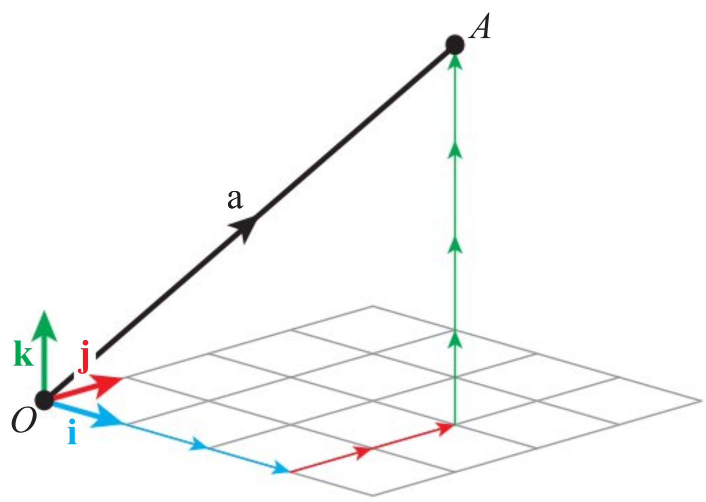
We often denote a vector by a single letter, just like variables in algebra. In printed text the letter is usually bold; when writing you should underline it (for example, $\underline{a}$) or use an arrow (for example, $\vec{a}$) to distinguish between vectors and scalars.

Another way to write a vector in components is to use **base vectors**, denoted **i** and **j** in two dimensions. These are vectors of length 1 in the directions 'right' and 'up'.

For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ can be written $3\mathbf{i} + 2\mathbf{j}$.



This approach can be extended to three dimensions. We need three base vectors, called **i, j, k**, all perpendicular to each other.



$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Tip

The base vectors written as column vectors are:

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Tip

You must be familiar with both base vector and column vector notation, as both are frequently used. When you write your answers, you can use whichever notation you prefer.

WORKED EXAMPLE 8.2

a Write the following using base vectors.

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$$

b Write the following as column vectors with three components.

$$\mathbf{d} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{e} = 2\mathbf{j} - \mathbf{k}, \quad \mathbf{f} = 3\mathbf{k} - \mathbf{i}$$

a

The components are the coefficients of **i** and **j**

$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$

The components are coefficients of **i**, **j** and **k**

$\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Notice that the **j**-component is zero

$\mathbf{c} = 4\mathbf{i} - 2\mathbf{k}$

b

The coefficients are the components of the vectors

$$\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Notice that the **i**-component is missing

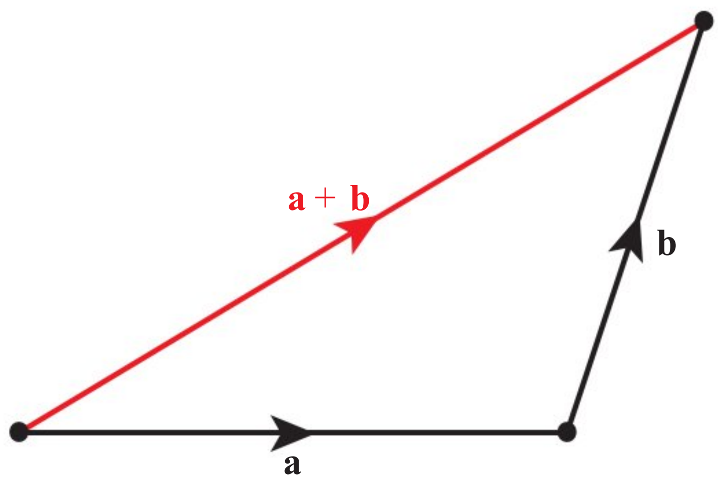
$$\mathbf{e} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Be careful – the components are not in the correct order!

$$\mathbf{f} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

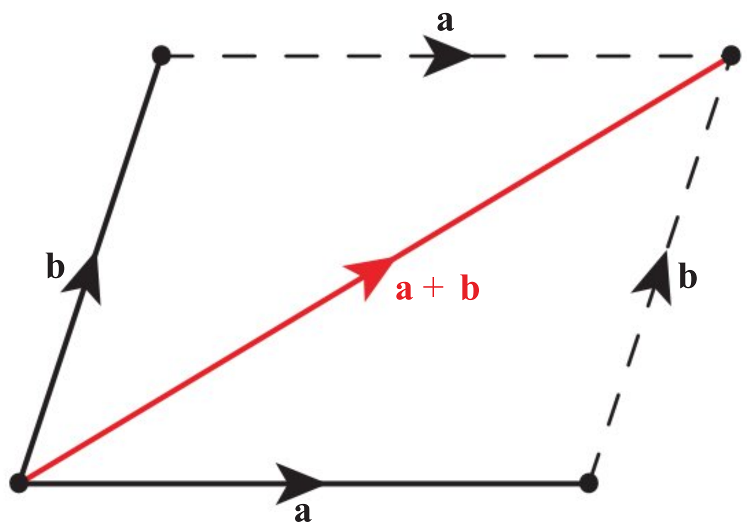
Addition and subtraction of vectors

On a diagram, vectors are added by joining the starting point of the second vector to the end point of the first. In component form, you just add the corresponding components.



$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

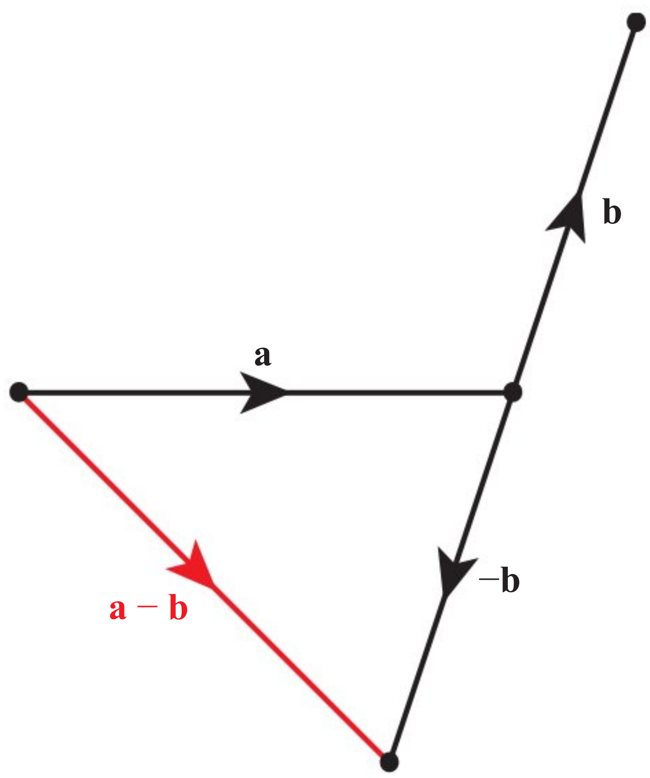
Another way of visualizing vector addition is as a diagonal of the parallelogram formed by the two vectors.



Tip

Equal vectors have the same magnitude and direction; they don't need to start or end at the same point.

To subtract vectors, reverse the direction of the second vector and add it to the first. Notice that subtracting a vector is the same as adding its negative. For example, if $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.



$$\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Tip

Subtracting a vector from itself gives the zero vector: $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

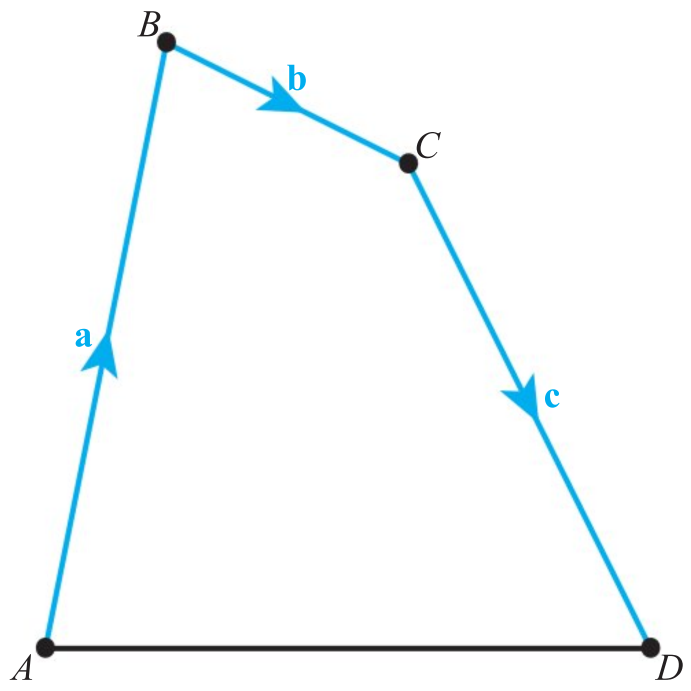
Another way of writing vectors is to label the end-points with capital letters. For example, $\overrightarrow{AB}$ is the vector in the direction from A to B, with magnitude equal to the length AB.

WORKED EXAMPLE 8.3

Express the following in terms of vectors **a**, **b** and **c**.

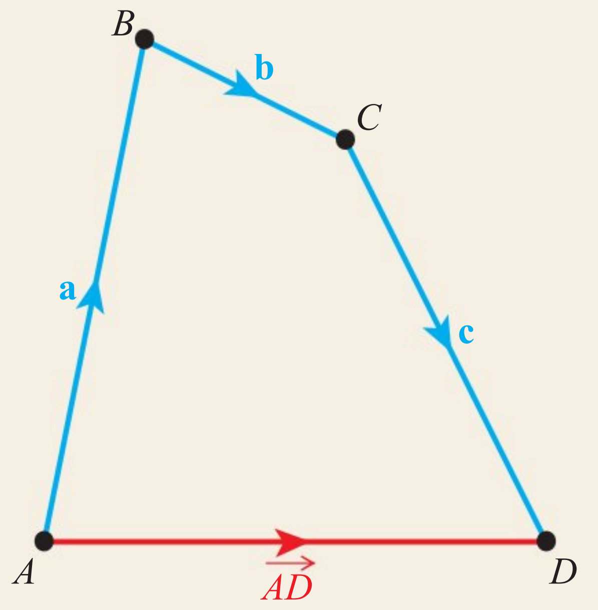
a $\overrightarrow{AD}$

b $\overrightarrow{DB}$



The vectors **a**, **b** and **c** are joined 'head to tail', starting at A and finishing at D

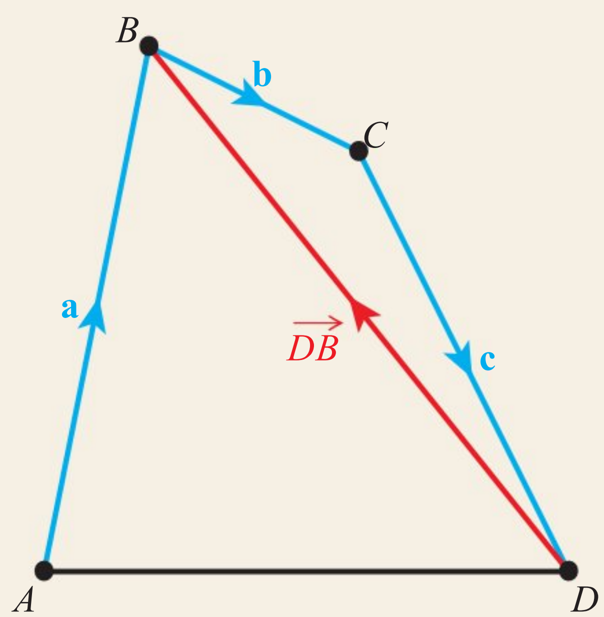
a



$\overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

You can see that $\mathbf{b} + \mathbf{c} = \overrightarrow{BD}$, and $\overrightarrow{DB}$ is the negative of this

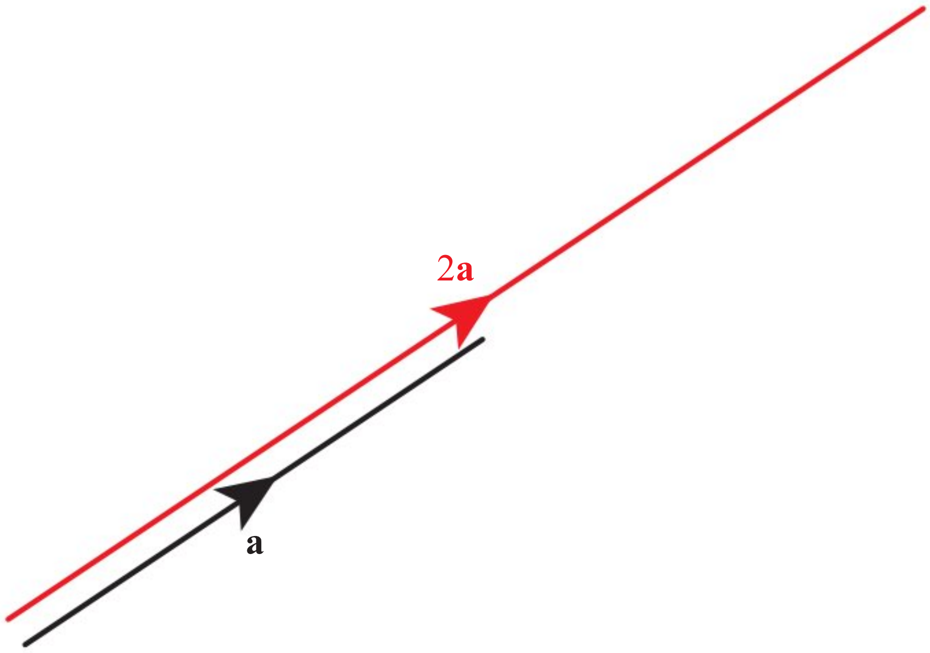
b



$$\begin{aligned}\overrightarrow{DB} &= -\overrightarrow{BD}\\ &= -(\mathbf{b} + \mathbf{c})\\ &= -\mathbf{b} - \mathbf{c}\end{aligned}$$

Scalar multiplication and parallel vectors

Multiplying by a scalar changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.



$$2\begin{pmatrix}3\\-5\\0\end{pmatrix}=\begin{pmatrix}6\\-10\\0\end{pmatrix}$$

Tip

Multiplying by a negative scalar reverses the direction.

WORKED EXAMPLE 8.4

Given vectors $\mathbf{a}=\begin{pmatrix}1\\2\\7\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-3\\4\\2\end{pmatrix}$, find $2\mathbf{a}-3\mathbf{b}$.

$$2\mathbf{a}-3\mathbf{b}=2\begin{pmatrix}1\\2\\7\end{pmatrix}-3\begin{pmatrix}-3\\4\\2\end{pmatrix}$$

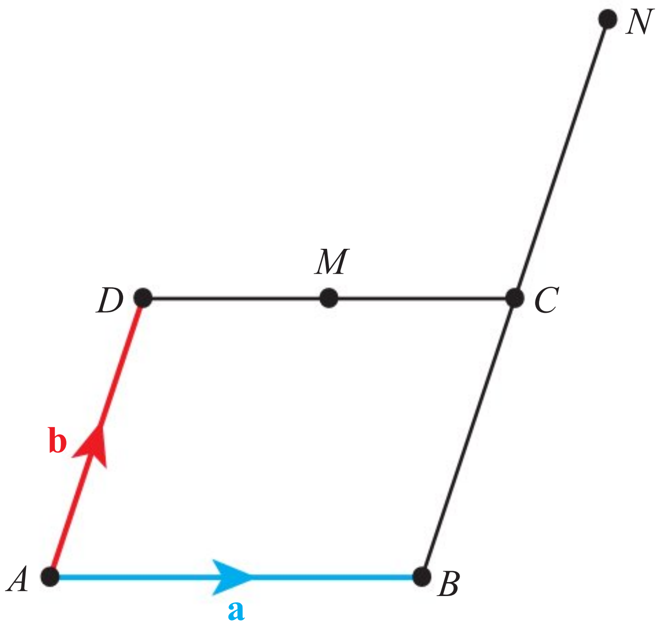
Multiply the scalar by each element of the relevant vector then subtract corresponding components of the vectors

$$=\begin{pmatrix}2\\4\\14\end{pmatrix}-\begin{pmatrix}-9\\12\\6\end{pmatrix}$$

$$=\begin{pmatrix}11\\-8\\8\end{pmatrix}$$

WORKED EXAMPLE 8.5

The diagram shows a parallelogram $ABCD$. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of CD and N is the point on BC such that $CN = BC$.



a Express vectors $\overrightarrow{CM}$ and $\overrightarrow{BN}$ in terms of **a** and **b**.

b Express $\overrightarrow{MN}$ in terms of **a** and **b**.

$\overrightarrow{CM}$ has the same direction as $\overrightarrow{BA}$, but half the length

a $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA}$

$\overrightarrow{BA} = -\overrightarrow{AB}$

$= -\frac{1}{2}\mathbf{a}$

$\overrightarrow{BN}$ is twice $\overrightarrow{BC}$…

$\overrightarrow{BN} = 2\overrightarrow{BC}$

… which is the same as $\overrightarrow{AD}$

$= 2\mathbf{b}$

b $\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$

We can think of $\overrightarrow{MN}$ as describing a way of getting from C to M moving only along the directions of **a** and **b**. This can be achieved by going from M to C and then from C to N

$= \frac{1}{2}\mathbf{a} + \mathbf{b}$

Two vectors are parallel if they have the same direction. This means that one is a scalar multiple of the other.

Tip

The scalar can be positive or negative.

KEY POINT 8.1

If vectors **a** and **b** are parallel, we can write $\mathbf{b} = t\mathbf{a}$ for some scalar t.

CONCEPTS – QUANTITY

We can do much more powerful mathematics with **quantities** than with drawings. Although the concept of parallel lines is a geometric concept, it is useful to quantify it in order to use it in calculations and equations. Key Point 8.1 gives us an equation to express the geometric statement ‘two lines are parallel’.

WORKED EXAMPLE 8.6

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$, find the values of p and q such that $\mathbf{c}$ is parallel to $\mathbf{a}$.

Write $\mathbf{c} = t\mathbf{a}$ for some scalar t.

Then,

If two vectors are parallel, we can write $v_1 = tv_2$

$$\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$$

If two vectors are equal, then all their components are equal

$$\begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$p = -4,\ q = -14$

Magnitude of a vector and unit vectors

The magnitude of a vector can be found from its components, using Pythagoras' theorem. The symbol for the magnitude is the same as the symbol for absolute value (modulus).

KEY POINT 8.2

The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

In some applications it is useful to make vectors have length 1. These are called **unit vectors**. The base vectors **i**, **j**, **k** are examples of unit vectors, but you can create a unit vector in any direction. You can take any vector in that direction and divide it by its magnitude; this will keep the direction the same but change the magnitude to 1.

KEY POINT 8.3

The unit vector in the same direction as vector $\mathbf{a}$ is $\frac{\mathbf{a}}{|\mathbf{a}|}$.

Tip

Note that there are two possible answers to part **b**, as you could divide **a** by –3 instead of 3, which reverses its direction.

WORKED EXAMPLE 8.7

a Find the magnitude of the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a unit vector parallel to **a**.

Use Pythagoras

a $|\mathbf{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

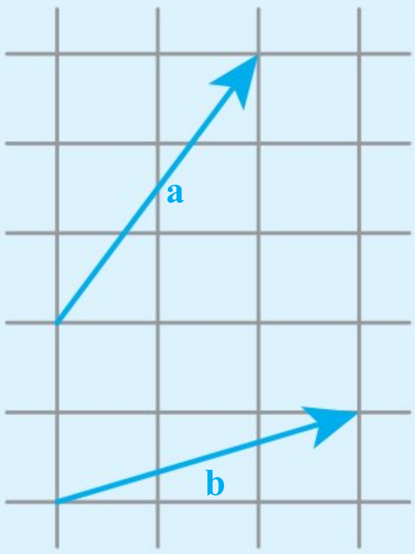
Divide by the magnitude of **a** to create a vector of length 1

b Unit vector is $\frac{1}{3}\mathbf{a} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

Exercise 8A

For questions 1 to 4, use the method demonstrated in Worked Example 8.1 to write vectors **a** and **b** as column vectors.

1



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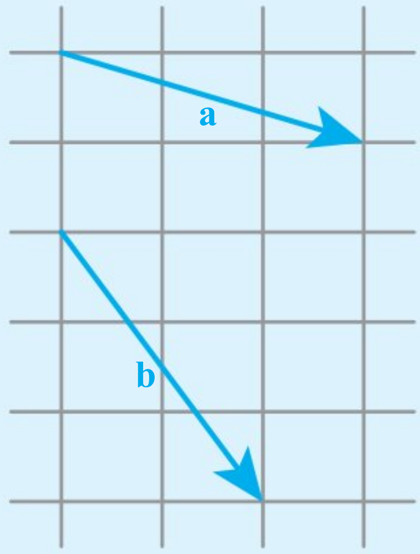

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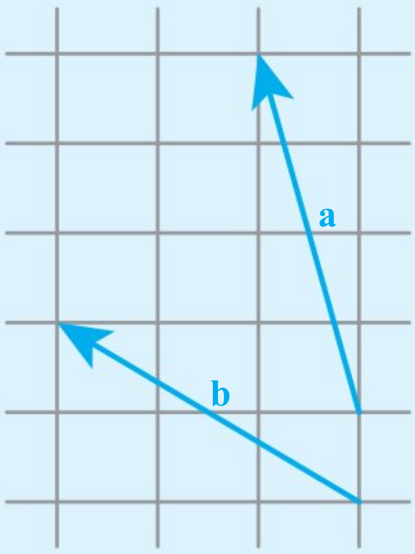

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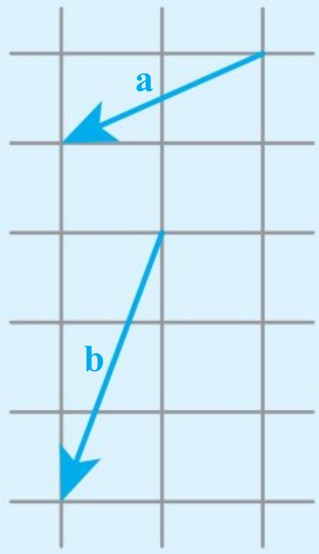


For questions 5 to 7, use the method demonstrated in Worked Example 8.2 to write the following using base vectors **i**, **j**, **k**.

5 a $\begin{pmatrix}1\\2\\3\end{pmatrix}$
b $\begin{pmatrix}4\\5\\6\end{pmatrix}$

6 a $\begin{pmatrix}-3\\1\\0\end{pmatrix}$
b $\begin{pmatrix}2\\-2\\0\end{pmatrix}$

7 a $\begin{pmatrix}3\\0\\0\end{pmatrix}$
b $\begin{pmatrix}0\\-5\\0\end{pmatrix}$

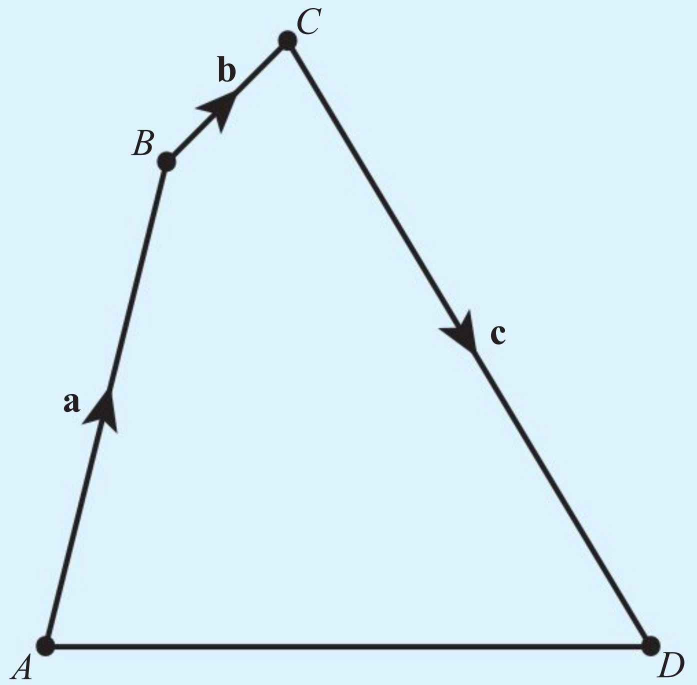
For questions 8 to 10, use the method demonstrated in Worked Example 8.2 to write the following as three-dimensional column vectors.

8 a $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
b $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

9 a $\mathbf{i} + 3\mathbf{k}$
b $2\mathbf{j} - \mathbf{k}$

10 a $4\mathbf{j} - \mathbf{i} - 2\mathbf{k}$
b $\mathbf{k} - 3\mathbf{i}$

In questions 11 and 12, use the method demonstrated in Worked Example 8.3 to express the following vectors in terms of **a**, **b** and **c**.



11 a $\overrightarrow{AC}$

b $\overrightarrow{BD}$

12 a $\overrightarrow{DA}$

b $\overrightarrow{DB}$

For questions 13 to 15, you are given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$. Use the method demonstrated in Worked Example 8.4 to write the following as column vectors.

13 a $\mathbf{a} - \mathbf{b}$

b $\mathbf{b} - \mathbf{a}$

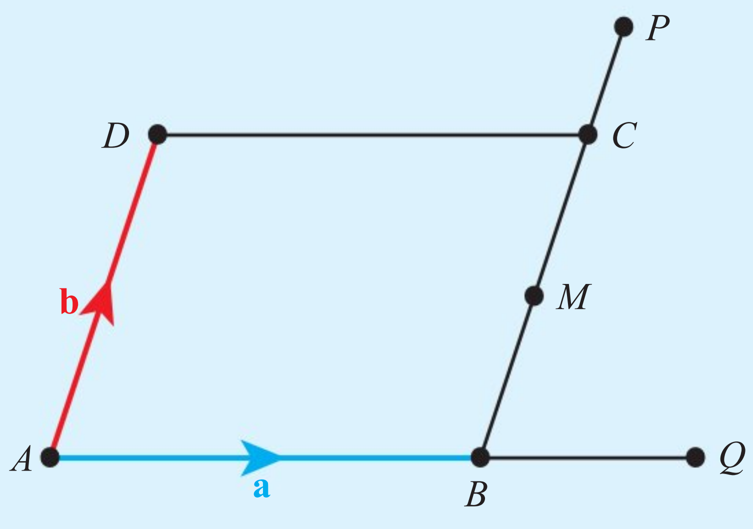
14 a $3\mathbf{a} + 2\mathbf{b}$

b $2\mathbf{a} + 5\mathbf{b}$

15 a $3\mathbf{b} - \mathbf{a}$

b $\mathbf{b} - 2\mathbf{a}$

For questions 16 to 18, $ABCD$ is a parallelogram, with $\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{a}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{a}$. M is the midpoint of BC, Q is the point on the extended line AB such that $BQ = \frac{1}{2}AB$ and P is the point on the extended line BC such that $CP = \frac{1}{3}BC$, as shown on the diagram.



Use the method demonstrated in Worked Example 8.5 to write the following vectors in terms of **a** and **b**.

16 a $\overrightarrow{AP}$

b $\overrightarrow{AM}$

17 a $\overrightarrow{QD}$

b $\overrightarrow{MQ}$

18 a $\overrightarrow{DQ}$

b $\overrightarrow{PQ}$

For questions 19 to 21, use the method demonstrated in Worked Example 8.6 to find the value of p and q such that the vectors **a** and **b** are parallel.

19 a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ p \\ q \end{pmatrix}$

20 a $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -9 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$

21 a $\mathbf{a} = \begin{pmatrix} 2 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 2 \\ 3 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -3 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 15 \\ 2 \end{pmatrix}$

22 a $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + 6\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + q\mathbf{j} + 6\mathbf{k}$

23 a $\mathbf{a} = 2\mathbf{i} + p\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 4\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = p\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + q\mathbf{k}$

For questions 24 to 27, use the method demonstrated in Worked Example 8.7 to find the unit vector in the same direction as vector **a**.

24 a $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

25 a $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

26 a $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

b $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

27 a $\mathbf{a} = \mathbf{i} - 4\mathbf{j}$

b $\mathbf{a} = 2\mathbf{j} - 3\mathbf{k}$

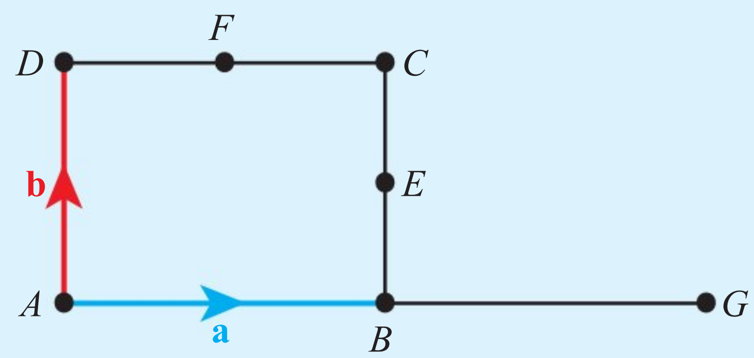
28 The diagram shows a rectangle $ABCD$. E is the midpoint of BC, F is the midpoint BC and G is the point on the extension of the side AB such that $BG = AB$.

Define vectors $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{a} = \overrightarrow{AD}$. Express the following vectors in terms of **a** and **b**.

a $\overrightarrow{AE}$

b $\overrightarrow{EF}$

c $\overrightarrow{DG}$



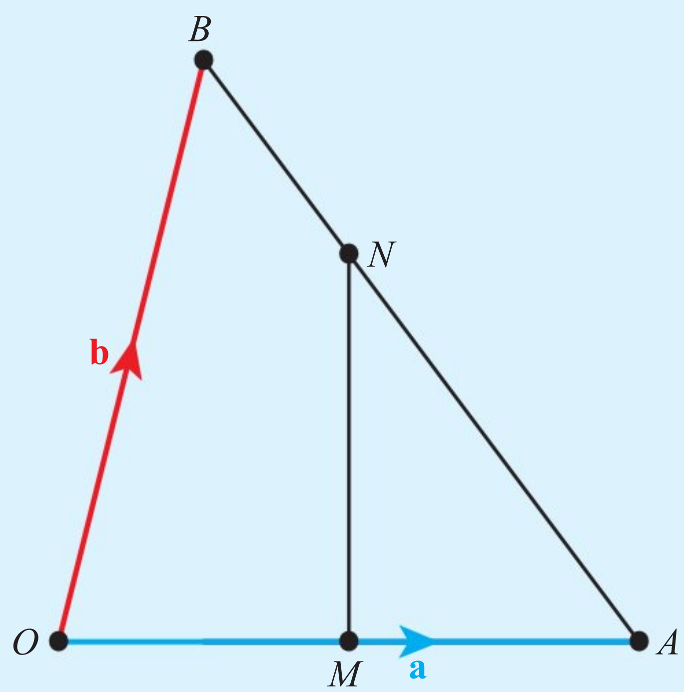
29 In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA and N is the point on AB such that $BN = \frac{1}{3}BA$.

Express the following vectors in terms of **a** and **b**.

a $\overrightarrow{BA}$

b $\overrightarrow{ON}$

c $\overrightarrow{MN}$



30 Given the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, find

a $3\mathbf{a} - \mathbf{c} + 5\mathbf{b}$

b $|\mathbf{b} - 2\mathbf{a}|$.

31 Find the possible values of the constant k such that the vector $\begin{pmatrix} 3k \\ -k \\ k \end{pmatrix}$ has magnitude 22.

32 The vector $2\mathbf{i} + 3t\mathbf{j} + (t-1)\mathbf{k}$ has magnitude 3. Find the possible values of t.

33 Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$ find vector x such that $3\mathbf{a} + 4\boldsymbol{x} = \mathbf{b}$.

34 Given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{k}$, find the value of the scalar t such that $\mathbf{a} + t\mathbf{b} = \mathbf{c}$.

35 a Find a unit vector parallel to $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$.

b Find a vector of magnitude 10 in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

36 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ find the value of the scalar p such that $\mathbf{a} + p\mathbf{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

37 Given that $\boldsymbol{x} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\boldsymbol{y} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ find the value of the scalar λ such that $\lambda\boldsymbol{x} = \boldsymbol{y}$ is parallel to vector $\mathbf{j}$.

38 Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

39 Let $\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$.

40 Find the smallest possible magnitude of the vector $(2t)\mathbf{i} + (t+3)\mathbf{j} - (2t+1)\mathbf{k}$, where t is a real constant.

41 a Show that the vector $\begin{pmatrix} 3\sin\theta \\ -3\cos\theta \\ 4 \end{pmatrix}$ has the same magnitude for all values of θ.

b Find the largest possible magnitude of the vector $\begin{pmatrix} 1+3\sin\theta \\ 1-3\cos\theta \\ 4 \end{pmatrix}$.

8B Vectors and geometry

Position and displacement vectors

In Section D you will find out how to use vectors in kinematics.

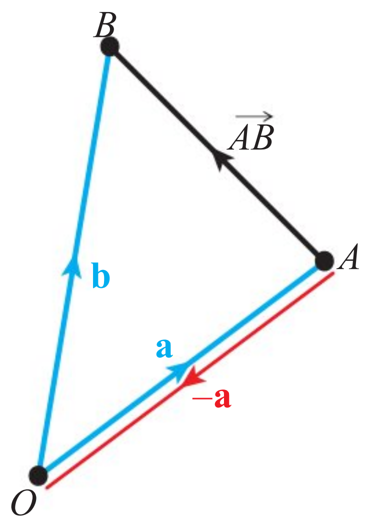
Vectors can be used to represent many different quantities, such as force, velocity or acceleration. They always obey the same algebraic rules you learnt in the previous section. One of the most common applications of vectors in pure mathematics is to represent positions of points in space, and thus describe geometrical figures.

You already know how to use coordinates to represent the position of a point, measured along the coordinate axes from the origin O. The vector from the origin to a point A is called the **position vector** of A. The base vectors **i**, **j**, and **k** are the unit vectors in the direction of x, y and z axes, respectively.

KEY POINT 8.4

- The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, where O is the origin.
- The components of **a** are the coordinates of A.

Position vectors describe positions of points relative to the origin, but you sometimes want to know the position of one point relative to another. This is described by a **displacement vector**.



KEY POINT 8.5

If points A and B have position vectors **a** and **b**, then the displacement vector from A to B is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

Tip

You can think of the equation $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ as saying: ‘to get from A to B, go from A to O and then from O to B’.

WORKED EXAMPLE 8.8

Points A and B have coordinates (3, −1, 2) and (5, 0, 3). Find the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point

$$\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

The displacement is the difference between the position vectors (end − start)

$$= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Distances

Tip

The displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have equal magnitude but opposite direction.

The distance between two points is equal to the magnitude of the displacement vector.

KEY POINT 8.6

The distance between the points A and B with position vectors **a** and **b** is

$$AB = \left|\overrightarrow{AB}\right| = |\mathbf{b} - \mathbf{a}|$$

WORKED EXAMPLE 8.9

Points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the exact distance AB.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

First find the displacement vector

$$= (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (3\mathbf{i} - \mathbf{j} - 4\mathbf{k})$$

$$= -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

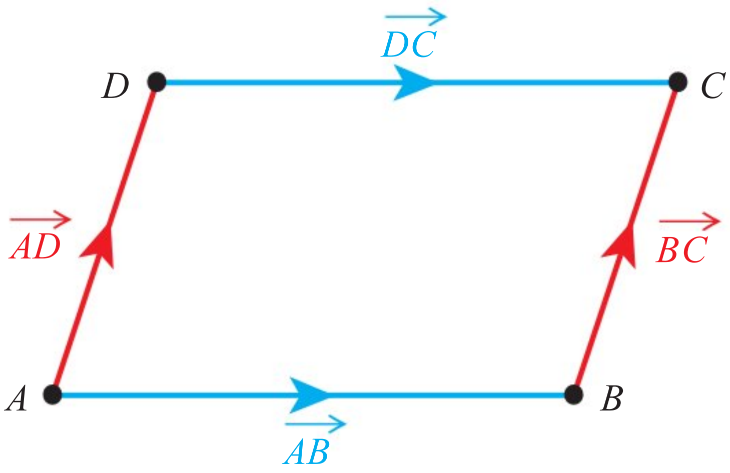
The distance is the magnitude of the displacement vector

$$\left|\overrightarrow{AB}\right| = \sqrt{4 + 1 + 9}$$

$$= \sqrt{14}$$

Using vectors to prove geometrical properties

One of the simplest geometrical figures to describe using vectors is a parallelogram. The opposite sides are parallel and equal length. In the diagram below, this means that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.



You can use the magnitudes of the vectors to check whether the shape is also a rhombus, which has all four sides equal length.

Tip

If one pair of sides are equal and parallel, then so are the other pair. You can check this in Worked Example 8.10 below.

KEY POINT 8.7

- If $\overrightarrow{AB} = \overrightarrow{DC}$, then *ABCD* is a parallelogram.
- If $|\overrightarrow{AB}| = |\overrightarrow{BC}|$, also, then *ABCD* is a rhombus.

WORKED EXAMPLE 8.10

Points *A*, *B*, *C* and *D* have position vectors
$\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j} - \mathbf{k}$, $\mathbf{c} = -\mathbf{j} + \mathbf{k}$, $\mathbf{d} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

a Show that *ABCD* is a parallelogram.

b Determine whether *ABCD* is a rhombus.

a

For a parallelogram, you need $\overrightarrow{AB} = \overrightarrow{DC}$

Remember that $\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$= (3\mathbf{i} + 6\mathbf{j} - \mathbf{k}) - (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$
$= -\mathbf{i} + 3\mathbf{j}$

$\overrightarrow{CD} = \mathbf{c} - \mathbf{d}$
$= (-\mathbf{j} + \mathbf{k}) - (\mathbf{i} - 4\mathbf{j} + \mathbf{k})$
$= -\mathbf{i} + 3\mathbf{j}$

$\overrightarrow{AB} = \overrightarrow{DC}$, so *ABCD* is a parallelogram.

b

You need to check whether *AB* and *BC* have equal length

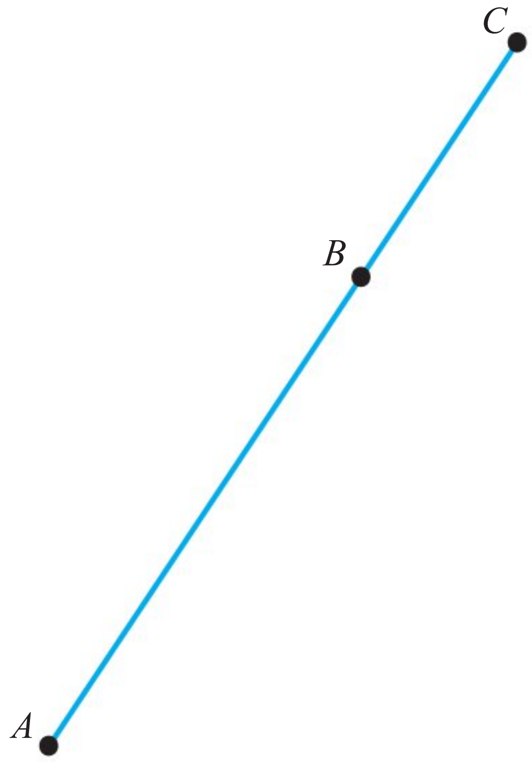
$|\overrightarrow{AB}| = \sqrt{1^3 + 3^2} = \sqrt{10}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{d}$
$= (-\mathbf{j} + \mathbf{k}) - (3\mathbf{i} + 6\mathbf{j} - \mathbf{k})$
$= -3\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$

So,
$|\overrightarrow{BC}| = \sqrt{3^2 + 7^2 + 2^2}$
$= \sqrt{62} \neq |\overrightarrow{AB}|$

ABCD is not a rhombus.

Vectors are also useful for checking whether three points are collinear (lie on the same straight line – pronounced co-linear).



In the diagram, A, B and C are collinear. This means that the vectors $\overrightarrow{AB} = \overrightarrow{BC}$ have the same direction. But you already know how to check whether two vectors are parallel.

KEY POINT 8.8

If points A, B and C are collinear, then $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

WORKED EXAMPLE 8.11

Find the values of p and q so that the points $A(4, 1, -2)$, $B(2, 2, 5)$ and $C(6, p, q)$ are collinear.

You need $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 6 \\ p \\ q \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ p-2 \\ q-5 \end{pmatrix}$$

Compare the first component, as this is known for both vectors

$$\overrightarrow{AB} = k\overrightarrow{BC} \Rightarrow -2 = 4k \Rightarrow k = -\frac{1}{2}$$

Use the same value of k for the remaining two components

Then,

$$1 = k(p-2) = -\frac{1}{2}(p-2) \Rightarrow p = 0$$

$$7 = k(q-5) = -\frac{1}{2}(q-5) \Rightarrow q = -9$$

You can also find the midpoint of the line segment with given endpoints.

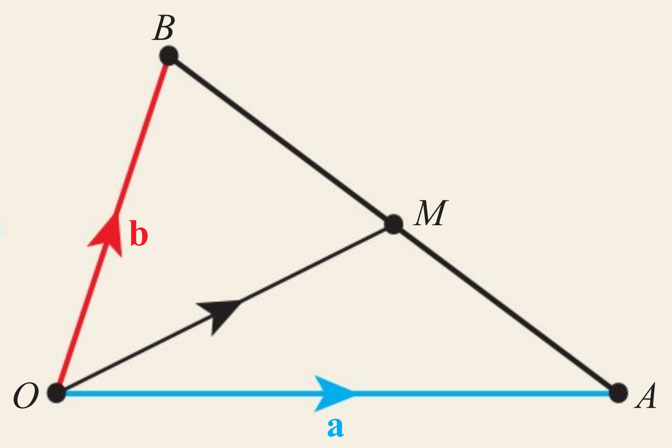
KEY POINT 8.9

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then the midpoint of AB has the position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Proof 8.1

Show that the midpoint of AB has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

The diagram shows the origin, O, the points A and B and their position vectors $\mathbf{a}$ and $\mathbf{b}$. M is the midpoint of AB



Express $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

M is half-way between A and B

$$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$

Use $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$

$$= \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a})$$

$$= \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$$

$$= \frac{1}{2}(\mathbf{a}+\mathbf{b})$$

WORKED EXAMPLE 8.12

Find the position vector of the midpoint of AB, where $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$.

The position vector of the midpoint is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$

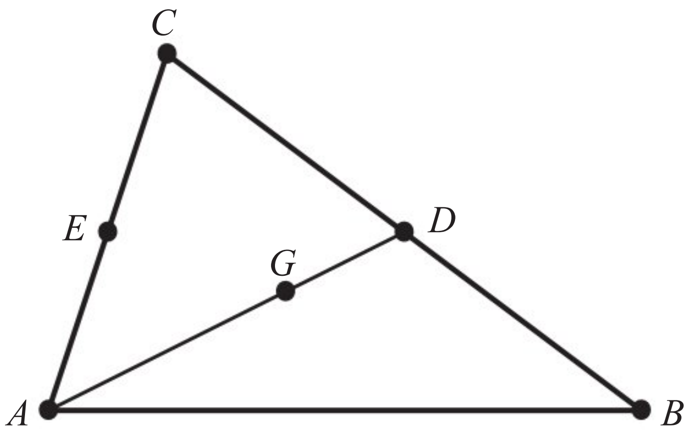
$$\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \frac{1}{2}\begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

You can use vector algebra to prove geometrical properties without knowing exact position vectors of the points.

WORKED EXAMPLE 8.13

The vertices of a triangle ABC have position vectors **a**, **b** and **c**. D is the midpoint of BC and E is the midpoint of AC. Point G (with position vector **g**) lies on AD such that $AG = \frac{2}{3}AD$.



a Express the position vector **g** in terms of **a**, **b** and **c**.

b Hence, show that the line BE passes through G and that $BG = \frac{2}{3}BE$.

a

Use $\overrightarrow{AG} = \mathbf{g} - \mathbf{a}$ to find the position vector of G

$$\mathbf{g} = \mathbf{a} + \overrightarrow{AG}$$
$$= \mathbf{a} + \frac{2}{3}\overrightarrow{AD}$$
$$= \mathbf{a} + \frac{2}{3}(\mathbf{d} - \mathbf{a})$$
$$= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{d}$$

D is the midpoint of BC, so $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$

$$= \frac{1}{3}\mathbf{a} + \frac{2}{3} \times \frac{1}{2}(\mathbf{b} + \mathbf{c})$$
$$= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$$

b

Find expressions for $\overrightarrow{BG}$ and $\overrightarrow{BE}$

$$\overrightarrow{BG} = \mathbf{g} - \mathbf{b}$$
$$= \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$$

$$\overrightarrow{BE} = \mathbf{e} - \mathbf{b}$$

E is the midpoint of AC, so $\mathbf{e} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$

$$= \frac{1}{2}(\mathbf{a} + \mathbf{c}) - \mathbf{b}$$
$$= \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}$$

If B, G and E lie on the same straight line then $\overrightarrow{BG}$ is a multiple of $\overrightarrow{BE}$

$$\overrightarrow{BG} = \frac{2}{3}\overrightarrow{BE}$$

This means that BE is parallel to BG, so B, G and E lie on the same line and $BG = \frac{2}{3}BE$.

Exercise 8B

For questions 1 to 3, points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Use the method demonstrated in Worked Example 8.8 to find the given displacement vectors.

1 a $\overrightarrow{AB}$
  b $\overrightarrow{AC}$

2 a $\overrightarrow{CB}$
  b $\overrightarrow{CA}$

3 a $\overrightarrow{BA}$
  b $\overrightarrow{BC}$

For questions 4 to 6, use the method demonstrated in Worked Example 8.9 to find the exact distance between the points A and B with the given position vectors.

4 a $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$
  b $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

5 a $\mathbf{a} = \mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$
  b $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$

6 a $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$
  b $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

For questions 7 to 9, you are given coordinates of points A, B, C and D. Use the method demonstrated in Worked Example 8.10 determine whether $ABCD$ is a parallelogram. If it is, determine whether it is a rhombus.

7 a $A(2, 0, 1)$, $B(4, 4, 2)$, $C(1, 2, 5)$, $D(-1,-2,4)$
  b $A(3, 1, 4)$, $B(4, 2, 8)$, $C(2, 2, 5)$, $D(1, 1, 1)$

8 a $A(1, 1, 2)$, $B(-1, 3, 5)$, $C(5, 1, 2)$, $D(2, 2, 3)$
  b $A(-1, 4, 5)$, $B(3, -3, 7)$, $C(1, 2, 5)$, $D(-1, 2, 2)$

9 a $A(1, 0, 2)$, $B(4, -4, 7)$, $C(-1, 1, 7)$, $D(-4, 5, 2)$
  b $A(3, 1, 6)$, $B(2, 2, 5)$, $C(1, 3, 6)$, $D(2, 2, 7)$

For questions 10 to 12, use the method demonstrated in Worked Example 8.11 to find the values of p and q such that the points A, B and C are collinear.

10 a $A(2, 5, 2)$, $B(1, 1, 6)$, $C(-3, p, q)$
   b $A(-1, 1, 5)$, $B(2, 1, 3)$, $C(11, p, q)$

11 a $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} p \\ 7 \\ q \end{pmatrix}$
   b $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -3 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 2 \\ p \\ q \end{pmatrix}$

12 a $\mathbf{a} = p\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + q\mathbf{k}$, $\mathbf{c} = 7\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$
   b $\mathbf{a} = \mathbf{i} + p\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + q\mathbf{k}$, $\mathbf{c} = -\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$

For questions 13 to 15, use the method demonstrated in Worked Example 8.12 to find the position vector of the midpoint of AB.

13 a $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -4 \\ 6 \\ 3 \end{pmatrix}$

14 a $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k},\ \mathbf{b} = \mathbf{i} - 3\mathbf{k}$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{j},\ \mathbf{b} = -5\mathbf{j} + 4\mathbf{k}$

15 a $A(-1, 0, 5),\ B(1, 1, 2)$

b $A(1, 1, 2),\ B(-3, 0, 5)$

16 Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. C is the midpoint of AB. Find the exact distance AC.

17 a Show that the points $A(2, 1, 2)$, $B(5, -4, 3)$ and $C(-4, 11, 0)$ are collinear.

b Find the ratio of the lengths $AB : BC$.

18 Points A and B have position vectors $\mathbf{a} = \mathbf{i} - 4\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

a Find the distance AB.

b Find the position vector of the midpoint of AB.

19 The vertices of a triangle have position vectors $\mathbf{a} = \begin{pmatrix} 4 \\ -11 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix}$. D is the midpoint of BC.

a Find the position vector of D.

b Find the length AD.

20 Point A has position vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and point D is such that $\overrightarrow{AD} = \mathbf{i} - \mathbf{j}$. Find the position vector of point D.

21 Given points A and B with position vectors $\mathbf{a} = \begin{pmatrix} 4 \\ -11 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, find the position vector of the point C such that $\overrightarrow{AB} = \overrightarrow{BC}$.

22 Points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$.

a Find the position vector of the point D such that $ABCD$ is a parallelogram.

b Determine whether $ABCD$ is a rhombus.

23 Three vertices of a parallelogram $ABCD$ have coordinates $A(5, -1, 2)$, $B(-1, 2, 8)$ and $C(7, 7, 0)$.

a Find the coordinates of D.

b Find the coordinates of the midpoint of AC.

c Show that this is also the midpoint of BD.

24 The vertices A, B and C of a triangle have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. M is the midpoint of AB and N is the midpoint of AC.

a Express $\overrightarrow{BC}$ and $\overrightarrow{MN}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

b What does this tell you about the lines BC and MN?

25 The vertices of a quadrilateral have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. M, N, P, Q are the midpoints of the sides. By expressing $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{q}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, prove that $MNPQ$ is a parallelogram.

26 Points A, B and C lie in a straight line and have coordinates $A(-3, 2, 5)$, $B(q, p, 2)$ and $C(12, 7, 0)$.

a Find the values of p and q.

b Find the ratio $AB : BC$.

27 Points A and B have position vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$. Point C lies between A and B so that $AC : CB = 2 : 3$. Find the position vector of C.

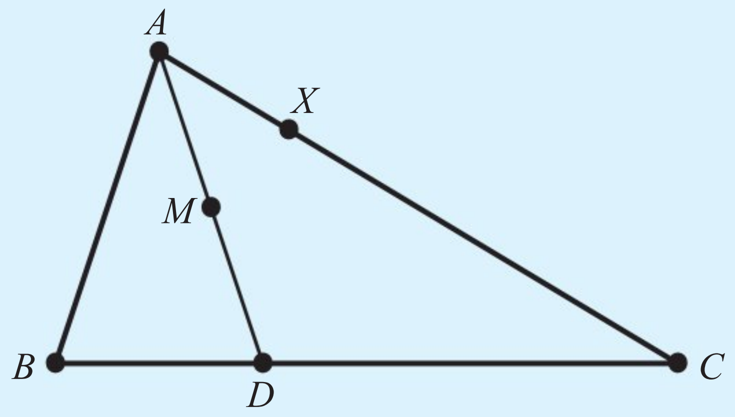
28 Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$, where O is the origin. Find the possible values of t such that $AB = 3$.

29 Points P and Q have position vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

a Find the position vector of the midpoint M of PQ.

b Point R is distinct from M and collinear with P and Q such that $QR = QM$. Find the coordinates of R.

30 The vertices of triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Point D lies on the side BC such that $BD = \frac{1}{3}BC$ and M is the midpoint of AD.



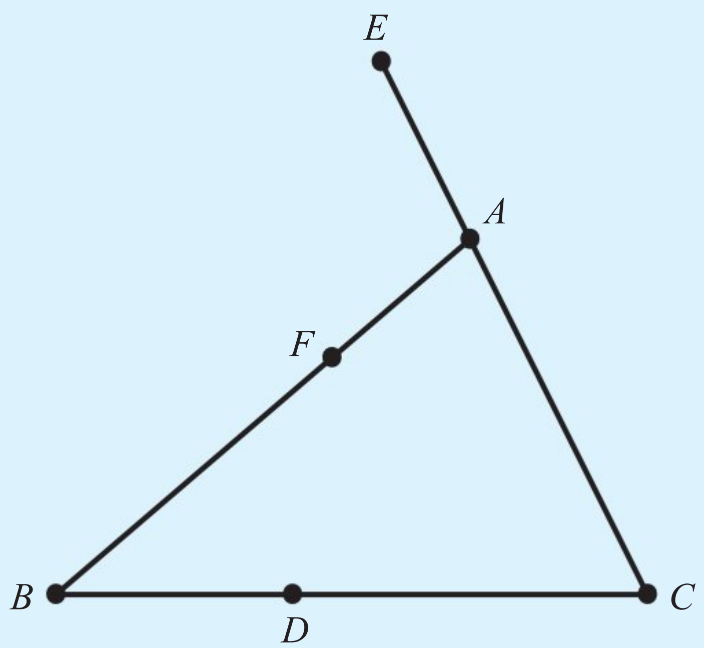
a Express the position vector of M in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Point X lies on AC and $AX = \frac{1}{4}AC$.

b Show that B, M and X lie in a straight line and find the ratio $BM : MX$.

31 Points D, E, F lie on the sides of the triangle ABC, as shown in the diagram, so that

$BD : CD = 2 : 3,\ CE : EA = 3 : 1,\ AF : FB = 1 : 2.$



$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of A, B and C.

a Express the position vectors of D, E and F in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

b Hence prove that the points D, E and F lie in a straight line.

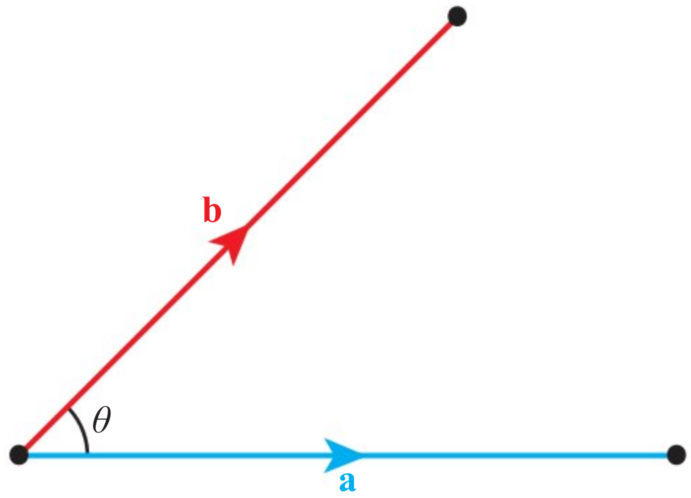
32 Four points have coordinates $A(2, -1)$, $B(k, k+1)$, $C(2k-3, 3k+2)$ and $D(k-1, 2k)$.

a Show that $ABCD$ is a parallelogram for all values of k.

b Show that there are no values of k for which $ABCD$ is rhombus.

8C Scalar product and angles

The diagram shows two lines with angle θ between them. **a** and **b** are vectors in the directions of the two lines. Notice that both arrows are pointing away from the intersection point.



It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors.

The definition of the scalar product

To start to find the link between vectors and angles we will define a way of multiplying vectors, called the scalar product.

Links to: Physics

The formula for the scalar product can be considered as the projection of one vector onto the other and it has many applications in physics. For example, if a force **F** acts on an object that moves from the origin to a point with position x, then the work done is $\mathbf{F} \bullet \boldsymbol{x}$.

KEY POINT 8.10

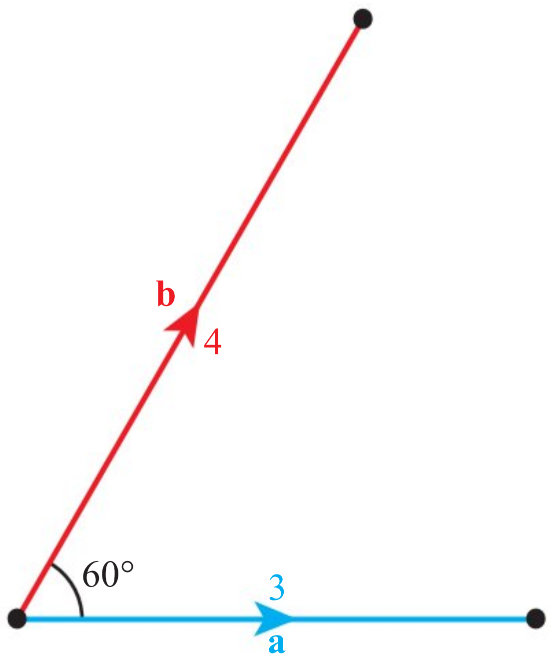
The **scalar product** (or **dot product**) of two vectors is defined by

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**.

WORKED EXAMPLE 8.14

The diagram shows vectors **a** and **b**. The numbers represent their lengths. Find the value of $\mathbf{a} \bullet \mathbf{b}$.



Use the formula for $\mathbf{a} \bullet \mathbf{b}$ $\quad$ $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}|\cos\theta$

The magnitudes are the lengths of the lines $\quad$ $= 3 \times 4 \times \cos 60°$

$= 6$

In Section 8F you will meet the cross product, for which the result is a vector.

Notice that the scalar product is a number (scalar).

Vectors are often given in terms of components, rather than by magnitude and direction. You can use the cosine rule to express the scalar product in terms of the components of the two vectors (see Proof 8.2).

KEY POINT 8.11

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Tip

The value of the scalar product can be negative.

WORKED EXAMPLE 8.15

Given that $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$, calculate $\mathbf{a} \cdot \mathbf{b}$.

Use $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (3)(-3) + (-2)(1) + (1)(4) \\ &= -9 - 2 + 4 \\ &= -7 \end{aligned}$$

Finding angles

Combining the results from Key Points 8.10 and 8.11 gives a formula for calculating the angle between two vectors given in component form.

Tip

The same formula can be used to find the angle between vectors in two dimensions – just set $a_3 = b_3 = 0$.

KEY POINT 8.12

If θ is the angle between vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

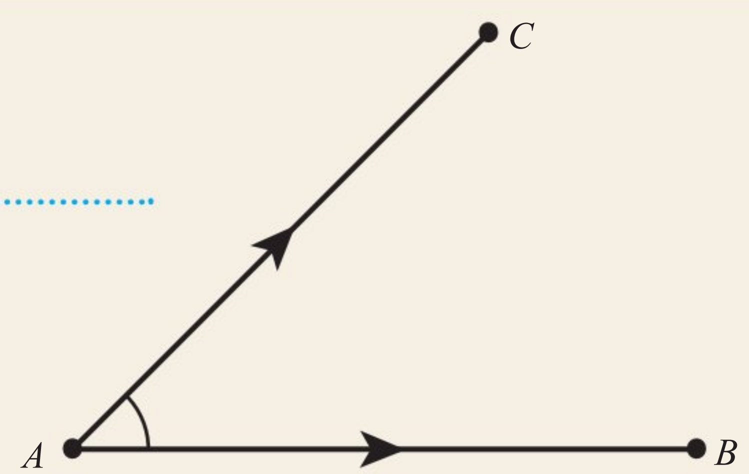
$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

WORKED EXAMPLE 8.16

Given points $A(3, -5, 2)$, $B(4, 1, 1)$ and $C(-1, 1, 2)$, find the size of angle $B\hat{A}C$ in degrees.

It is always a good idea to draw a diagram to see which vectors you need to use



You can see that the required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$.

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

You need to find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{1\times(-4) + 6\times 6 + (-1)\times 0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\theta = \cos^{-1}(0.7199) = 44.0°$$

Be the Examiner 8.1

Given points $A(-1, 4, 2)$, $B(3, 3, 1)$ and $C(2, -5, 3)$, find the size of angle $A\hat{B}C$ in degrees.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$
$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$
$\cos\theta = \frac{-4+8-2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$
$= 0.142$	$= -0.142$	$= -0.142$
$\theta = \cos^{-1}(0.142) = 81.8°$	$\theta = \cos^{-1}(-0.142) = 98.2°$	$\theta = \cos^{-1}(-0.142) = 98.2°$
	angle $= 180 - 98.2 = 81.8°$	

TOOLKIT: Problem Solving

There is more than one solution to $\cos x = 0.7199$ in the worked example above, but we have only given one answer. What do the other solutions represent?

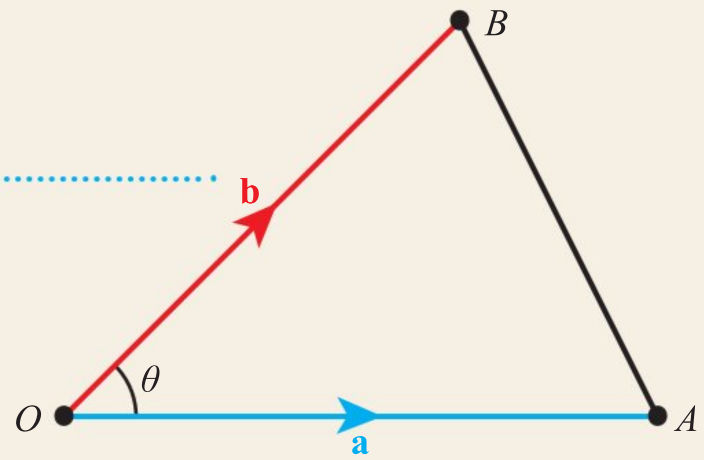
Proof 8.2

Here is a proof of the formula $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ for vectors in two dimensions.

Consider the triangle with vertices O, A and B and angle $AOB = \theta$

Let the position vectors of A and B be

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$



Use the cosine rule to link the lengths of the sides to $\cos\theta$

$$AB^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Now express all the lengths in terms of the components of **a** and **b**

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2},\ |b| = \sqrt{b_1^2 + b_2^2},$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

So,

$$AB^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$$
$$= b_1^2 + a_1^2 - 2a_1b_1 + b_2^2 + a_2^2 - 2a_2b_2$$

Hence,

Substitute all the lengths into the cosine rule

$$b_1^2 + a_1^2 - 2a_1b_1 + b_2^2 + a_2^2 - 2a_2b_2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}\cos\theta$$

All the square terms cancel

$$2a_1b_1 + 2a_2b_2 = 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}\cos\theta$$

Divide by 2 and rewrite the square root as $|\mathbf{a}||\mathbf{b}|$

$$a_1b_1 + a_2b_2 = |\mathbf{a}||\mathbf{b}|$$

$$\therefore \cos\theta = \frac{a_1b_1 + a_2b_2}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

as required.

Can you produce a similar proof for vectors with three components?

Algebraic properties of scalar product

Scalar product has many properties similar to multiplication of numbers.

KEY POINT 8.13

- $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
- $(-\mathbf{a})\cdot\mathbf{b} = -(\mathbf{a}\cdot\mathbf{b})$
- $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})$
- $(k\mathbf{a})\cdot\mathbf{b} = k(\mathbf{a}\cdot\mathbf{b})$

Tip

Some properties of multiplication of numbers do not apply to scalar product. For example, it is not possible to calculate the scalar product of three vectors, for example, the expression $(\mathbf{a} \bullet \mathbf{b}) \bullet \mathbf{c}$. Can you see why?

TOOLKIT: Proof

Try proving the rules from Key Point 8.13 by writing the vectors in components, for example, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Does it matter how many components the vectors have?

It is worth remembering the special case of finding the scalar product of a vector with itself, which follows from the formula $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ with $\mathbf{a} = \mathbf{b}$ and $\theta = 0$.

KEY POINT 8.14

$\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$

WORKED EXAMPLE 8.17

$\mathbf{a}$ and $\mathbf{b}$ are unit vectors and the angle between them is 60°. Find the value of $\mathbf{a} \bullet (2\mathbf{a} + 3\mathbf{b})$.

Expand the brackets using $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b}) + (\mathbf{a} \bullet \mathbf{c})$

$\mathbf{a} \bullet (2\mathbf{a} + 3\mathbf{b}) = \mathbf{a} \bullet (2\mathbf{a}) + \mathbf{a} \bullet (3\mathbf{b})$

Use $(k\mathbf{a}) \bullet \mathbf{b} = k(\mathbf{a} \bullet \mathbf{b})$

$= 2(\mathbf{a} \bullet \mathbf{a}) + 3(\mathbf{a} \bullet \mathbf{b})$

Use $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$ and $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= 2|\mathbf{a}|^2 + 3|\mathbf{a}||\mathbf{b}|\cos 60°$

$\mathbf{a}$ and $\mathbf{b}$ are unit vectors, so $|\mathbf{a}| = |\mathbf{b}| = 1$

$= 2 + 3\cos 60°$

$= \frac{7}{2}$

TOK Links

All the operations with vectors work in the same way in two and three dimensions. If there were a fourth dimension, so that the position of each point is described using four numbers, we could use analogous rules to calculate 'distances' and 'angles'. Does this mean that we can acquire knowledge about a four-dimensional world which we can't see, or even imagine?

Perpendicular and parallel vectors

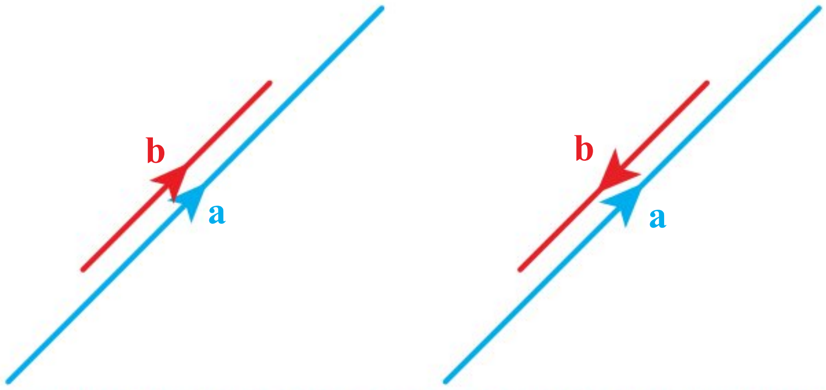
Two further important properties of the scalar product concern perpendicular and parallel vectors. They are derived using the facts that $\cos 90° = 0$, $\cos 0° = 1$ and $\cos 180° = -1$.

KEY POINT 8.15

- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors, then $\mathbf{a} \bullet \mathbf{b} = 0$.
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Tip

The angle between two parallel vectors can be either 0° or 180°.



WORKED EXAMPLE 8.18

Find the value of t such that the vector $\mathbf{a} = \begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix}$ is perpendicular to the vector $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

For perpendicular vectors, $\mathbf{a} \cdot \mathbf{b} = 0$

$$\begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 0$$

Express the scalar product in terms of the components

$$(3t)(1) + (1+t)(-2) + (2-5t)(2) = 0$$

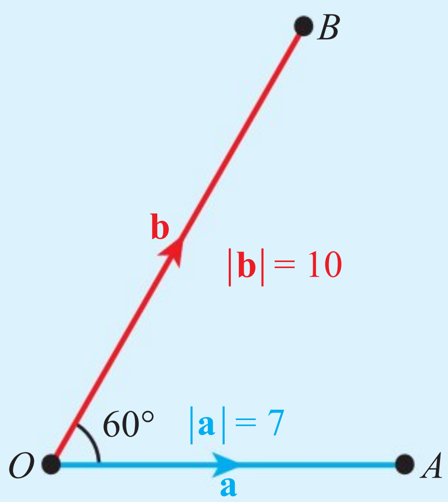
$$-9t + 2 = 0$$

$$t = \frac{2}{9}$$

Exercise 8C

For questions 1 to 3, use the method demonstrated in Worked Example 8.14 to find $\mathbf{a} \cdot \mathbf{b}$ for the vectors in each diagram.

1 a



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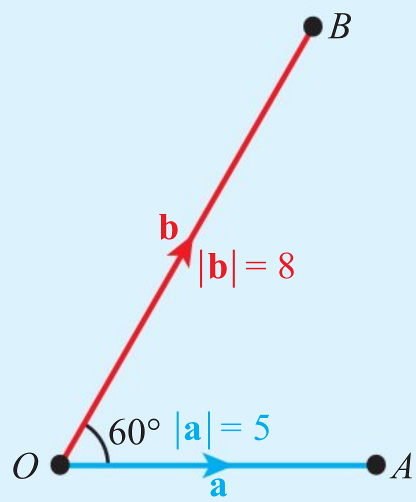
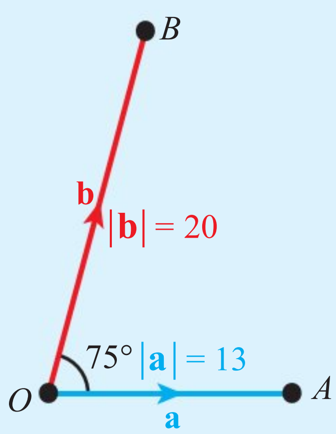


2 a



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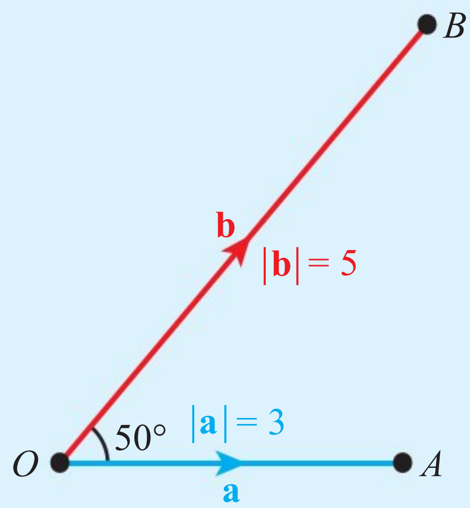
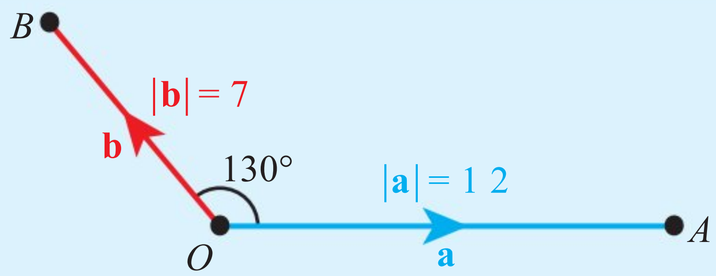
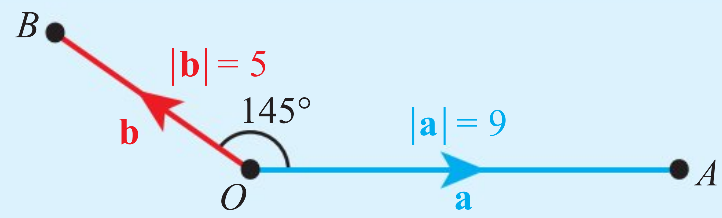


3 **a** 

b 

For questions 4 to 6, use the method demonstrated in Worked Example 8.15 to find **a** • **b** for the two given vectors.

4 **a** $\mathbf{a}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}5\\2\\2\end{pmatrix}$ **b** $\mathbf{a}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-12\\4\\-8\end{pmatrix}$

5 **a** $\mathbf{a}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}0\\0\\-8\end{pmatrix}$ **b** $\mathbf{a}=\begin{pmatrix}0\\-3\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}4\\0\\5\end{pmatrix}$

6 **a** $\mathbf{a}=4\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+\mathbf{j}+3\mathbf{k}$ **b** $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$

For questions 7 to 10, use the method demonstrated in Worked Example 8.16 to find the required angle, giving your answer to the nearest degree.

7 **a** Angle between vectors $\begin{pmatrix}5\\1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ **b** Angle between vectors $\begin{pmatrix}3\\0\\2\end{pmatrix}$ and $\begin{pmatrix}0\\-1\\1\end{pmatrix}$

8 **a** Angle between vectors $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ **b** Angle between vectors $\mathbf{i}-\mathbf{j}$ and $2\mathbf{i}+3\mathbf{j}$

9 **a** Angle $B\hat{A}C$ where $A(2, 1, 0)$, $B(3, 1, 2)$, $C(4, 4, 1)$ **b** Angle $B\hat{A}C$ where $A(2, 1, 0)$, $B(3, 0, 0)$, $C(2, -2, 4)$

10 **a** Angle $A\hat{B}C$ where $A(3, 6, 5)$, $B(2, 3, 6)$, $C(4, 0, 1)$ **b** Angle $A\hat{B}C$ where $A(8, -1, 2)$, $B(3, 1, 2)$, $C(0, -2, 0)$

For questions 11 to 14, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and the angle between **a** and **b** is 60°. Use the method demonstrated in Worked Example 8.17 to find the value of the given expression.

11 **a** $\mathbf{a}\cdot(2\mathbf{a}+5\mathbf{b})$
b $\mathbf{a}\cdot(4\mathbf{a}+3\mathbf{b})$

12 **a** $3\mathbf{a}\cdot(4\mathbf{a}-5\mathbf{b})$
b $2\mathbf{a}\cdot(4\mathbf{a}-5\mathbf{b})$

13 **a** $(2\mathbf{a}+\mathbf{b})\cdot(3\mathbf{a}+2\mathbf{b})$
b $(\mathbf{a}+4\mathbf{b})\cdot(2\mathbf{a}+5\mathbf{b})$

14 **a** $(2\mathbf{a}-\mathbf{b})\cdot(3\mathbf{a}-2\mathbf{b})$
b $(\mathbf{a}-4\mathbf{b})\cdot(2\mathbf{a}-5\mathbf{b})$

For questions 15 to 18, use the method demonstrated in Worked Example 8.18 to find the value t such that the two given vectors are perpendicular.

15 **a** $\begin{pmatrix}t+1\\2t-1\\2t\end{pmatrix}$ and $\begin{pmatrix}2\\6\\0\end{pmatrix}$
b $\begin{pmatrix}2t\\1\\-3t\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\2\end{pmatrix}$

16 **a** $\begin{pmatrix}t+1\\-2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\t\\2\end{pmatrix}$
b $\begin{pmatrix}1\\t+1\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\5\\3-7t\end{pmatrix}$

17 **a** $(5-t)\mathbf{i}+3\mathbf{j}-(10-t)\mathbf{k}$ and $-3\mathbf{i}+6\mathbf{j}+2\mathbf{k}$
b $(2t)\mathbf{i}+(t+1)\mathbf{j}-5\mathbf{k}$ and $4\mathbf{i}-\mathbf{j}+3\mathbf{k}$

18 **a** $5t\mathbf{i}-(2+t)\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+4\mathbf{j}-t\mathbf{k}$
b $t\mathbf{i}-3\mathbf{k}$ and $2\mathbf{i}+(t+4)\mathbf{j}$

19 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ calculate

a $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

b $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{c})$

c $(\mathbf{b} + \mathbf{d}) \cdot (2\mathbf{a})$

20 Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$.

21 Four points are given with coordinates $A(2, -1, 3)$, $B(1, 1, 2)$, $C(6, -1, 2)$ and $D(7, -3, 3)$. Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

22 Given that $\mathbf{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, and that θ is the angle between $\mathbf{p}$ and $\mathbf{q}$, find the exact value of $\cos\theta$.

23 Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \cdot \mathbf{b} = 10$ find, in degrees, the angle between $\mathbf{a}$ and $\mathbf{b}$.

24 Given that $|\mathbf{c}| = 9$, $|\mathbf{d}| = 12$ and $\mathbf{c} \cdot \mathbf{d} = -15$ find, in degrees, the angle between $\mathbf{c}$ and $\mathbf{d}$.

25 Given that $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = 5\mathbf{i} - 3\mathbf{k}$ and $\mathbf{d} = -2\mathbf{j} + \mathbf{k}$ verify that

a $\mathbf{b} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{b}$

b $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

c $(\mathbf{c} - \mathbf{d}) \cdot \mathbf{c} = |\mathbf{c}|^2 - \mathbf{c} \cdot \mathbf{d}$

d $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

26 Given that $|\mathbf{a}| = 8$, $\mathbf{a} \cdot \mathbf{b} = 12$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$, find the exact value of $|\mathbf{b}|$.

27 Find, in degrees, the angles of the triangle with vertices $(1, 1, 3)$, $(2, -1, 1)$ and $(5, 1, 2)$.

28 The vertices of a triangle have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 7 \\ 1 \\ -2 \end{pmatrix}$. Find, in degrees, the angles of the triangle.

29 Vertices of a triangle have position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i}$.

a Show that the triangle is right angled.

b Calculate the other two angles of the triangle.

c Find the area of the triangle.

30 Given that $\mathbf{p}$ is a unit vector making a $45°$ angle with vector $\mathbf{q}$, and that $\mathbf{p} \cdot \mathbf{q} = 3\sqrt{2}$, find $|\mathbf{q}|$.

31 Given that $\mathbf{p} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, find the value of the scalar t such that $\mathbf{p} + t\mathbf{q}$ is perpendicular to $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

32 Given that the vectors $t\mathbf{i} - 3\mathbf{k}$ and $2t\mathbf{i} + \mathbf{j} + t\mathbf{k}$ are perpendicular, find the possible values of t.

33 $\mathbf{a}$ is a unit vector perpendicular to $\mathbf{b}$. Find the value of $\mathbf{a} \cdot (2\mathbf{a} - 3\mathbf{b})$.

34 Points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - 19\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\lambda\mathbf{i} + (\lambda + 2)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -6\mathbf{i} - 15\mathbf{j} + 7\mathbf{k}$.

a Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above

b find the angles of the triangle ABC

c find the area of the triangle ABC.

35 Given that $\mathbf{a}$ and $\mathbf{b}$ are two vectors of equal magnitude such that $(3\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - 3\mathbf{b})$, prove that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

36 $ABCD$ is a parallelogram with $AB \parallel DC$. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

a Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Simplify $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{b}-\mathbf{a})$.

c Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular.

37 Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

a Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

b Find the value of λ for which CBA is a right angle.

c For the value of λ found above, calculate the exact distance from C to the line OA.

8D Equation of a line in three dimensions

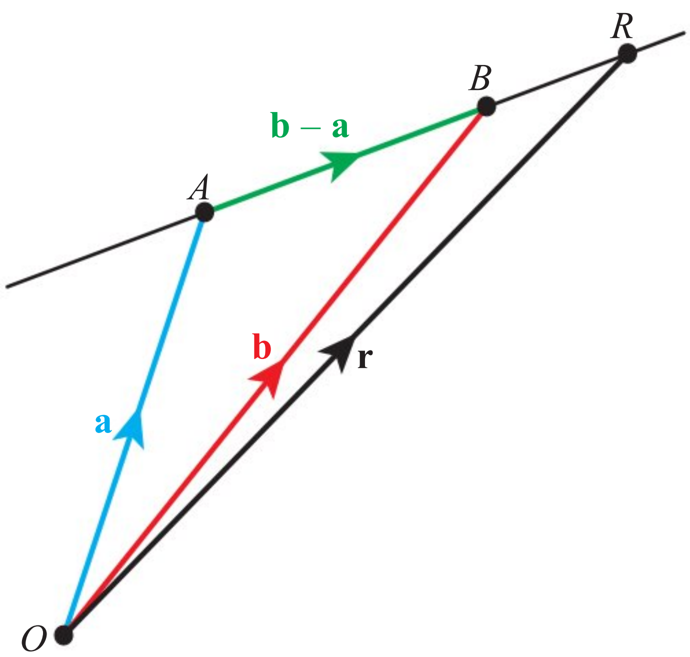
Vector equation of a line

In the previous section, you learnt how to check that three points are collinear. You can use the same idea to find a vector equation of a straight line. This is an equation that gives the position vector of any point on the line.

Consider a straight line through points A and B, with position vectors $\mathbf{a}$ and $\mathbf{b}$. For any other point R on the line, the vector $\overrightarrow{AR}$ is in the same direction as $\overrightarrow{AB}$, so you can write $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ for some scalar λ. Using position vectors, this equation becomes:

$$\mathbf{r}-\mathbf{a} = \lambda(\mathbf{b}-\mathbf{a})$$

This can be rearranged to express the position vector $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.



KEY POINT 8.16

The equation of the line through points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$$

Tip

You can write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to find the coordinates of a point on the line.

Different values of λ give position vectors of different points on the line. For example, you can check $\lambda = 0$ gives point A, $\lambda = 1$ gives point B, and $\lambda = 0.5$ gives $\mathbf{r} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, which is the midpoint of AB.

WORKED EXAMPLE 8.19

a Find the equation of the straight line through the point $A(2, -1, 3)$ and $B(1, 1, 5)$.

b Determine whether the point $C(1.5, 0, 4.5)$ lies on this line.

a

First find the vector $\mathbf{b} - \mathbf{a}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

The equation of line is:

Use $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

b

Find λ such that

Is there a value of λ which gives $\mathbf{r} = \mathbf{c}$?

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0 \\ 4.5 \end{pmatrix}$$

This represents three equations, one for each component

$$\begin{cases} 2 - \lambda = 1.5 & (1) \\ -1 + 2\lambda = 0 & (2) \\ 3 + 2\lambda = 4.5 & (3) \end{cases}$$

The same value of λ needs to satisfy all three equations. Find λ from the first equation and check in the other two

(1): $\lambda = 0.5$

(2): $-1 + 2(0.5) = 0$

(3): $3 + 2(0.5) = 4 \neq 4.5$

There is no value of λ which satisfies all three equations

Point C does not lie on the line.

You will learn more about recognizing which equations describe the same line in the next section.

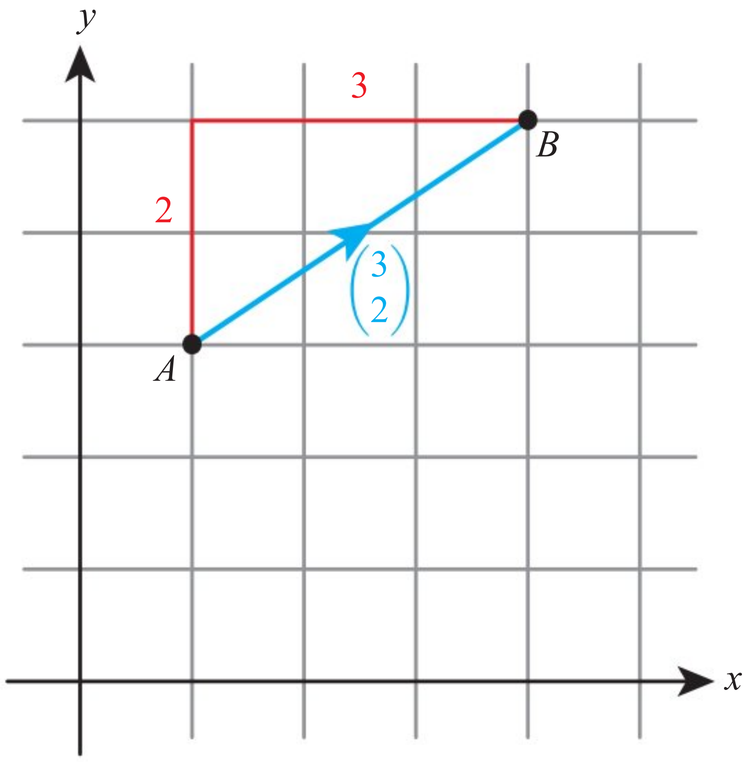
Tip

You could have used point B instead of A to write the equation in Worked Example 8.14 as $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$. Or you could use vector $\overrightarrow{BA}$ instead of $\overrightarrow{AB}$ to get $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

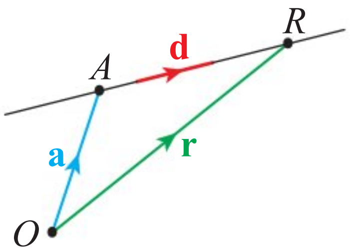
All of those equations represent the same line, but the value of λ for a given point on the line is different for different equations. For example, the point $(0, 3, 7)$ corresponds to $\lambda = 1$ in the first equation, and $\lambda = -2$ in the second equation.

The vector equation of a line takes the same form in two dimensions. For example, the equation of the line through the points $(1, 3)$ and $(4, 5)$ can be written as $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

If you draw a diagram, you can see how the vector **b** – **a** is related to the gradient of the line:



The vector **b** – **a** is a **direction vector** of the line. You can find the equation of a line if you know only one point and a direction vector.



KEY POINT 8.17

A vector equation of the line with direction vector **d** passing through the point **a** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$.

Tip

You can use any multiple of $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ as a direction vector.

WORKED EXAMPLE 8.20

Write down a vector equation of the line with direction vector $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ passing through the point (–3, 3, 5).

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where **a** is the position vector of a point on the line and **d** is a direction vector

$$\mathbf{r} = \begin{pmatrix}-3\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\2\end{pmatrix}$$

You saw above that, in two dimensions, the direction vector is related to the gradient of the line. In three dimensions, it is not possible to replace the direction vector by a single number. You will see later how to convert from vector to Cartesian equation, in both two and three dimensions.

Parametric form of the equation of a line

You can rewrite the vector equation of a line as three separate equations for x, y and z.

To do this, just remember that $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

KEY POINT 8.18

The **parametric form** of the equation of a line is found by expressing x, y and z in terms of λ.

WORKED EXAMPLE 8.21

Write the parametric form of the equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$.

Write r as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$$

Write a separate equation for each component

$$\Rightarrow \begin{cases} x = 3 - 4\lambda \\ y = -2 \\ z = 5\lambda \end{cases}$$

Cartesian form of the equation of a line

A Cartesian equation is a relationship between x and y (and z in three dimensions). You can get a Cartesian equation of a line by eliminating λ from the parametric equations. In two dimensions, this results in a familiar form of the equation of a line, $ax + by = c$.

Tip

If a line is parallel to the x-axis, the direction vector will be of the form $\begin{pmatrix} p \\ 0 \end{pmatrix}$ equation will be of the form $y = c$. If it is parallel to the y-axis, the direction vector will be of the form $\begin{pmatrix} 0 \\ q \end{pmatrix}$ and the equation will be of the form $x = c$.

WORKED EXAMPLE 8.22

A line has vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Find the Cartesian equation of the line in the form $ax + by = c$.

Use $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ to write x and y in terms of λ

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = -1 + 5\lambda \\ y = 2 - 3\lambda \end{cases}$$

Eliminate λ: make λ the subject of both equations…

$$\Rightarrow \begin{cases} \lambda = \dfrac{x+1}{5} \\ \lambda = \dfrac{2-y}{3} \end{cases}$$

… then equate the two expressions for λ

$$\frac{x+1}{5} = \frac{2-y}{3}$$

Rearrange into the required form

$$\Leftrightarrow \quad 3x + 3 = 10 - 5y$$

$$\Leftrightarrow 3x + 5y = 7$$

In three dimensions, it is impossible to write a single equation relating x, y and z. The above method can still be used to eliminate λ, but you end up with two equations.

WORKED EXAMPLE 8.23

A line has a vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$. Find the Cartesian equation of the line.

Use $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to write x, y and z in terms of λ

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = 4 - 3\lambda \\ y = -3 + 5\lambda \\ z = 2 + \lambda \end{cases}$$

Eliminate λ: make λ the subject of each equation…

$$\Rightarrow \begin{cases} \lambda = \dfrac{4 - x}{3} \\ \lambda = \dfrac{y + 3}{5} \\ \lambda = z - 2 \end{cases}$$

… then equate the three expressions for λ

$$\therefore \frac{4 - x}{3} = \frac{y + 3}{5} = z - 2$$

Tip

The equation from Worked Example 8.23 can be written in the form

$$\frac{x - 4}{-3} = \frac{y + 3}{5} = \frac{z - 2}{1}$$

which allows you to 'read off' the components of the direction vector from the denominators of the fractions.

KEY POINT 8.19

To find the Cartesian equation of a line given its vector equation

- write $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of λ, giving three equations
- make λ the subject of each equation
- equate the three expressions for λ to get an equation of the form $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$.

If the direction vector has any components equal to 0, you need to adjust the form of the Cartesian equation slightly. You would still follow the first two steps from Key Point 8.19.

WORKED EXAMPLE 8.24

Find the Cartesian equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$.

Use $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to write x, y and z in terms of λ

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = 3 + 2\lambda \\ y = -1 - 3\lambda \\ z = 5 \end{cases}$$

Eliminate λ: you can only make λ the subject of the first two equations

$$\Rightarrow \begin{cases} \lambda = \dfrac{x-3}{2} \\ \lambda = \dfrac{y+1}{-3} \\ z = 5 \end{cases}$$

Equate the two expressions for λ. z does not depend on λ, so leave it as a separate equation

$$\therefore \frac{x-3}{2} = \frac{y+1}{-3}, \quad z = 5$$

You can reverse the procedure to go from Cartesian to vector equation.

KEY POINT 8.20

To find a vector equation of a line from a Cartesian equation in the form

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

- set each of the three expressions equal to λ
- express x, y and z in terms of λ
- write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to obtain $\mathbf{r}$ in terms of λ.

WORKED EXAMPLE 8.25

Find a vector equation of the line with Cartesian equation

a $\dfrac{x+1}{3} = \dfrac{y-4}{-7} = \dfrac{2z+1}{5}$

b $\dfrac{2x-1}{-5} = \dfrac{z+1}{3}, \; y = -2$

Set each expression equal to λ

a $\dfrac{x+1}{3} = \dfrac{y-4}{-7} = \dfrac{2z+1}{5} = \lambda$

Express x, y and z in terms of λ

$$\Rightarrow \begin{cases} x = 3\lambda - 1 \\ y = 4 - 7\lambda \\ z = \frac{5}{2}\lambda - \frac{1}{2} \end{cases}$$

Write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$...

$$\mathbf{r} = \begin{pmatrix} 3\lambda - 1 \\ 4 - 7\lambda \\ \frac{5}{2}\lambda - \frac{1}{2} \end{pmatrix}$$

... and then split into the part without λ and part containing λ

$$\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ \frac{-1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -7 \\ \frac{5}{2} \end{pmatrix}$$

Set the two expressions equal to λ

b $$\frac{2x-1}{-5} = \frac{z+1}{3} = \lambda$$

Express x and y in terms of λ

$$\Rightarrow \begin{cases} x = \frac{1}{2} - \frac{5}{2}\lambda \\ z = 3\lambda - 1 \end{cases}$$

Write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, remembering that $y = -2$

$$\mathbf{r} = \begin{pmatrix} \frac{1}{2} - \frac{5}{2}\lambda \\ -2 \\ 3\lambda - 1 \end{pmatrix}$$

Split into the part without λ and part containing λ

$$\therefore \mathbf{r} = \begin{pmatrix} \frac{1}{2} \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{-5}{2} \\ 0 \\ 3 \end{pmatrix}$$

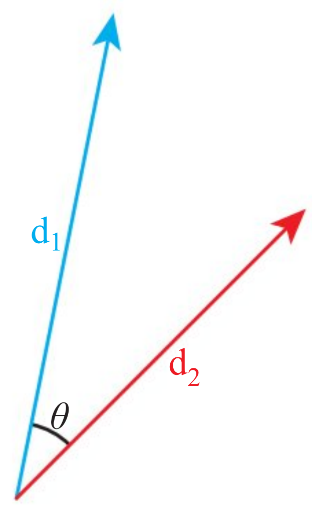
Angle between two lines

You can find the angle between two lines by using their direction vectors.

You learnt how to use the dot product to find the angle between two vectors in Section 8C.

KEY POINT 8.21

The angle between two lines is equal to the angle between their direction vectors.



WORKED EXAMPLE 8.26

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Identify the direction vectors

Direction vectors:

$$\mathbf{d}_1 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}, \ \mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Use scalar product to find the angle between the direction vectors

$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}$$

$$= \frac{(-4+0-6)}{\sqrt{4^2+0^2+3^2}\sqrt{1^2+1^2+2^2}}$$

$$= -0.816$$

$$\theta = \cos^{-1}(-0.816) = 144.7°$$

The question asks for the acute angle

$$180 - 144.7 = 35.3°$$

Be the Examiner 8.2

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{d}_1 \cdot \mathbf{d}_2 = 20+0+6 = 26$	$\mathbf{d}_1 \cdot \mathbf{d}_2 = -4-10+3 = -11$	$\mathbf{d}_1 \cdot \mathbf{d}_2 = -4-10+3 = -11$
$\lvert\mathbf{d}_1\rvert = \sqrt{25+0+4} = \sqrt{29}$	$\lvert\mathbf{d}_1\rvert = \sqrt{1+4+9} = \sqrt{14}$	$\lvert\mathbf{d}_1\rvert = \sqrt{1+4+9} = \sqrt{14}$
$\lvert\mathbf{d}_2\rvert = \sqrt{16+1+9} = \sqrt{26}$	$\lvert\mathbf{d}_2\rvert = \sqrt{16+25+1} = \sqrt{42}$	$\lvert\mathbf{d}_2\rvert = \sqrt{16+25+1} = \sqrt{42}$
$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$	$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$	$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$
$= \frac{26}{\sqrt{29\times 26}} = 0.947$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$
$\theta = 18.8°$	$\theta = 117°$	$\theta = 117°$
	So, acute angle $= 180 - 117 = 63.0°$	So, acute angle $= 117 - 90 = 27.0°$

Tip

Remember that you can use the scalar product to identify perpendicular vectors. This is particularly useful for finding perpendicular distances.

WORKED EXAMPLE 8.27

Line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, −2).

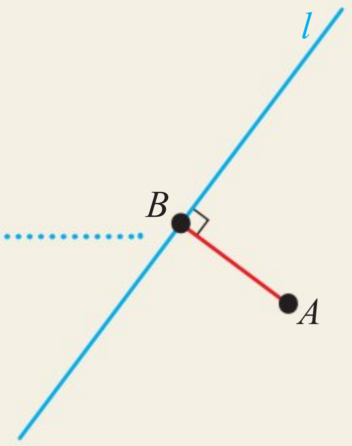
Point B lies on the line l and AB is perpendicular to l.

a Find the coordinates of B.

b Hence, find the shortest distance from A to l.

a

Draw a diagram. The line AB should be perpendicular to the direction vector of l



$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

We know that B lies on l, so its position vector is given by the equation for $\mathbf{r}$

$$\mathbf{b} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ t \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$= \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

We can now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$(\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\lambda = -4$$

Using this value of λ in the equation of the line gives the position vector of B

$$\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

B has coordinates (−1, 3, −4)

b

The shortest distance from a point to a line is the perpendicular distance, in other words, the distance AB

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{16+36+4} = 2\sqrt{14}$$

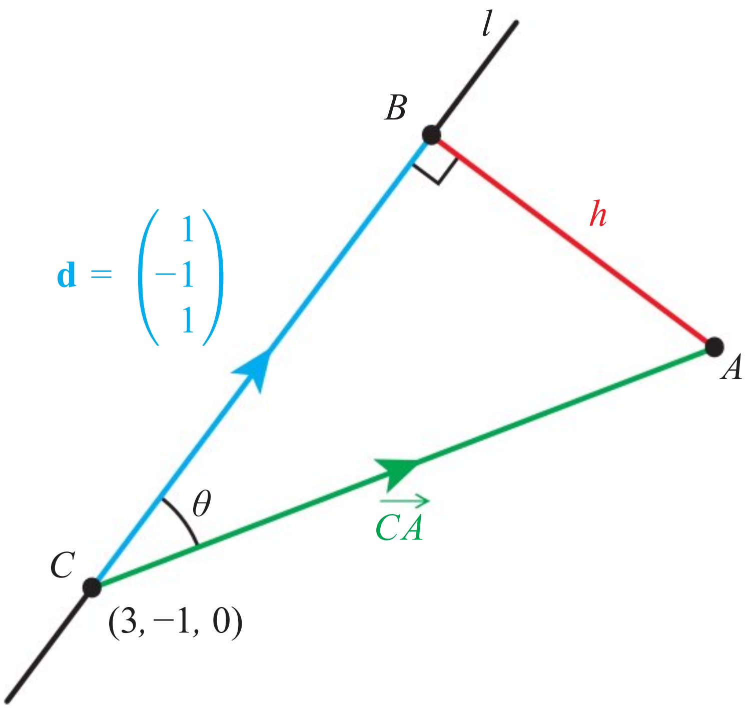
TOOLKIT: Problem Solving

Here are two alternative methods to solve the problem in Worked Example 8.27.

1 You found that $\overrightarrow{AB} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$. Show that $AB^2 = 3\lambda^2 + 24\lambda + 104$.

Hence, find the smallest possible value of AB.

2 In the diagram below, h is the required shortest distance from A to the line. You know, from the equation of the line, that the point $C(3, -1, 0)$ lies on the line. θ is the angle between $\overrightarrow{CA}$ and the direction vector of the line.



a Find the length AC.

b Find the exact value of $\cos\theta$. Hence, show that $\sin\theta = \frac{6}{\sqrt{78}}$.

c Use the right-angled triangle ABC to find the length h.

Applications to kinematics

You have probably used the equation $s = vt$ for an object moving in a straight line in one dimension. In this equation, v is the constant velocity and s is the displacement from the starting point at time t.

When an object moves in two or three dimensions, vectors are needed to describe its displacement and velocity. If the velocity is constant, the object will move in a straight line and the displacement from the starting position will still be $t\mathbf{v}$, where $\mathbf{v}$ is the velocity vector. The actual position of the object (relative to the origin) can be found by adding this displacement to the initial position vector.

KEY POINT 8.22

For an object moving with constant velocity $\mathbf{v}$ from the starting position $\mathbf{r}_0$, the position after time t is given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

The speed of the object is $v = |\mathbf{v}|$.

Notice that the equation in Key Point 8.22 represents a straight line with direction vector **v**, where t plays the role of the parameter λ. It is indeed the equation of the line along which the object moves.

WORKED EXAMPLE 8.28

An object moves with constant velocity $\mathbf{v} = (2\mathbf{i} - \mathbf{j} + 5\mathbf{k})\,\text{m s}^{-1}$. Its position vector when $t = 0$ is $\mathbf{r}_0 = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\,\text{m}$.

a Find the speed of the object.

b Write down an equation for the position of the object at time t seconds.

c Find the distance of the object from the origin when $t = 5$ seconds.

Speed is the modulus of the velocity

a $v = |\mathbf{v}| = \sqrt{4+1+25}$

$= 5.48\,\text{m s}^{-1}$

Use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

b $\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

First find the position vector when $t = 5$

c When $t = 5$:

$\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + 5(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

$= (14\mathbf{i} - 2\mathbf{j} + 21\mathbf{k})$

Distance is the magnitude of the displacement vector

Distance $= |\mathbf{r}| = \sqrt{14^2 + 2^2 + 21^2}$

$= 25.3\,\text{m}$

TOOLKIT: Modelling

The velocity of an aeroplane is modelled by the constant vector $(p\mathbf{i} + q\mathbf{j} + 0\mathbf{k})\,\text{km h}^{-1}$.

a Suggest suitable directions for the unit base vectors **i**, **j** and **k**.

b What assumptions have been made in this model? Are those assumptions reasonable?

c Can you suggest other ways of modelling the motion of an aeroplane?

Exercise 8D

For questions 1 to 3, use the method demonstrated in Worked Example 8.19 to find the equation of the line through A and B, and determine whether point C lies on the line.

1 **a** $A(2, 1, 5)$, $B(1, 3, 7)$, $C(0, 5, 9)$

b $A(-1, 0, 3)$, $B(3, 1, 8)$, $C(-5, -1, 3)$

2 **a** $A(4, 0, 3)$, $B(8, 0, 6)$, $C(0, 0, 2)$

b $A(-1, 5, 1)$, $B(-1, 5, 8)$, $C(-1, 3, 8)$

3 **a** $A(4, 1)$, $B(1, 2)$, $C(5, -2)$

b $A(2, 7)$, $B(4, -2)$, $C(1, 11.5)$

For questions 4 to 6, use the method demonstrated in Worked Example 8.2 to write down a vector equation of the line with the give direction vector passing through the given point.

4 **a** Point (1, 0, 5), direction $\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ **b** Point (–1, 1, 5), direction $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

5 **a** Direction $\mathbf{i} - 3\mathbf{k}$, point (0, 2, 3) **b** Direction $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, point (4, –3, 0)

6 **a** Direction $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, point (4, –1) **b** Direction $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, point (4, 1)

For questions 7 to 9, use the method demonstrated in Worked Example 8.2 to write the parametric form of the equation of the line with the given vector equation.

7 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

8 **a** $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j}) + \lambda(\mathbf{i} - 4\mathbf{k})$ **b** $\mathbf{r} = (3\mathbf{j} - \mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j})$

9 **a** $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

For questions 1to 12, use the method demonstrated in Worked Example 8.22 to write the equation of each line in the form $ax + by = c$.

10 **a** $\mathbf{r} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

11 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

12 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

For questions 13 to 15, use the method demonstrated in Worked Example 8.23 to find the Cartesian equation of each line.

13 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$

14 **a** $\mathbf{r} = \begin{pmatrix} 1/2 \\ -2 \\ 4/3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 2/3 \\ 2 \\ -1/2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

15 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1/2 \\ 1/3 \\ -3 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1/2 \\ 2/3 \end{pmatrix}$

For questions 16 to 18, use the method demonstrated in Worked Example 8.24 to find the Cartesian equation of each line.

16 **a** $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

17 **a** $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

18 a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For questions 19 to 22, use the method demonstrated in Worked Example 8.25 to find a vector equation of each line.

19 a $\frac{x-2}{5} = \frac{y+1}{-3} = \frac{z-3}{1}$ b $\frac{x+5}{2} = \frac{y-2}{7} = \frac{z+2}{1}$

20 a $\frac{2x-3}{2} = \frac{4-y}{2} = \frac{z-1}{3}$ b $\frac{3x+1}{4} = y-3 = \frac{4-z}{2}$

21 a $\frac{x-2}{3} = \frac{y+1}{5},\ z = 4$ b $\frac{x+1}{3} = \frac{y-1}{-4},\ z = -2$

22 a $x = 3, y = -4$ b $y = 2, z = -2$

For questions 23 to 25, use the method demonstrated in Worked Example 8.26 to find the acute angle between the two given lines.

23 a $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

24 a $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

25 a $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

26 a Find a vector equation of the line passing through the points (3, –1, 5) and (–1, 1, 2).

b Determine whether the point (0, 1, 5) lies on the line.

27 Find the Cartesian equation of the line passing through the point (–1, 1, 2) parallel to the line with vector equation $\mathbf{r} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{i}) + \lambda(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$.

28 Find the acute angle between lines with equations $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

29 Show that the lines with equations $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ are perpendicular.

30 Determine whether the point $A(3, -2, 2)$ lies on the line with equation $\frac{x+1}{2} = \frac{4-y}{3} = \frac{2z}{3}$.

31 A particle moves with constant velocity $\mathbf{v} = (0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})\,\text{m s}^{-1}$. At $t = 0$ seconds the particle is at the point with the position vector $(12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k})\,\text{m}$.

a Find the speed of the particle.

b Write down an equation for the position vector of the particle at time t seconds.

c Does the particle pass through the point (16, 8, 14)?

32 Two particles move so that their position vectors at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 10 \\ 5 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0.5 \\ 2 \\ -0.5 \end{pmatrix}$$

The distance is measured in metres. Find the distance between the particles when $t = 3$ seconds.

33 A line is given by parametric equations $x = 3 - \lambda,\ y = 4\lambda,\ z = 2 + \lambda$.

a The point $(0,\ p,\ q)$ lies on the line. Find the values of p and q.

b Find the angle the line makes with the z-axis.

34 a Show that the points $A(4, -1, -8)$ and $B(2, 1, -4)$ lie on the line l with equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

b Find the coordinates of the point C on the line l such that $AB = BC$.

35 a Find the vector equation of line l through points $P(7, 1, 2)$ and $Q(3, -1, 5)$.

b Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R.

36 a Write down the vector equation of the line l through the point $A(2, 1, 4)$ parallel to the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

b Calculate the magnitude of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

c Find the possible coordinates of point P on l such that $AP = 35$.

37 a Find the Cartesian equation of the line with parametric equation $x = 3\lambda + 1,\ y = 4 - 2\lambda,\ z = 3\lambda - 1$.

b Find the unit vector in the direction of the line.

38 a Find a vector equation of the line with Cartesian equation

$$\frac{(3-x)}{2} = \frac{(3z+1)}{4},\quad y = -1.$$

b Find the value of p so that the point $(2, -1,\ p)$ lies on the line.

39 A line has Cartesian equation

$$\frac{(2x-1)}{3} = \frac{2-z}{4},\quad y = 7.$$

a Find a vector equation of the line.

b Find the angle that the line makes with the x-axis.

40 Find, in degrees, the acute angle between the lines $\frac{x-3}{5} = y - 2 = \frac{3-2z}{2}$ and $\frac{x+1}{3} = 3 - z,\ y = 1$.

41 Line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates $(7, 2, 3)$.

Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

42 An object moves with a constant velocity. Its position vector at time t seconds is given by

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

where distance is measured in metres.

a Find the initial position of the object.

b Find the speed of the object.

c Find the distance of the object from the origin after 3 seconds.

43 Two particles move in a plane, each with a constant velocity. At $t = 0$, Particle 1 starts from the point $(3, 0)$ and its velocity is $(-2\mathbf{i} + 5\mathbf{j})$; Particle 2 starts from the point $(0, 5)$ and its velocity is $(4\mathbf{i} + \mathbf{j})$.

a Write down the position vector of each particle at time t.

b Find and simplify an expression for the distance between the two particles at time t.

c Hence find the minimum distance between the particles.

44 In this question, the distance is measured in km and the time in hours.

An aeroplane, initially at the point $(2, 0, 0)$, moves with constant speed $894\,\mathrm{km\,h^{-1}}$ in the direction of the vector $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. Find an equation for the position vector of the aeroplane at time t hours.

45 Given line $l : \mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and point $P(21, 5, 10)$,

a find the coordinates of point M on l such that PM is perpendicular to l

b show that the point $Q(15, -14, 17)$ lies on l

c find the coordinates of point R on l such that $|PR| = |PQ|$.

46 Two lines have equations $l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

a Show that the point $P\left(\frac{5}{6}, \frac{19}{6}, \frac{9}{2}\right)$ lies on both lines.

b Find, in degrees, the acute angle between the two lines.

Point Q has coordinates $(-1, 5, 10)$.

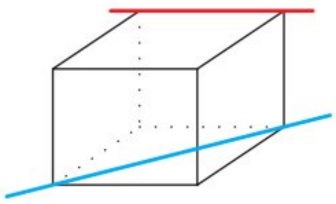
c Show that Q lies on l_2.

d Find the distance PQ.

e Hence find the shortest distance from Q to the line l_1.

47 Find the distance of the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

48 Find the shortest distance from the point $(-1, 1, 2)$ to the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.



Tip

Any pair of skew lines can be envisaged as running along an edge and a diagonal of opposite sides of a cuboid as shown here; skew lines lie in parallel planes but are not parallel to each other.

8E Intersection of lines

In two dimensions, two distinct lines either intersect or are parallel. In three dimensions there is one additional possibility: the two lines can be **skew**. These are lines which are neither intersecting nor parallel. They do not lie in the same plane. You need to be able to distinguish between the different cases and find the coordinates of the point of intersection in the case when the lines intersect.

CONCEPTS – SPACE

Some properties of objects depend on the dimension they occupy in **space**. One of the most interesting examples of this is diffusion, which is very important in physics and biology. If a large number of particles move randomly (performing a so-called random walk) in three dimensions, on average they will keep moving away from the starting point. This is, however, not the case in one or two dimensions where, on average, the particles will return to the starting point.

Tip

When working with two different lines, use two different letters (such as λ and μ) for the parameters.

Finding the point of intersection of two lines

Suppose two lines have vector equations $\mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{d}_1$ and $\mathbf{r}_2 = \mathbf{b} + \mu \mathbf{d}_2$. If they intersect, then there is a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\mathbf{r}$, this means that we need to find the values of λ and μ which make $\mathbf{r}_1 = \mathbf{r}_2$.

WORKED EXAMPLE 8.29

Find the coordinates of the point of intersection of the following pair of lines.

$$\mathbf{r} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

You need to make $\mathbf{r}_1 = \mathbf{r}_2$

$$\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

Write three separate equations, one for each component

$$\Rightarrow \begin{cases} 0 + \lambda = 1 + 4\mu \\ -4 + 2\lambda = 3 - 2\mu \\ 1 + \lambda = 5 - 2\mu \end{cases}$$

$$\Rightarrow \begin{cases} \lambda - 4\mu = 1 & (1) \\ 2\lambda + 2\mu = 7 & (2) \\ \lambda + 2\mu = 4 & (3) \end{cases}$$

Pick two equations to solve, then check the answers in the third. This case, subtracting (1) from (3) eliminates λ

$$(3) - (1) \Rightarrow 6\mu = 3$$

$$\mu = \frac{1}{2}, \ \lambda = 3$$

The values of λ and μ you have found must also satisfy equation (2)

$$(2): \ 2 \times 3 + 2 \times \frac{1}{2} = 7$$

The lines intersect.

The position of the intersection point is given by the vector $\mathbf{r}_1$ (or $\mathbf{r}_2$ – they should be the same – you should always check this)

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The lines intersect at the point (3,2 4).

Skew lines

If two lines are skew, it is impossible to find the values of λ and μ which solve all three equations.

WORKED EXAMPLE 8.30

Show that the lines with equations $\mathbf{r} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ are skew.

Try to make $\mathbf{r}_1 = \mathbf{r}_2$ and then show that this is not possible

$$\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} t - 2\lambda = 6 & (1) \\ t + 3\lambda = -2 & (2) \\ 4t - 2\lambda = -2 & (3) \end{cases}$$

Find t and λ from the first two equations

(1) and (2) $\Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$

The values found must also satisfy the third equation

(3): $4\left(\frac{14}{5}\right) - 2\left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$

This tells you that it is impossible to find t and λ to make $\mathbf{r}_1 = \mathbf{r}_2$

The two lines do not intersect.

The lines are not parallel:

The two lines don't intersect, so they could be parallel or skew. Parallel lines have parallel direction vectors

$$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \neq k \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

Hence, the two lines are skew.

Parallel and coincident lines

KEY POINT 8.23

If two lines with direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are parallel, then $\mathbf{d}_1 = k\mathbf{d}_2$ for some scalar k.

However, if the direction vectors are parallel, the lines could be parallel, but the two equations could also represent the same line. You can check whether this is the case by checking whether the position vector '**a**' which lies on one line also lies on the other line.

WORKED EXAMPLE 8.31

a Show that the equations $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$ represent the same straight line.

b Show that the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ \sqrt{6} \\ \sqrt{2} \end{pmatrix}$ are parallel.

a

First check that the lines have the same direction (parallel direction vectors)

$$\begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

So, the lines have the same direction.

Now check that the point (2, 2, 1) lies on the second line

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2 = 5 - 2\mu & \Rightarrow \mu = 1.5 \\ 2 = -1 + 2\mu & \Rightarrow \mu = 1.5 \\ 1 = 10 - 6\mu & \Rightarrow \mu = 1.5 \end{cases}$$

The point (2, 2, 1) lies on the second line.

The two lines have a common point and same direction, so they are the same line.

b

Check that the lines have the same direction

$$\begin{pmatrix} 2 \\ \sqrt{6} \\ \sqrt{2} \end{pmatrix} = \sqrt{2}\begin{pmatrix} \sqrt{2} \\ \sqrt{3} \\ 1 \end{pmatrix}$$

So, the lines have the same direction.

Now check whether the point (1, 5, 2) lies on the second line

$$\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ \sqrt{6} \\ \sqrt{2} \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1 = 2\mu & \Rightarrow \mu = \frac{1}{2} \\ 5 = \sqrt{6}\mu & \Rightarrow \mu = \frac{5}{\sqrt{6}} \neq \frac{1}{2} \end{cases}$$

The point (1, 5, 2) is on the first line but not the second line, so the two lines are different.

Therefore, the lines are parallel.

Exercise 8E

For questions 1 to 3, use the method demonstrated in Worked Example 8.29 to find the coordinates of the point of intersection of the two lines.

1 a $\mathbf{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

2 a $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

3 a $\mathbf{r} = (3\mathbf{i} + \mathbf{j}) + \lambda(2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$

b $\mathbf{r} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (8\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

For questions 4 and 5, use the method demonstrated in Worked Example 8.30 to show that the two lies are skew.

4 a $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ -11 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

5 a $\mathbf{r} = (\mathbf{i}+5\mathbf{j}+4\mathbf{k})+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = (3\mathbf{j}+5\mathbf{k})+\mu(\mathbf{i}+\mathbf{k})$

b $\mathbf{r} = (2\mathbf{i}+3\mathbf{j}-\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j}-4\mathbf{k})$ and $\mathbf{r} = (3\mathbf{i}+2\mathbf{k})+\mu(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})$

For questions 6 to 8, use the method demonstrated in Worked Example 8.31 to determine whether the two equations describe the same line, two parallel lines or non-parallel lines.

6 a $\mathbf{r} = (\mathbf{i}+3\mathbf{j}-\mathbf{k})+\lambda(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ and $(\mathbf{i}-2\mathbf{j}-\mathbf{k})+\mu(2\mathbf{i}-4\mathbf{j}-4\mathbf{k})$

b $\mathbf{r} = \begin{pmatrix}-1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1.5\\-0.5\\-1\end{pmatrix}$

7 a $\mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}0\\0\\1\end{pmatrix}$

b $\mathbf{r} = (4\mathbf{i}+\mathbf{j}+2\mathbf{k})+\lambda(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ and $(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})+\mu(4\mathbf{i}+\mathbf{j}+2\mathbf{k})$

8 a $\mathbf{r} = (-\mathbf{i}+3\mathbf{k})+\lambda(2\mathbf{i}-\mathbf{j})$ and $\mathbf{r} = (-3\mathbf{i}+\mathbf{j}+3\mathbf{k})+\mu(0.5\mathbf{j}-\mathbf{i})$

b $\mathbf{r} = (3\mathbf{j}+\mathbf{k})+\lambda(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = (-\mathbf{i}+5\mathbf{j})+\mu(-2\mathbf{i}+4\mathbf{j}-2\mathbf{k})$

9 Points A, B, C and D have coordinates $A(1, 0, 0)$, $B(6, 5, 5)$, $C(8, 3, 3)$, $D(6, 3, 3)$. Find the point of intersection of the lines AB and CD.

10 Determine whether these two lines intersect, are parallel or are skew.

l_1: $x=-3-\lambda,\ y=5-2\lambda,\ z=2-4\lambda$

l_2: $x=8-\mu,\ y=5-3\mu,\ z=1+3\mu$

11 Two lines have Cartesian equations

$\frac{x-7}{2}=\frac{y-1}{2}=z-5$ and $\frac{4-x}{3}=y+6=\frac{z+3}{5}$.

a For each line, express x and y in terms of z.

b Hence show that the lines intersect and find the coordinates of the intersection point.

12 Show that these two lines intersect, and find the coordinates of the intersection point.

$\mathbf{r} = \begin{pmatrix}1\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}7\\2\\7\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix}$

13 a Show that the equations $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\frac{x-5}{6}=\frac{y-7}{6}=\frac{z-5}{3}$ represent the same straight line.

b Show that the equation $\mathbf{r} = (4t-5)\mathbf{i}+(4t-3)\mathbf{j}+(1+2t)\mathbf{k}$ represents a different straight line.

14 a Find the coordinates of the point where the line with equation $\frac{x-6}{2}=\frac{y+1}{7}=\frac{z+9}{-3}$ intersects the y-axis.

b Show that the line does not intersect the z-axis.

15 a Find the coordinates of the point of intersection of the lines with Cartesian equations $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z+1}{1}$ and $5-x=\frac{y+2}{-3}=\frac{z-7}{2}$.

b Show that the line with equation $\mathbf{r} = \begin{pmatrix}7\\8\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ passes through the intersection point found in part a.

16 Find the value of p for which the lines with equations $\mathbf{r} = (\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}) + \lambda(\mathbf{i} + p\mathbf{k})$ intersect. Find the point of intersection in this case.

17 Two toy helicopters are flown, each in a straight line. The position vectors of the two helicopters at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Distance is measured in metres.

a Show that the paths of the helicopters cross.

b Determine whether the helicopters collide.

18 Two particles move so that their position vectors at time t are given by

$\mathbf{r}_1 = (1 + 2t)\mathbf{i} + (t - 3)\mathbf{j} + (3 + 7t)\mathbf{k}$ and $\mathbf{r}_2 = (9 - 2t)\mathbf{i} + (t - 2)\mathbf{j} + (22 + 2t)\mathbf{k}$.

a Find the speed of each particle.

b Determine whether the particles meet.

19 Two flies move so that their position vectors at time t seconds are given by

$\mathbf{r}_1 = (0.7\mathbf{j} + 3\mathbf{k}) + t(1.2\mathbf{i} + 0.8\mathbf{j} - 0.1\mathbf{k})$ and $\mathbf{r}_2 = (7.7\mathbf{i} + \mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + 0.3\mathbf{k})$

where distance is measured in metres. The base vectors **i** and **j** are in the horizontal plane and vector **k** points upwards.

a Show that there is a time when one fly is vertically above the other.

b Find the distance between the flies at that time.

20 In this question, distance is measured in kilometres and time in hours.

A boat is moving with the constant velocity $(64\mathbf{i})\,\text{km h}^{-1}$. At time $t = 0$, it is located at the origin. A small submarine is located at the point (0, 0.5, –0.02). At time $t = 0$, it starts moving with a constant velocity in the direction of the vector $(40\mathbf{i} - 25\mathbf{j} + c\mathbf{k})$. Given that the submarine reaches the boat

a find the value of c

b find the speed of the submarine.

21 Lines l_1 and l_2 have equations

$l_1 : \mathbf{r} = (\mathbf{i} - 10\mathbf{j} + 12\mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

$l_2 : \mathbf{r} = (4\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{k})$.

P is a point on l_1 and Q is a point on l_2 such that $\overrightarrow{PQ}$ is perpendicular to both lines.

a Show that $26\lambda + 7\mu = 64$ and find another equation for λ and μ.

b Hence find the shortest distance between the lines l_1 and l_2.

8F Vector product and areas

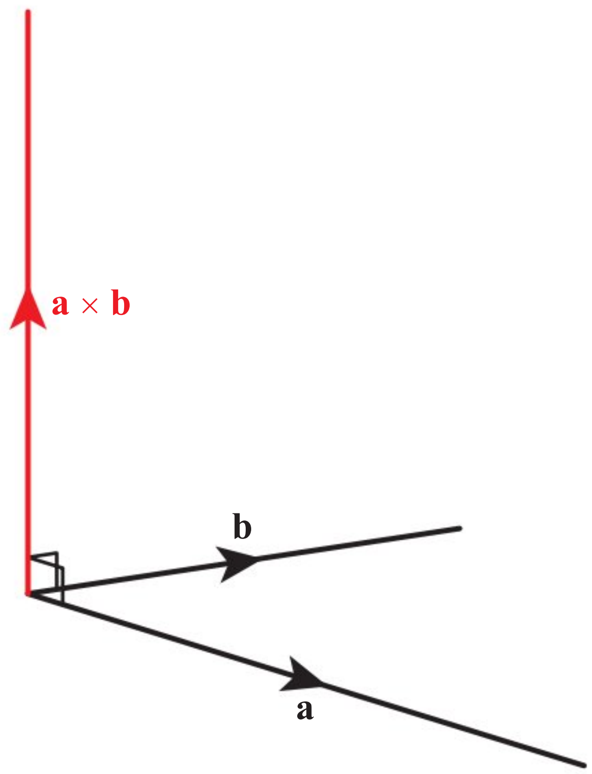
The definition of the vector product

One way to define the vector product is to give its magnitude and direction.

KEY POINT 8.24

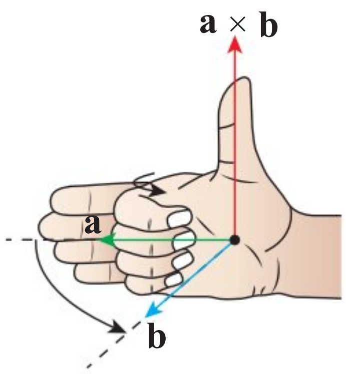
The **vector product** (or **cross product**) of vectors **a** and **b** is a vector denoted by $\mathbf{a} \times \mathbf{b}$.

- The magnitude is equal to $|\mathbf{a}||\mathbf{b}|\sin\theta$ (where θ is the angle between **a** and **b**).
- The direction is perpendicular to both **a** and **b** (as shown in the diagram).



Tip

The direction of the vector $\mathbf{a} \times \mathbf{b}$ is given by the right-hand screw rule.

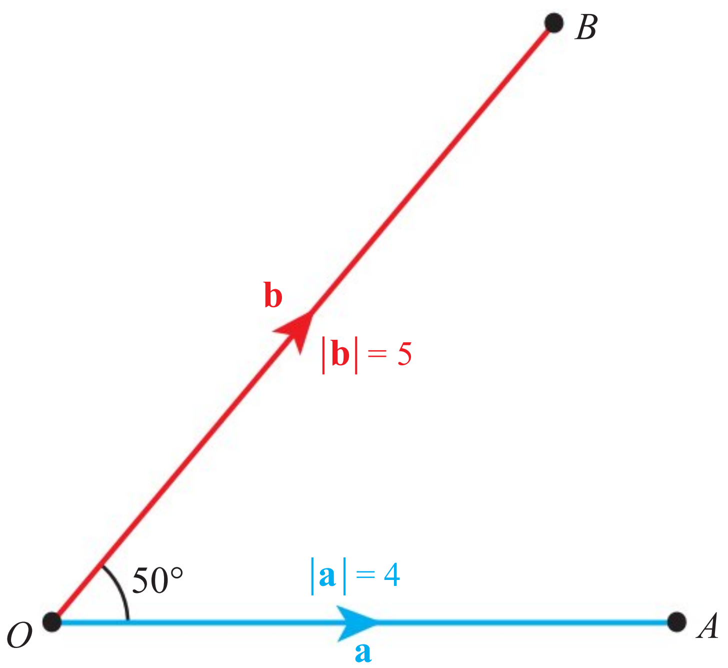


Links to: Physics

Vectors are used to model quantities in both mathematics and physics. In geometry, the vector product is used to find the direction which is perpendicular to two given vectors. In physics, it is used in many equations involving quantities which are modelled as vectors. For example, the angular momentum of a particle moving in a circle is given by $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ is the position vector and $\mathbf{v}$ the velocity of the particle. In electrodynamics, the Lorentz force acting on a charge q moving with velocity $\boldsymbol{v}$ in magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

WORKED EXAMPLE 8.32

For the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in the diagram, find the magnitude of $\mathbf{a} \times \mathbf{b}$.



Use $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$|\mathbf{a} \times \mathbf{b}| = (4)(5)\sin 50°$$
$$= 15.3$$

Tip

This formula is given in the Mathematics: analysis and approaches formula booklet.

The vector product can also be expressed in component form.

KEY POINT 8.25

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

WORKED EXAMPLE 8.33

Given that $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, find the vector $\mathbf{a} \times \mathbf{b}$.

Use the formula for the component form of $\mathbf{a} \times \mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (-3)(5) - (2)(1) \\ (2)(2) - (1)(5) \\ (1)(1) - (-3)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$$

You can check that $\begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

Be the Examiner 8.3

Find $\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\begin{pmatrix} 6-5 \\ 20+6 \\ 8+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 26 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 20-6 \\ 8-2 \\ 6-5 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 20+6 \\ -(8+2) \\ 6-5 \end{pmatrix} = \begin{pmatrix} 26 \\ -10 \\ 1 \end{pmatrix}$

Algebraic properties of vector product

Vector product has many properties similar to multiplication of numbers. The most important difference is that $\mathbf{a} \times \mathbf{b}$ is not the same as $\mathbf{b} \times \mathbf{a}$. For example, try calculating $\mathbf{b} \times \mathbf{a}$ in Worked Example 8.33.

KEY POINT 8.26

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

Tip

Notice that, since the vector product produces a vector, each zero in Key Point 8.27 is the zero *vector*, not a scalar value.

It is worth remembering the special result for parallel and perpendicular vectors, which follows from the fact that $\sin 0° = \sin 180° = 0$ and $\sin 90° = 1$.

KEY POINT 8.27

- If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

WORKED EXAMPLE 8.34

Given that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors with $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$, find the magnitude of $(\mathbf{a} + \mathbf{b}) \times (3\mathbf{a} + 5\mathbf{b})$.

Expand the brackets to try and simplify the expression

$$(\mathbf{a} + \mathbf{b}) \times (3\mathbf{a} + 5\mathbf{b})$$
$$= \mathbf{a} \times (3\mathbf{a}) + \mathbf{a} \times (5\mathbf{b}) + \mathbf{b} \times (3\mathbf{a}) + \mathbf{b} \times (5\mathbf{b})$$

Use $\mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$

$$= 3(\mathbf{a} \times \mathbf{a}) + 5(\mathbf{a} \times \mathbf{b}) + 3(\mathbf{b} \times \mathbf{a}) + 5(\mathbf{b} \times \mathbf{b})$$

$\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$

$$= 3(\mathbf{0}) + 5(\mathbf{a} \times \mathbf{b}) - 3(\mathbf{a} \times \mathbf{b}) + 5(\mathbf{0})$$
$$= 2(\mathbf{a} \times \mathbf{b})$$

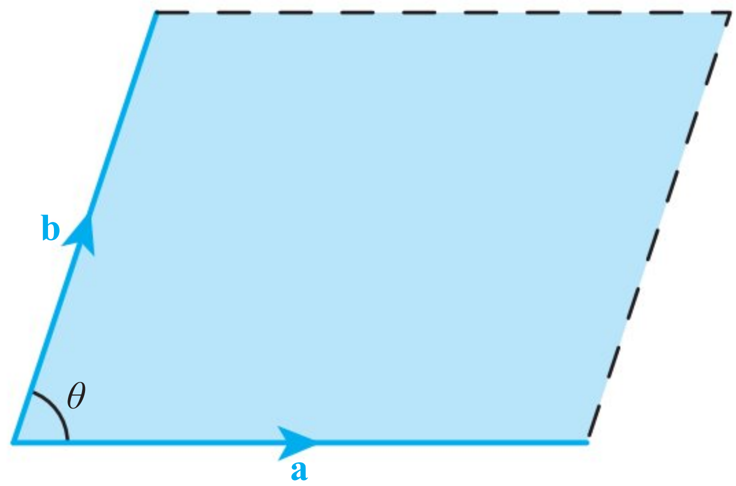
For perpendicular vectors, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$

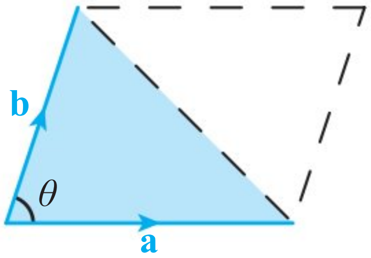
Hence,

$$|(\mathbf{a} + \mathbf{b}) \times (3\mathbf{a} + 5\mathbf{b})| = 2\,|\mathbf{a}||\mathbf{b}|$$
$$= 24$$

Areas of parallelograms and triangles

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}| \sin\theta$. But this is also the area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$.





A parallelogram can be divided into two triangles, so you can also use the vector product to find the area of a triangle.

KEY POINT 8.28

The area of the triangle with two sides defined by vectors **a** and **b** is equal to $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

When you are given the coordinates of the vertices of a triangle, sketch a diagram to identify which two vectors to use.

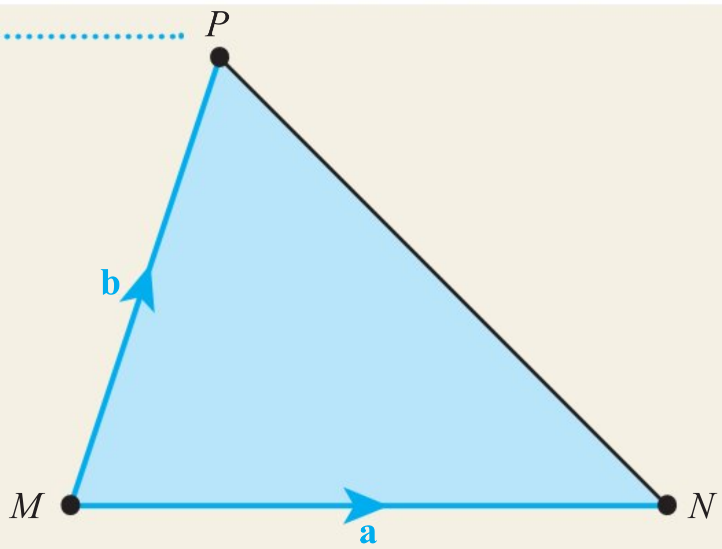
Tip

It does not matter which two sides of the triangle you use.

WORKED EXAMPLE 8.33

Find the area of the triangle with vertices $M(1, 4, 2)$, $N(3, -3, 0)$ and $P(-1, 8, 9)$.

Sketch a diagram to see which vectors to use



Two of the sides of the triangle are vectors $\overrightarrow{MN}$ and $\overrightarrow{MP}$

$$\mathbf{a} = \overrightarrow{MN} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{MP} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

The area of the triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. Find the vector $\mathbf{a}\times\mathbf{b}$ first…

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -49+8 \\ 4-14 \\ 8-14 \end{pmatrix} = \begin{pmatrix} -41 \\ -10 \\ -6 \end{pmatrix}$$

… then find half of its magnitude

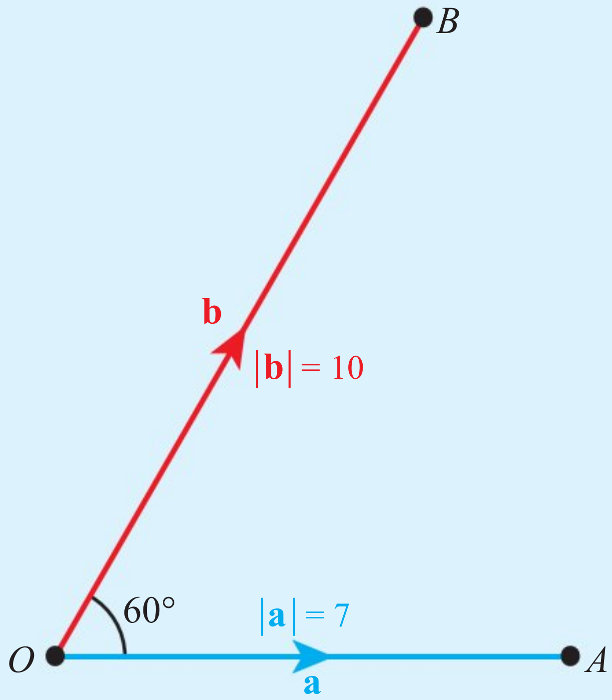
$$|\mathbf{a}\times\mathbf{b}| = \sqrt{41^2+10^2+6^2} = 42.6$$

$$\text{Area} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = 21.3$$

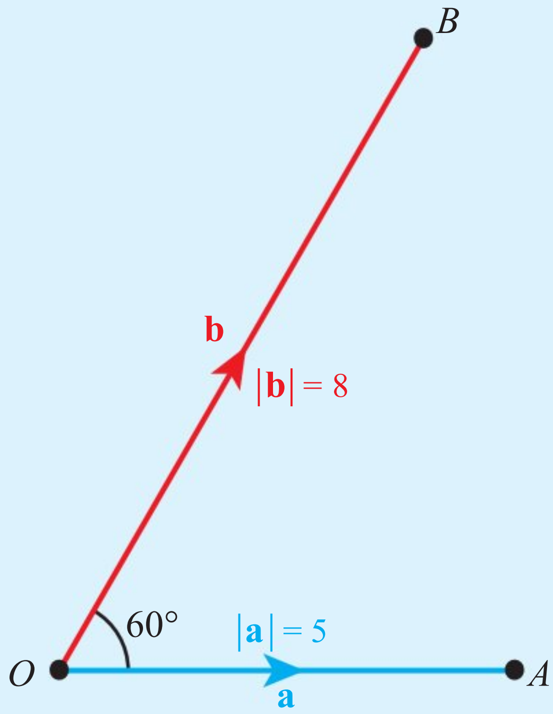
Exercise 8F

For questions 1 to 3, use the method demonstrated in Worked Example 8.32 to find the magnitude of $\mathbf{a} \times \mathbf{b}$ for the vectors in each diagram.

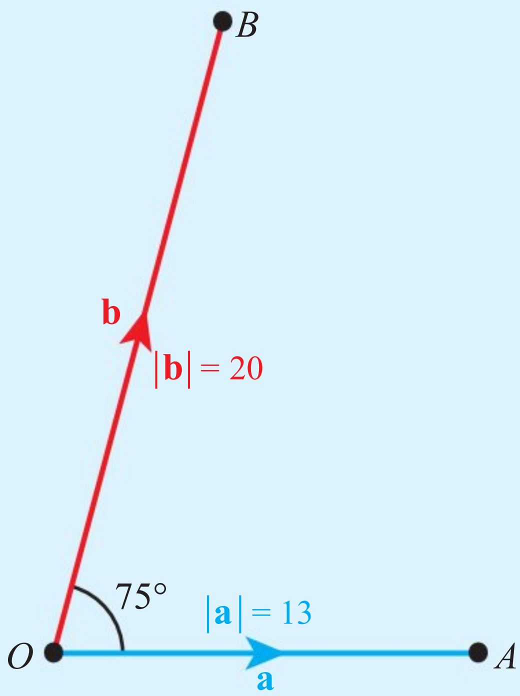
1 a



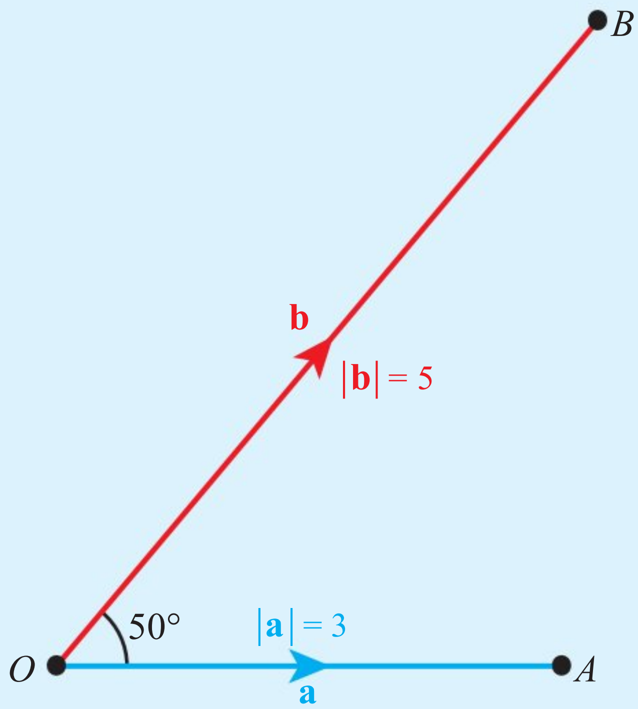
b



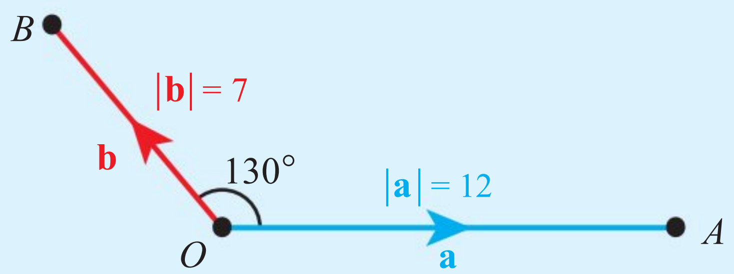
2 a



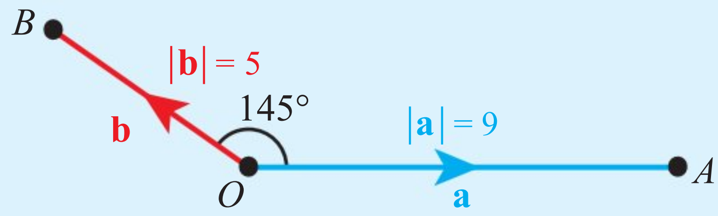
b



3 **a**



b



For questions 4 to 6, use the method demonstrated in Worked Example 8.33 to find $\mathbf{a}\times\mathbf{b}$ for the two given vectors.

4 **a** $\mathbf{a}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}5\\2\\2\end{pmatrix}$ **b** $\mathbf{a}=\begin{pmatrix}4\\1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}$

5 **a** $\mathbf{a}=\begin{pmatrix}3\\-1\\4\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-1\\1\\5\end{pmatrix}$ **b** $\mathbf{a}=\begin{pmatrix}1\\7\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}$

6 **a** $\mathbf{a}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$ **b** $\mathbf{a}=-3\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-4\mathbf{k}$

For questions 7 to 10, **a** and **b** are perpendicular vectors with $|\mathbf{a}|=2$ and $|\mathbf{b}|=5$. Use the method demonstrated in Worked Example 8.34 to find the magnitude of the following vectors:

7 **a** $\mathbf{a}\times(2\mathbf{a}+5\mathbf{b})$ **b** $\mathbf{a}\times(4\mathbf{a}+3\mathbf{b})$

8 **a** $3\mathbf{b}\times(4\mathbf{a}-5\mathbf{b})$ **b** $2\mathbf{b}\times(4\mathbf{a}-5\mathbf{b})$

9 **a** $(2\mathbf{a}+\mathbf{b})\times(3\mathbf{a}+2\mathbf{b})$ **b** $(\mathbf{a}+4\mathbf{b})\times(2\mathbf{a}+5\mathbf{b})$

10 **a** $(2\mathbf{a}-\mathbf{b})\times(3\mathbf{a}+2\mathbf{b})$ **b** $(\mathbf{a}-4\mathbf{b})\times(2\mathbf{a}+5\mathbf{b})$

For questions 11 to 13, use the method demonstrated in Worked Example 8.35 to find the area of the triangle with given vertices.

11 **a** $(1, 3, 3)$, $(-1, 1, 2)$ and $(1, -2, 4)$ **b** $(3, -5, 1)$, $(-1, 1, 3)$ and $(-1, -5, 2)$

12 **a** $(-3, -5, 1)$, $(4, 7, 2)$ and $(-1, 2, 2)$ **b** $(4, 0, 2)$, $(4, 1, 5)$ and $(4, -3, 2)$

13 **a** $(1, 5, 2)$, $(8, 4, 6)$ and $(0, 6, 7)$ **b** $(2, 1, 2)$, $(3, 8, 4)$ and $(1, 3, -1)$

14 Given that $|\mathbf{a}|=5$, $|\mathbf{b}|=7$ and the angle between **a** and **b** is 30° find the exact value of $|\mathbf{a}\times\mathbf{b}|$.

15 Given that $|\mathbf{a}|=2$, $|\mathbf{b}|=5$ and $|\mathbf{a}\times\mathbf{b}|=7$ find, in radians, the acute angle between the directions of vectors **a** and **b**.

16 Given that $|\mathbf{a}|=7$, $|\mathbf{b}|=1$ and $\mathbf{a}\times\mathbf{b}=2\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ find, in radians, the acute angle between the directions of vectors **a** and **b**.

17 Find $|\mathbf{p}\times\mathbf{q}|$, where $\mathbf{p}=\begin{pmatrix}4\\1\\2\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

18 Find a vector perpendicular to both $\mathbf{a}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

19 **a** Find $\begin{pmatrix}2\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}$.

b Find a unit vector perpendicular to both $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

20 Find a unit vector perpendicular to both $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\5\end{pmatrix}$.

21 a Prove that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2\mathbf{a}\times\mathbf{b}$.

b Simplify $(2\mathbf{a}-3\mathbf{b})\times(3\mathbf{a}+2\mathbf{b})$.

22 a Explain why $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a}=0$.

b Evaluate $(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})$.

23 Given points A, B and C with coordinates $(3, -5, 1)$, $(7, 7, 2)$ and $(-1, 1, 3)$,

a calculate $\mathbf{p}=\overrightarrow{AB}\times\overrightarrow{AC}$ and $\mathbf{q}=\overrightarrow{BA}\times\overrightarrow{BC}$.

b What can you say about vectors $\mathbf{p}$ and $\mathbf{q}$?

24 For the vectors $\mathbf{a}=\begin{pmatrix}3\\1\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\\3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}2\\0\\1\end{pmatrix}$ check whether $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

25 A quadrilateral has vertices $A(1, 4, 2)$, $B(3, -3, 0)$, $C(1, 1, 7)$ and $D(-1, 8, 9)$.

a Show that the quadrilateral is a parallelogram.

b Find the area of the quadrilateral.

26 Find the area of the triangle with vertices $(2, 1, 2)$, $(5, 0, 1)$ and $(-1, 3, 5)$.

27 The points $A(3, 1, 2)$, $B(-1, 1, 5)$ and $C(7, -2, 3)$ are vertices of a parallelogram $ABCD$.

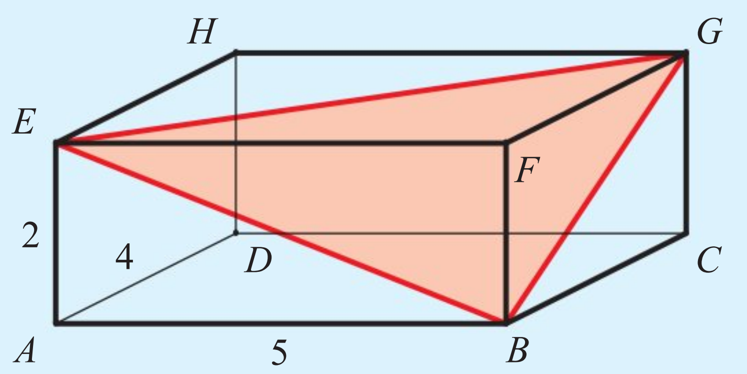
a Find the coordinates of D.

b Calculate the area of the parallelogram.

28 Prove that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\bullet\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2$.

29 Given that $\mathbf{p}+\mathbf{q}+\mathbf{r}=0$ show that $\mathbf{p}\times\mathbf{q}=\mathbf{q}\times\mathbf{r}=\mathbf{r}\times\mathbf{p}$.

30 A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are $A(0, 0, 0)$, $B(5, 0, 0)$, $D(0, 4, 0)$ and $E(0, 0, 2)$.



a Find the coordinates of the remaining four vertices.

Face diagonals BE, BG and EG are drawn as shown.

b Find the area of the triangle BEG.

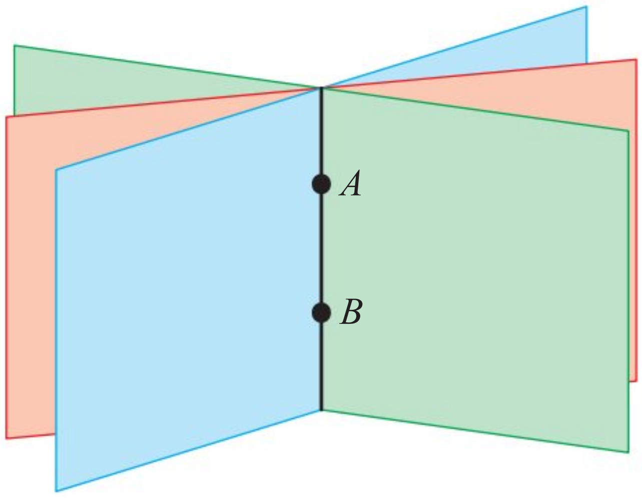
8G Equation of a plane

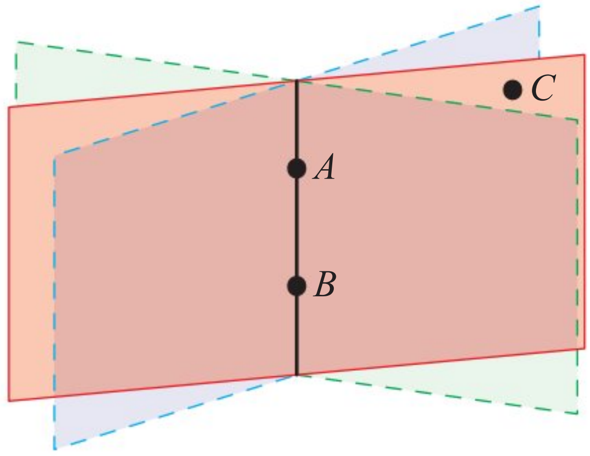
You saw in Section 8D that a line is determined by two points – if you have two distinct points A and B, there is only one straight line that passes through them, and its vector equation is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

However, there are infinitely many planes which contain those two points. You can pick one of those planes by specifying a third point, C, that lies in it.

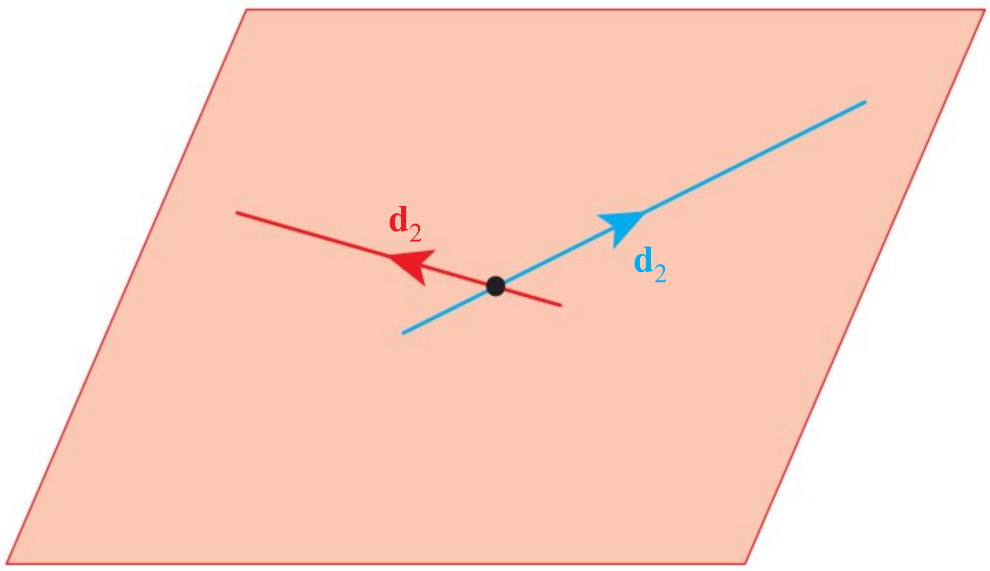
Tip

A plane is a flat surface which extends indefinitely in all directions.



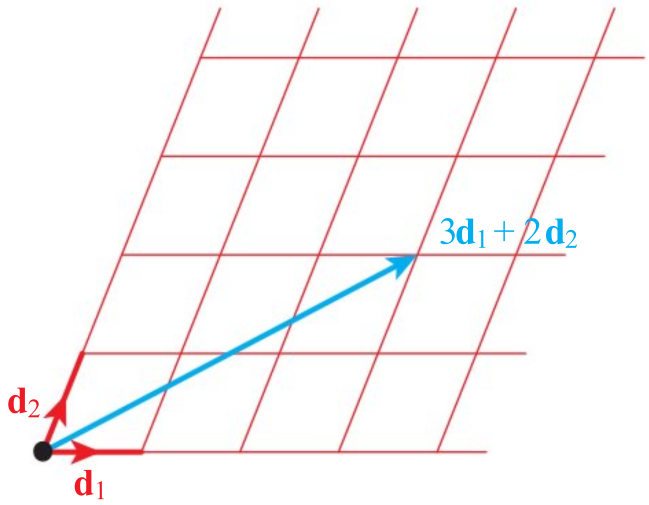


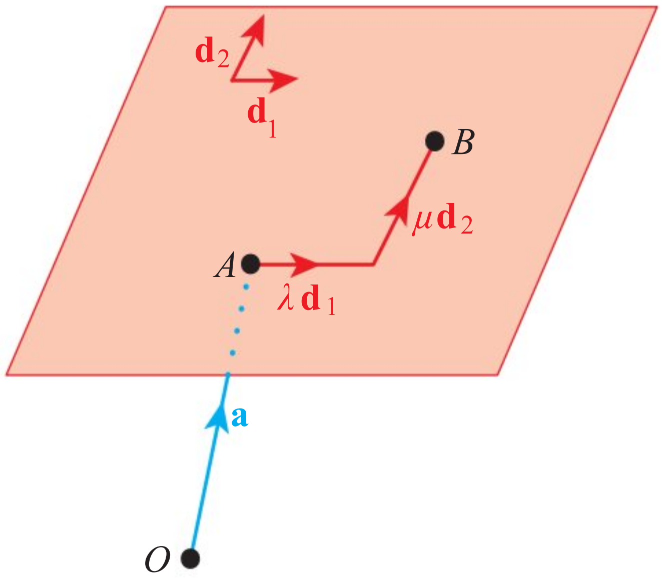
This suggests that a plane is uniquely determined by three points, or a line and a point outside the line. A plane can also be determined by two intersecting lines. You can use the two direction vectors of the lines to find a vector equation of the plane.



Vector equation of a plane

Suppose you have two non-parallel direction vectors in a plane, $\mathbf{d}_1$ and $\mathbf{d}_2$. Starting from a point in the plane you can reach any other point by travelling along the directions of $\mathbf{d}_1$ and $\mathbf{d}_2$, as illustrated in the first diagram below. If, instead, you start at the origin, you need to go to a point A in the plane first and then travel along the two directions, as in the second diagram.





KEY POINT 8.29

A vector equation of the plane containing point $\mathbf{a}$ and parallel to the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$$

WORKED EXAMPLE 8.36

a Write down a vector equation of a plane which contains the point (3, 1, 4) and is parallel to the vectors $\begin{pmatrix}-1\\1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\0\\5\end{pmatrix}$.

b Determine whether the point (3, –2, 5) lies in the plane.

a

Use the position vector of the point as $\mathbf{a}$ and the two vectors as direction $\mathbf{d}_1$ and $\mathbf{d}_2$

$$\mathbf{a} = \begin{pmatrix}3\\1\\4\end{pmatrix},\ \mathbf{d}_1 = \begin{pmatrix}-1\\1\\3\end{pmatrix},\ \mathbf{d}_2 = \begin{pmatrix}2\\0\\5\end{pmatrix}$$

A vector equation has the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$

$$\mathbf{r} = \begin{pmatrix}3\\1\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\0\\5\end{pmatrix}$$

You need to check whether there are values of λ and μ such that $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

b If the point (3, −2, 5) lies in the plane, then

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$$

Write one equation for each component

$$\Rightarrow \begin{cases} -\lambda + 2\mu = 0 & (1) \\ \lambda = -3 & (2) \\ 3\lambda + 5\mu = 1 & (3) \end{cases}$$

Solve the first two equations, then check in the third

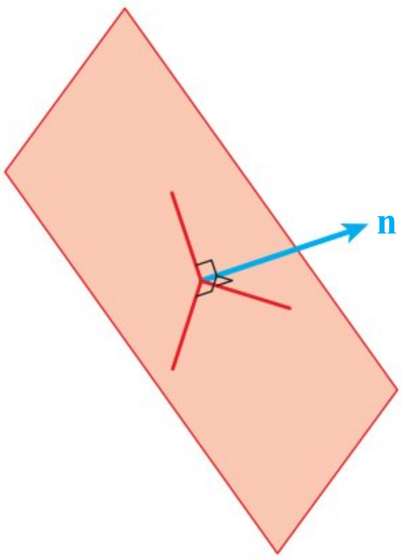
(1), (2) $\Rightarrow \lambda = -3,\ \mu = 1.5$

(3): $3(-3) + 5(1.5) = -1.5 \neq 1$

There are no values of λ and μ which make all three coordinates correct

The point does not lie in the plane.

Scalar product form of the equation of a plane



The vector equation of the plane can be difficult to work with because it contains two parameters. It is also impossible to see at a glance whether two vector equations represent the same plane. Luckily, there is another form of equation of a plane which helps with both of these problems.

The diagram shows a plane and a vector **n** perpendicular to it. This vector is perpendicular to every line in the plane and it is called the **normal vector** of the plane. All planes with the same normal vector are parallel to each other. You can pick out one of those planes by specifying that it contains a given point A.

If R is any other point on the plane, then the line AR lies in the plane, so it is perpendicular to the normal vector **n**. You can express this using the scalar product: $\overrightarrow{AR} \cdot \mathbf{n} = 0$. Using position vectors, this equation becomes $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$. Expanding the brackets gives another form of the equation of a plane.

KEY POINT 8.30

The **scalar product form** of the equation of the plane with normal vector **n** and that contains the point with position vector **a** is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Cartesian equation of a plane

The scalar product form of the equation can be re-written explicitly in terms of coordinates x, y and z by expanding the scalar product.

KEY POINT 8.31

The **Cartesian equation of the plane** with normal vector **n** and containing point with position vector **a** is

$$n_1 x + n_2 y + n_3 z = d$$

where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and $d = \mathbf{a} \cdot \mathbf{n}$.

WORKED EXAMPLE 8.37

A plane has normal vector $\mathbf{n} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ and contains the point $A(4, 8, -1)$. Find the equation of the plane in

a scalar product form

b Cartesian form.

The scalar product form of the equation is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

a $$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

Evaluate the scalar product on the right

$$\begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = 12 - 16 - 2 = -6$$

$$\therefore\ \mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = -6$$

Replace $\mathbf{r}$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

b $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = -6$$

Expand the scalar product on the left

$$3x - 2y + 2z = -6$$

CONCEPTS – RELATIONSHIPS

You have now met three different forms of the equation of a line and three different forms of the equation of a plane. Each expresses the **relationship** between the points on the line (or in the plane) in a different way and each is useful in different situations. When would you choose to use a vector equation and when a **Cartesian equation of a plane**?

Converting from the vector equation of a plane to the Cartesian equation of a plane

You learnt about vector product in Section 8F.

If you know a **vector equation of a plane**, to find the Cartesian equation of the plane you need to find the normal vector first. The normal vector is perpendicular to all lines in the plane; in particular, it is perpendicular to the two direction vectors. You can use vector product (cross product) to find such a vector.

KEY POINT 8.32

A plane with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$ has normal vector $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$.

WORKED EXAMPLE 8.38

Find the Cartesian equation of the plane with vector equation $\mathbf{r} = \begin{pmatrix}1\\-2\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\5\end{pmatrix}$.

To find the Cartesian equation we need the normal vector and one point

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Point (1, −2, 5) lies in the plane

$$\mathbf{a} = \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$\mathbf{n}$ is perpendicular to the two direction vectors $\begin{pmatrix}1\\1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\5\end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix}1\\1\\3\end{pmatrix} \times \begin{pmatrix}2\\-3\\5\end{pmatrix}$$

The cross product of two vectors is perpendicular to both of them

$$= \begin{pmatrix}14\\1\\-5\end{pmatrix}$$

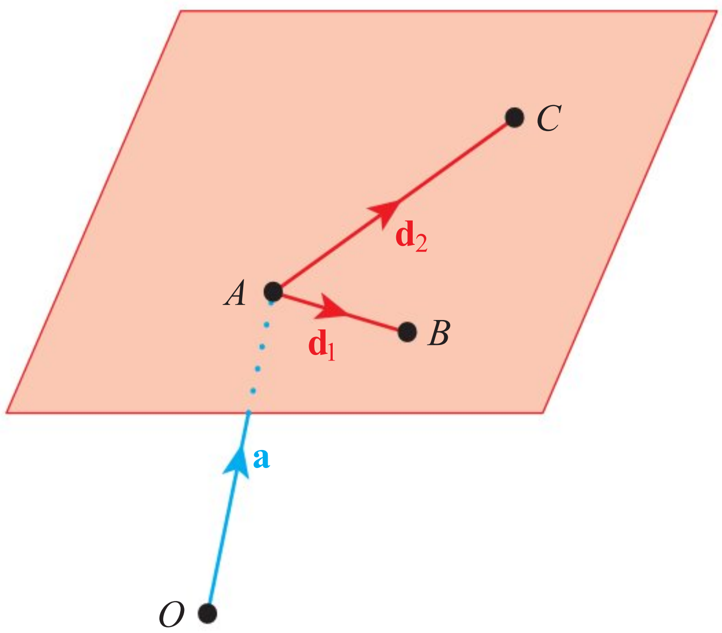
To get the Cartesian equation, write $\mathbf{r}$ as $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}14\\1\\-5\end{pmatrix} = \begin{pmatrix}1\\-2\\5\end{pmatrix} \cdot \begin{pmatrix}14\\1\\-5\end{pmatrix}$$

$$14x + y - 5z = -13$$

Equation of a plane containing three points

You saw at the start of this section that a plane is uniquely determined by three non-collinear points. To find the equation of the plane, consider the following diagram.



KEY POINT 8.33

For a plane containing points with position vectors **a, b** and **c**, two direction vectors parallel to the plane are (**b** – **a**) and (**c** – **a**).

WORKED EXAMPLE 8.39

A plane contains points $A(3, 4, -2)$, $B(1, -1, 3)$ and $C(5, 0, 2)$. Find the equation of the plane in

a vector form

b Cartesian form.

a

You need one point and two vectors parallel to the plane: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$

You can choose any of the three given points as they all lie in the plane: $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane: $\mathbf{d}_1 = \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$

$$\mathbf{d}_2 = \overrightarrow{AC} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

You can now write down a vector equation: $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$

b

To find the Cartesian equation you first need the normal vector, which is the cross product of the two direction vectors: $\mathbf{n} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \\ 18 \end{pmatrix}$

Write the equation in scalar product form first: $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 18 \\ 18 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 18 \\ 18 \end{pmatrix}$$

Expand the scalar products and evaluate the right-hand side: $0x + 18y + 18z = 0 + 72 - 36 = 36$

Simplify the equation: $\therefore\ y + z = 2$

Tip

Vector **n** determines the normal direction, so its magnitude is not important. In Worked Example 8.39 you could have used $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ instead of $\begin{pmatrix} 0 \\ 18 \\ 18 \end{pmatrix}$. Check that this leads to the same Cartesian equation.

TOOLKIT: Problem Solving

$y + z = 2$ looks like the equation of a line in two dimensions, (y and z). Why is it here an equation of a plane?

Exercise 8G

For questions 1 to 3, use the method demonstrated in Worked Example 8.36 to write down a vector equation of the plane containing point A and parallel to the vectors $\mathbf{d_1}$ and $\mathbf{d_2}$. Also determine whether point B lies in the plane.

1 **a** $A(3, 4, -2)$, $\mathbf{d}_1 = \begin{pmatrix} -2 \\ 5 \\ 5 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $B(3, 4, -2)$

b $A(4, -1, 2)$, $\mathbf{d}_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $B(-2, 1, 3)$

2 **a** $\mathbf{d}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $A(1, 0, 2)$, $B(1, 1, 7)$

b $\mathbf{d}_1 = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$, $A(0, 2, 0)$, $B(0, 6, 1)$

3 **a** $A(0, 1, 1)$, $B(1, 1, 2)$, $\mathbf{d}_1 = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{d}_2 = \mathbf{i} - 3\mathbf{j}$

b $A(1, -6, 2)$, $B(0, -3, 1)$, $\mathbf{d}_1 = 5\mathbf{i} - 6\mathbf{j}$, $\mathbf{d}_2 = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

For questions 4 to 6, use the method demonstrated in Worked Example 8.37 to find the equation of the plane with the given normal vector $\mathbf{n}$ and passing through the given point A

i in scalar product form

ii in Cartesian form.

4 **a** $\mathbf{n} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$, $A(3, 3, 1)$

b $\mathbf{n} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$, $A(4, 3, -1)$

5 **a** $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $A(4, 1, -1)$

b $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $A(1, -2, 2)$

6 **a** $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $A(-3, 0, 2)$

b $\mathbf{n} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$, $A(0, 0, 2)$

For questions 7 to 9, use the method demonstrated in Worked Example 8.38 to find the Cartesian equation with the given vector equation.

7 **a** $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

8 **a** $\mathbf{r} = (7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + \lambda(-5\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{k})$

b $\mathbf{r} = (3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) + \lambda(6\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

9 **a** $\mathbf{r} = (7 - 8\lambda - 2\mu)\mathbf{i} + (1 + 3\lambda + \mu)\mathbf{j} + (2 + 5\lambda + \mu)\mathbf{k}$

b $\mathbf{r} = (12 + \lambda + 3\mu)\mathbf{i} + (4 - 8\mu)\mathbf{j} + (10 - 5\lambda - 10\mu)\mathbf{k}$

For questions 10 to 12, use the method demonstrated in Worked Example 8.39a to find a vector equation of the plane containing the three given points.

10 **a** $A(12, 4, 10)$, $B(13, 4, 5)$, $C(15, -4, 0)$

b $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$

11 **a** $A(3, -1, 3)$, $B(1, 1, 2)$, $C(4, -1, 2)$

b $A(-1, -1, 5)$, $B(4, 1, 2)$, $C(-7, 1, 1)$

12 **a** $A(9, 0, 0)$, $B(-2, 1, 0)$, $C(1, -1, 2)$

b $A(11, -7, 3)$, $B(1, 14, 2)$, $C(-5, 10, 0)$

13 Determine whether the point $(-4, 8, 13)$ lies in the plane with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$.

14 Plane Π contains the points $A(3, -1, 2)$, $B(3, 3, 4)$ and $C(-1, 3, 6)$.

a Write down a vector equation of Π.

b Determine whether the point $D(4, 4, 0)$ lies in Π.

15 A plane has normal vector $\mathbf{n} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ and contains the point with position vector $\mathbf{a} = \mathbf{i} + 5\mathbf{k}$.

a Find the Cartesian equation of the plane.

b Determine whether the point with position vector $\mathbf{b} = 5\mathbf{i} + 11\mathbf{j} - \mathbf{k}$ lies in the plane.

16 Plane Π has Cartesian equation $5x + y - 4z = 20$.

a Write down a normal vector of Π.

b Find the value of c so that Π contains the point $(2, c, 1)$.

17 Plane Π has Cartesian equation $x + 5y - 8z = 37$.

a Find a unit vector perpendicular to the Π.

b Find the values of p and q such that the points $(p, 3, 1)$ and $(48, q, 2)$ lie in Π.

c Hence show that Π contains the line with Cartesian equation $\frac{x-30}{12} = \frac{y-3}{2} = \frac{z-1}{-1}$.

18 Two lines have equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 26 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

a Show that the lines intersect and find the coordinates of the point of intersection.

b Calculate $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

c Hence find the Cartesian equation of the plane containing l_1 and l_2.

19 **a** Find the coordinates of the point of intersection of lines

$$l_1 : \frac{x-1}{3} = \frac{y+1}{4} = \frac{3-z}{3}$$

and

$$l_2 : \frac{x+12}{2} = \frac{y}{1} = \frac{z+17}{1}.$$

b Find a vector perpendicular to both lines.

c Hence find the Cartesian equation of the plane containing l_1 and l_2.

20 Find a vector equation of the plane containing the line $l : \mathbf{r} = \begin{pmatrix} 9 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$ and the point $P(11, 12, 13)$.

21 Plane Π contains the line $l : \mathbf{r} = t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and the point $P(4, 0, 2)$.

a Show that the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is parallel to Π.

b Calculate $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

c Hence find the Cartesian equation of Π.

22 Show that the plane with equation $5x + y - 2z = 15$ contains the line with equation $\frac{x-4}{1} = \frac{y+1}{1} = \frac{z-2}{3}$.

23 Show that the line with equation $\mathbf{r} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ lies in the plane with equation $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

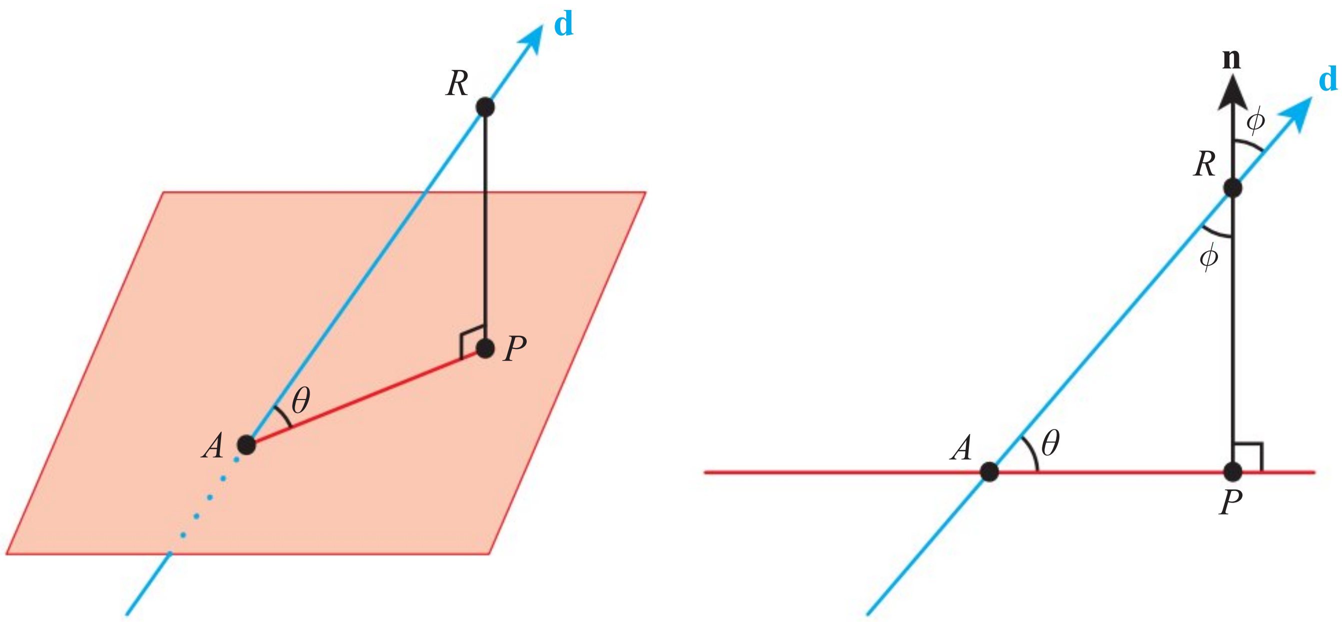
24 Show that the points $A(11, 0, 6)$, $B(9, -4, 0)$, $C(-4, 6, 0)$ and $D(3, 8, 7)$ lie in the same plane.

8H Angles and intersections between lines and planes

You already know (from Sections 8D and 8E) how to find the intersection and the angle between two lines. In this section you will apply similar ideas to planes. You will then be able to combine various techniques from this chapters to solve a variety of geometrical problems in three-dimensional space.

Angle between a line and a plane

When a line intersects a plane, the angle between them is defined as the smallest possible angle that the line makes with any of the lines in the plane. This is the angle θ shown in the first diagram below. You can think of P as the projection of the point R onto the plane. Imagine a light vertically above R – AP would then be the shadow of AR.



You know that the line AR is in the direction of the line, but you do not know the direction of the line AP. However, notice that the line PR is in the direction of the normal to the plane. The second diagram above shows a two-dimensional sketch of triangle APR. You can see that angle ϕ (at the top of the triangle) is the angle between the plane's normal and the line's direction vector.

Tip

The modulus sign around $\mathbf{d} \cdot \mathbf{n}$ in the $\cos\phi$ formula ensures that the angle is acute.

KEY POINT 8.34

The angle between a line with direction vector **d** and a plane with normal vector **n** is

$$\theta = 90° - \phi, \text{ where } \cos\phi = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$$

WORKED EXAMPLE 8.40

Find the angle between line l with equation $\mathbf{r} = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ and the plane with equation $3x + 3y + 2z = 11$.

$$\cos\phi = \frac{\mathbf{d} \cdot \mathbf{n}}{|\mathbf{d}||\mathbf{n}|}$$

First find the angle between the direction of the line and the normal of the plane

$$= \frac{\begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{25+1+1}\sqrt{9+9+4}}$$

$$= \frac{-14}{\sqrt{22}\sqrt{27}}$$

$$\phi = 125.1°$$

You want the acute angle between **d** and **n**. We could have taken the modulus of the cosine before finding ϕ, as in Key Point 8.34, or we could just subtract our answer from 180°

$$\phi_1 = 180 - 125.1 = 54.9°$$

Remember the diagram above Key Point 8.34

$$\theta = 90° - \phi_1 = 35.1°$$

Be the Examiner 8.4

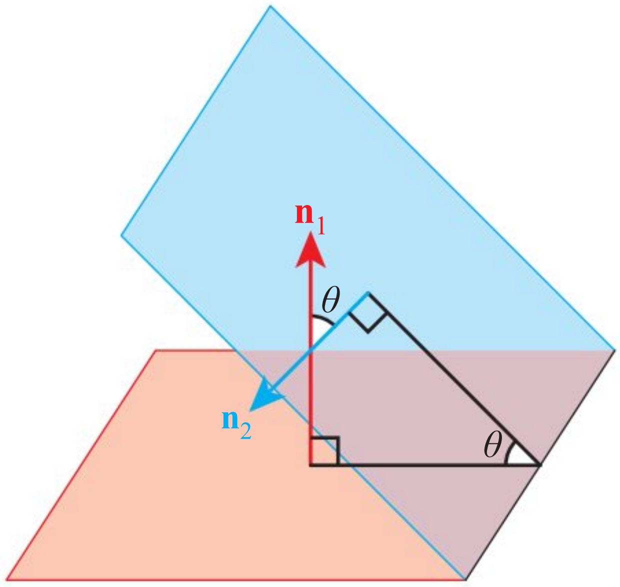
Find the angle between the line $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and the plane $4x - y + z = 22$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\cos\theta = \frac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1+1+9}\sqrt{16+1+1}}$	$\cos\theta = \frac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1+1+9}\sqrt{16+1+1}}$	$\cos\theta = \frac{\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{9+1+4}\sqrt{16+1+1}}$
$= \frac{8}{\sqrt{198}} = 0.569$	$= \frac{8}{\sqrt{198}} = 0.569$	$= \frac{9}{\sqrt{252}} = 0.567$
$\theta = \cos^{-1}(0.569) = 55.4°$	$\theta = \cos^{-1}(0.569) = 55.4°$	$\cos^{-1}(0.567) = 55.5°$
	$90 - 55.4 = 34.6°$	$90 - 55.5 = 34.6°$

Angle between two planes

The diagram shows two planes and their normals. Using the fact that the sum of the angles in a quadrilateral is 360°, you can show that the two angles marked θ are equal.



KEY POINT 8.35

The angle between two planes is the angle between their normals.

WORKED EXAMPLE 8.41

Find the acute angle between planes with equations $4x - y + 5z = 11$ and $x + y - 3z = 3$.

You need to find the angle between the two normal vectors

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

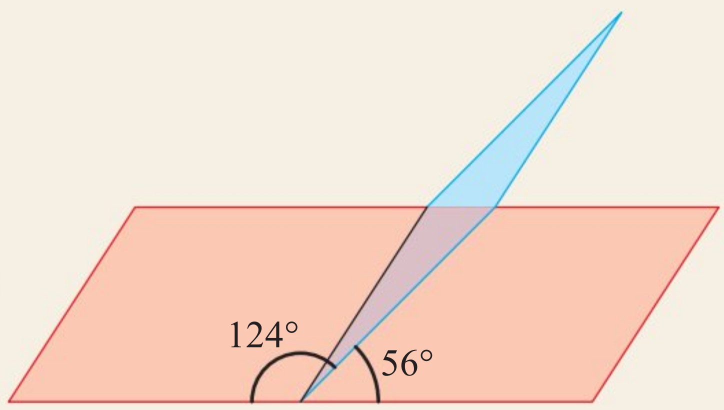
The components of the normal vector are the coefficients in the Cartesian equation

$$= \frac{\begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}}{\sqrt{16+1+25}\sqrt{1+1+9}}$$

$$= \frac{-12}{\sqrt{42}\sqrt{11}}$$

$$\theta = 123.9°$$

The question asks for the acute angle

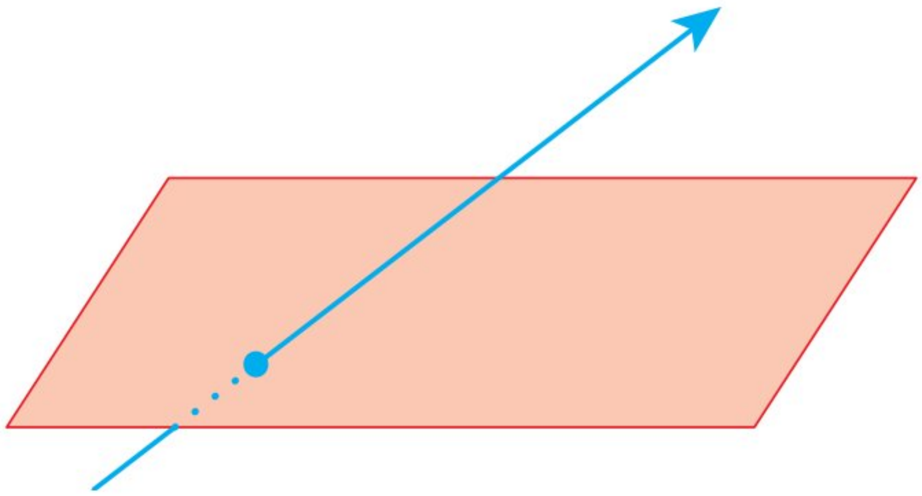


$$180° - 123.9° = 56.1°$$

The angle between the planes is 56.1°.

Intersection between a line and a plane

A line which is not parallel to a plane intersects it at a single point. The coordinates of this point need to satisfy both the equation of the line and the equation of the plane. The easiest way to find them is to substitute the expressions for x, y and z in terms of λ (from the vector equation of the line) into the Cartesian equation of the plane.



WORKED EXAMPLE 8.42

Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and the plane $x - 2y + z = 1$.

Express x, y and z in terms of λ…

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-\lambda \\ -1+2\lambda \\ 2+2\lambda \end{pmatrix}$$

… then substitute into the equation of the plane

$$(3-\lambda) - 2(-1+2\lambda) + (2+2\lambda) = 1$$
$$7 - 3\lambda = 1$$
$$\lambda = 2$$

Use this value of λ in the equation of the line to find the coordinates. You should also check that (1, 3, 6) is indeed on the plane

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-2 \\ -1+4 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

The point of intersection is (1, 3, 6).

Intersection of two planes

Two planes intersect along a line (unless they are parallel). You can find the equation of this line by treating the Cartesian equations of two planes as simultaneous equations and finding the general solution.

You learnt about finding the general solution of a system of three equations in Section 2C.

Tip

Which variable you choose to replace with λ will depend on the equation you see; the end answer will always be equivalent, but your choice may affect how 'nice' the numbers are during the working.

WORKED EXAMPLE 8.43

Find the equation of the line of intersection of the planes $4x - y + 3z = 12$ and $x + 2y + z = 39$.

$$\begin{cases} 4x - y + 3z = 12 & (1) \\ x + 2y + z = 39 & (2) \end{cases}$$

Eliminate y. (You can eliminate any of the three variables, but in this case y is the simplest)

$2 \times (1) + (2)$:

$$\begin{cases} 4x - y + 3z = 12 & (1) \\ 9x \qquad + 7z = 63 & (4) \end{cases}$$

Choose one of the variables and express the others in terms of it. In this case, choose z

Let $z = \lambda$. Then,

$$(4) \Rightarrow x = \frac{63 - 7\lambda}{9} = 7 - \frac{7}{9}\lambda$$

$$(1) \Rightarrow y = 4\left(7 - \frac{7}{9}\lambda\right) + 3(\lambda) - 12$$

$$= 16 - \frac{1}{9}\lambda$$

This general solution represents a line. You can find its vector equation by writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 - \frac{7}{9}\lambda \\ 16 - \frac{1}{9}\lambda \\ \lambda \end{pmatrix}$$

Write the equation in the more conventional way by separating the direction vector

$$= \begin{pmatrix} 7 \\ 16 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7/9 \\ -1/9 \\ 1 \end{pmatrix}$$

The equation of the intersection line is

Remember that the magnitude of the direction vector is unimportant, so you can multiply it by 9 to remove the fractions

$$\mathbf{r} = \begin{pmatrix} 7 \\ 16 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -1 \\ 9 \end{pmatrix}$$

TOOLKIT: Problem Solving

There is more than one way to find the line of intersection of two planes.

Since the line of intersection lies on both planes, it is perpendicular to both normal vectors. In the example above, you could have found the direction vector of the line by calculating

$$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

What other information do you need in order to find the equation of the line? Question 29 in Exercise 8H is an application of this method.

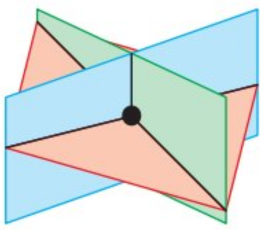
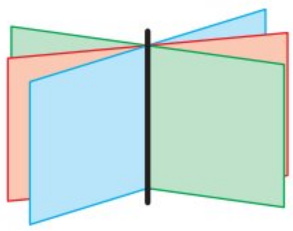
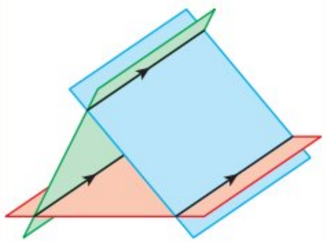
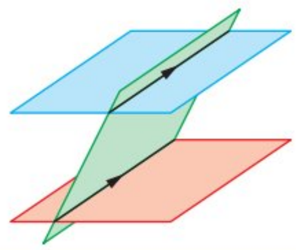
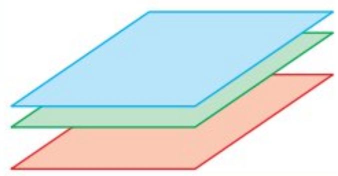
Intersection of three planes

Tip

Two planes are parallel if their normals are parallel.

For three planes, there are more possible configurations. You need to determine whether the system of three equations has a unique solution, infinitely many solutions or no solutions, and also whether any of the three planes are parallel.

This table shows how three distinct planes can intersect.

Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
				
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

TOOLKIT: Problem Solving

If the planes are not necessarily distinct, there are three more possible configurations. Can you describe them, and say how you would recognize each situation?

WORKED EXAMPLE 8.44

Three planes have equations

$\Pi_1:\ x + 2y + kz = 8$

$\Pi_2:\ 2x + 5y + 2z = 7$

$\Pi_3:\ 5x + 12y + z = 2.$

a For $k = 2$, the three planes intersect at a single point. Find its coordinates.

b For $k = -3$, show that the three planes do not intersect and describe their geometrical configuration.

Tip

If you try using your GDC to solve a system that does not have a unique solution, you may get an error message.

You are told that there is a single intersection point. This means that the system of three equations has a unique solution,

a Using $k = 2$:

$$\begin{cases} x + 2y + 2z = 8 \\ 2x + 5y + 2z = 7 \\ 5x + 12y + z = 2 \end{cases}$$

which can be found using the simultaneous equation solver on your GDC

From GDC, the intersection point is (2, –1, 4).

If the planes do not intersect, you will not be able to use GDC to show this. Instead, try solving the simultaneous equations by elimination

b Using $k = -3$:

$$\begin{cases} x + 2y - 3z = 8 & (1) \\ 2x + 5y + 2z = 7 & (2) \\ 5x + 12y + z = 2 & (3) \end{cases}$$

$(4) = 3 \times (3) + (1)$

$(5) = 2 \times (3) - (2)$

It seems simplest to eliminate z from equations (1) and (2)

$$\begin{cases} 16x + 38y = 14 & (4) \\ 8x + 17y = -3 & (5) \\ 5x + 12y + z = 2 & (3) \end{cases}$$

$(6) = 2 \times (5) - (4)$

Now eliminate x from equation (4)

$$\begin{cases} 0 = -20 & (6) \\ 8x + 17y = -3 & (5) \\ 5x + 12y + z = 2 & (3) \end{cases}$$

Equation (6) is false, so the system is inconsistent

Inconsistent system, so no solutions. The planes don't intersect.

You need to check whether any of the three planes are parallel: are any of the normal vectors multiples of each other?

None of the normal vectors are multiples of each other, so no two planes are parallel.
The planes form a triangular prism.

WORKED EXAMPLE 8.45

Show that the following three planes intersect in a line, and find its vector equation.

Π_1: $x + 4y - 2z = -3$

Π_2: $2x - y + 5z = 12$

Π_3: $8x + 5y + 11z = 30$

The question suggests that the solution is not unique, so you cannot use your GDC. Solve the system by elimination

$$\begin{cases} x + 4y - 2z = -3 & (1) \\ 2x - y + 5z = 12 & (2) \\ 8x + 5y + 11z = 30 & (3) \end{cases}$$

$(4) = 2 \times (1) - (2)$

$(5) = 8 \times (1) - (3)$

Eliminate x from equations (2) and (3)

$$\begin{cases} x + 4y - 2z = -3 & (1) \\ 9y - 9z = -18 & (4) \\ 27y - 27z = -54 & (5) \end{cases}$$

$(6) = 3 \times (4) - (5)$

Eliminate z from equations (4) and (5)

$$\begin{cases} x + 4y - 2z = -3 & (1) \\ 9y - 9z = -18 & (4) \\ 0 = 0 & (6) \end{cases}$$

Equation (6) is always true, so the equations are consistent

There are infinitely many solutions.

Let $z = \lambda$.
Then,

You can express y in terms of z from equation (4)

$$(4): y = \frac{-18 + 9z}{9} = -2 + \lambda$$

Use equation (1) to express x in terms of λ

$$(1): x = -3 - 4(-2 + \lambda) + 2\lambda$$
$$x = 5 - 2\lambda$$

The solutions form a line. To find its equations, write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - 2\lambda \\ -2 + \lambda \\ \lambda \end{pmatrix}$$

Write the equation of the line in the usual form by separating the direction vector

The intersection is the line with equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Exercise 8H

For questions 1 to 3, use the method demonstrated in Worked Example 8.40 to find the angle between line l and plane Π.

1 **a** $l: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$, $\Pi: x - y + 3z = 1$

b $l: \mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\Pi: x - 3y + 3z = 13$

2 **a** $l: \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\Pi: 3x - y - z = 22$

b $l: \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\Pi: 4x - y + z = 7$

3 **a** $l: \mathbf{r} = \begin{pmatrix} 2 \\ -8 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$, $\Pi: 3x - 7y + z = 11$

b $l: \mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$, $\Pi: 2x + 5y + 5z = 20$

For questions 4 to 6, use the method demonstrated in Worked Example 8.41 to find the acute angle between the two planes.

4 **a** $x + y + 2z = 10$ and $2x + y + 2z = 27$ **b** $3x - y + 2z = 1$ and $x + y + z = -3$

5 **a** $x - y - 2z = 0$ and $x + y + 2z = 22$ **b** $x - y - z = 7$ and $x + 2y + 3z = 27$

6 **a** $x - z = 5$ and $y + z = 11$ **b** $x - y = 3$ and $x - z = 2$

For questions 7 to 9, use the method demonstrated in Worked Example 8.42 to find the coordinates of the point where line l intersects plane Π.

7 **a** $l: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\Pi: x + y + 2z = 32$

b $l: \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\Pi: 3x + y + z = 24$

8 **a** $l: \mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$, $\Pi: x + 3y + 4z = 30$

b $l: \mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, $\Pi: 2x + y + 2z = 18$

9 **a** $l: \mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, $\Pi: x - y + 2z = -1$

b $l: \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$, $\Pi: 2x - 3y + z = -16$

For questions 10 to 12, use the method demonstrated in Worked Example 8.43 to find the equation of the line of intersection of two planes.

10 **a** $2x - y + z = 1$ and $x + 2y - z = 0$ **b** $3x + 2y + z = 1$ and $5x + 3y + 3z = 2$

11 **a** $2x + y + z = 7$ and $3x - y + 2z = 17$ **b** $x + 4y - z = 11$ and $3x + y + 2z = 5$

12 **a** $x - y + 2z = 5$ and $2x - 2y + 3z = 9$ **b** $3x - 6y + z = 3$ and $x - 2y + z = 1$

For questions 13 to 15, the three planes either intersect at a single point or do not intersect at all. Use the method demonstrated in Worked Example 8.44 to find the coordinates of the point of intersection, or describe the geometrical configuration of the planes in cases when they do not intersect.

13 **a** $2x - 5y + 4z = 10$, $x + 3y - 2z = 5$ and $4x + 2y - z = -4$

b $x - 3y + z = -5$, $3x - y - z = -7$, $x + 3y + 5y = 1$

14 **a** $\Pi_1: x + 5y - z = -1$, $\Pi_2: 2x - 2y + z = 2$, $\Pi_3: 5x - 3y + 2z = 4$

b $\Pi_1: 2x + 3y + 8z = 6$, $\Pi_2: 2x + y + 2z = 4$, $\Pi_3: 6x - y - 6z = 5$

15 **a** $\Pi_1: x - y + 2z = 11$, $\Pi_2: 3x - 3y + 6z = 15$, $\Pi_3: 2x + y + z = 16$

b $\Pi_1: x + y - 2z = 6$, $\Pi_2: 2x - y + z = 5$, $\Pi_3: 3x + 3y - 6z = 2$

For questions 16 to 18, use the method demonstrated in Worked Example 8.45 to find a vector equation of the line of intersection of the three planes.

16 **a** $\Pi_1: x - y + 2z = 1$, $\Pi_2: x + 2y - z = 4$, $\Pi_3: 2x + y + z = 5$

b $\Pi_1: 2x - y - 3z = 3$, $\Pi_2: x + y - 3z = 0$, $\Pi_3: x + 2y - 4z = -1$

17 **a** $\Pi_1: 3x + 2y + z = 1$, $\Pi_2: 7x + 4y + 5z = 3$, $\Pi_3: 5x + 3y + 3z = 2$

b $\Pi_1: 2x + y + z = 11$, $\Pi_2: x - y + z = 5$, $\Pi_3: x + 5y - z = 7$

18 **a** $\Pi_1: -x + y + z = -2$, $\Pi_2: x - y + 2z = 5$, $\Pi_3: 2x - 2y + 3z = 9$

b $\Pi_1: 3x - 6y + z = 3$, $\Pi_2: x - 2y + z = 1$, $\Pi_3: -x + 2y + 2z = -1$

19 Plane Π_1 has Cartesian equation $3x - y + z = 7$.

a Write down a normal vector of Π_1.

Plane Π_1 has equation $x - 5y + 5z = 11$.

b Find, correct to the nearest degree, the acute angle between Π_1 and Π_2.

20 Plane Π has equation $2x + 2y - z = 11$. Line l is perpendicular to Π and passes through the point $P(-3, -3, 4)$. It intersects Π at the point N.

a Write down the direction vector of l.

b Write down a vector equation for l.

c Find the coordinates of N.

d Hence find the perpendicular distance on P from the plane.

21 Plane Π_2 contains points $A(2, -2, 1)$, $B(5, 1, 2)$ and $C(-2, 2, 5)$.

a Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

b Hence find the Cartesian equation of Π_1.

Plane Π_2 has equation $\mathbf{r} \bullet (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 12$.

c Find the acute angle between Π_1 and Π_2.

22 Line L passes through the point $A(4, 1, -3)$ and has direction vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Plane Π has equation $\mathbf{r} \bullet \mathbf{n} = 20$, where $\mathbf{n} = 3\mathbf{i} - \mathbf{k}$. L and Π intersect at the point M.

a Find the coordinates of M.

b Find the angle between L and Π.

c Point N lies in Π so that AN is perpendicular to Π. Find the distance MN.

23 Plane Π has equation $x - 5y + 4z = 16$ and line l has equation $\mathbf{r} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} - 3\mathbf{k})$.

a Find the angle between l and Π.

b Find the coordinate of the point where l intersects Π.

c Hence find the perpendicular distance from the point $A(3, 1, -1)$ to Π.

24 Plane Π has equation $x - 3y + 5z = -24$ and point A has coordinates $(4, 1, 2)$.

a Write down a vector equation of the line through A which is perpendicular to Π.

b Find the coordinates of the point B where this line intersects Π.

c Hence find the coordinate of the reflection of A in Π.

25 a Calculate $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

b Plane Π_1 has normal vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and contains point $A(3, 4, -2)$. Find the Cartesian equation of the plane.

c Plane Π_2 has equation $3x + y - z = 15$. Show that Π_2 contains point A.

d Write down the vector equation of the line of intersection of the two planes.

e A third plane, Π_3, has equation $\mathbf{r} \bullet \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 12$. Find the coordinates of the point of intersection of all three planes.

f Find the angle between Π_1 and Π_3 in degrees.

26 a Find the value of a for which the following system of equations is consistent.

$$\begin{cases} 3x - 5y + z = 7 \\ x + 3y - 4z = 22 \\ 7x - 21y + 11z = a \end{cases}$$

b The three equations represent planes. Describe the geometrical configuration of the planes when a takes the value found in part **a**.

27 Three planes have equations

Π_1: $x - y - 2z = 2$

Π_2: $2x - 2y + z = 0$

Π_3: $3x - 3y + 4z = a$

a For $a = 1$, describe the geometrical configuration of the three planes.

b Find the value of a for which the three planes intersect in a line, and find the equation of this line.

28 Line l is the intersection of the planes Π_1: $x - 3y + z = 7$ and Π_2: $2x + y + z = 10$

a Find a vector equation of l.

Another plane has equation Π_3: $5x - 7y + 3z = 16$.

b Show that l is parallel to Π_3.

c Describe the geometrical configuration of three planes, justifying your answer.

29 Two planes have equations Π_1: $x - 4z + 7 = 0$ and Π_2: $4x + 5y - z = 7$.

a Show that there is a value of c, which you should find, such that the point $(-3, 4, c)$ lies in both planes.

b Calculate $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$.

c Hence write down the Cartesian equation of the line of intersection of Π_1 and Π_2.

30 Four points have coordinates $A(3, 11, 5)$, $B(-4, 7, 4)$, $C(1, 1, 3)$ and $D(-5, 2, 11)$.

a Find $\overrightarrow{AB} \times \overrightarrow{BC}$

b Find the area of triangle ABC.

c Find the equation of the plane Π containing the points A, B and C.

d Show that the point D does not lie in Π.

Line l is perpendicular to Π and passes through D.

e Find the coordinates of the point of intersection of l and Π.

f Hence find the volume of the tetrahedron $ABCD$.

31 **a** Show that the planes Π_1: $6x - 9y + 15z = 20$ and Π_2: $-4x + 6y - 10z = 3$ are parallel.

b Find the value of k such that the point $A\ (2, 0, k)$ lies in Π_1.

c Write down the equations of line L which passes through A and is perpendicular to both planes.

d Hence find the perpendicular distance between Π_1 and Π_2.

32 Planes Π_1 and Π_2 have equations Π_1: $5x + y + z = 21$ and Π_2: $x - 3y - 2 = 3$.

a Show that Π_1 and Π_2 are perpendicular.

b Calculate $(5\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$.

c Show that the point $P(1, -1, 5)$ does not lie in either of the two planes.

d Find a Cartesian equation of the line through P which is parallel to both Π_1 and Π_2.

33 The equation of three planes are

Π_1: $3x - y + 5z = 2$

Π_2: $2x + 4y + z = 1$

Π_3: $x + y + kz = c$.

Find the set of values of k and c for which the planes

a intersect at a single point

b intersect along a line

c do not intersect.

Checklist

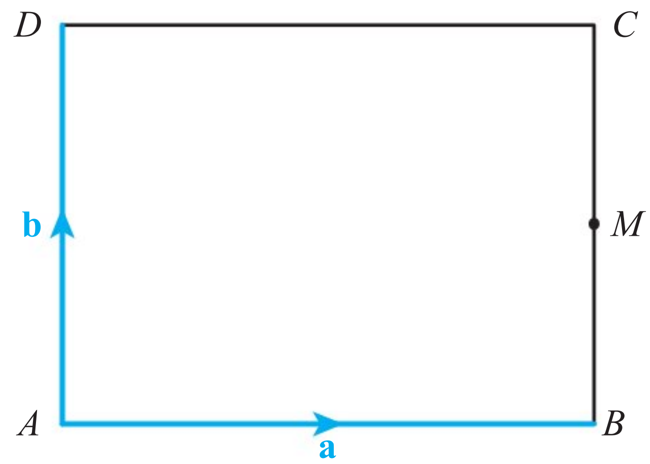
- You should be able to represent vectors as either directed line segments or by their components (as column vectors or using **i**, **j**, **k** base vectors).
  - You should be able to add, subtract and multiply vectors by a scalar using both representations.
  - The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and the unit vector in the same direction as vector **a** is $\frac{\mathbf{a}}{|\mathbf{a}|}$.
  - The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, the displacement vector from A to B is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and the distance between A and B is $|\mathbf{b} - \mathbf{a}|$.
  - If points A and B have position vectors **a** and **b**, then the midpoint of AB has the position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.
- You should be able to use vectors to solve geometrical problems. In particular:
  - If points A, B and C are collinear, then $\overrightarrow{AB} = k\,\overrightarrow{BC}$ for some scalar k.
  - Line segments $[AB]$ and $[DC]$ are equal and parallel if $\overrightarrow{AB} = \overrightarrow{DC}$. You can use this to identify a parallelogram.
- You should know that the scalar product (dot product) is linked to the angle between two vectors:
  - $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$
- You should be able to use the following algebraic properties of the scalar product:
  - $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
  - $(-\mathbf{a}) \bullet \mathbf{b} = -(\mathbf{a} \bullet \mathbf{b})$
  - $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b}) + (\mathbf{a} \bullet \mathbf{c})$
  - $(k\mathbf{a}) \bullet \mathbf{b} = k(\mathbf{a} \bullet \mathbf{b})$
  - $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$
- You should know that vectors **a** and **b** are:
  - parallel if $\mathbf{b} = t\mathbf{a}$ for some scalar t
  - perpendicular if $\mathbf{a} \bullet \mathbf{b} = 0$.
- You should be able to find and use various forms of the equation of a line.
  - The vector equation of a line has the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where **d** is the direction vector and **a** is the position vector of one point on the line. The components of the position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are the coordinates of a general point on the line.
  - Expressing x, y and z in terms of λ gives the parametric equations of a line.
  - Eliminating λ from the parametric equations, by making all three equations equal to λ, gives the Cartesian equation of the line in the form, $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$.
- You should know the vector equation of a line, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ can be used to represent the position of a particle moving with a constant velocity in two or three dimensions. In that case, λ represents time, **d** velocity and $|\mathbf{d}|$ speed of the particle.
- You should know the angle between two lines is the angle between their direction vectors.
- You should be able to find the point of intersection of two lines, or show that two lines are skew or parallel.
- You should know that the vector product (cross product) is a vector perpendicular to both **a** and **b**.
  - The component form is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.
  - The magnitude of the cross product is $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, which equals the area of the parallelogram formed by the vectors **a** and **b**.

You should be able to use the following properties of the vector product:
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
- If vectors **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- If **a** and **b** are perpendicular vectors, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

- You should be able to use various forms of the equation of a plane.
  - The vector equation of a plane has the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1 + \mu \mathbf{d}_2$, where $\mathbf{d}_1$ and $\mathbf{d}_2$ are two direction vectors parallel to the plane and **a** is a position vector of one point in the plane.
  - The scalar product form of the equation of the plane is $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$ where **n** is the normal vector.
  - This can be expanded to give the Cartesian equation of the plane, $n_1 x + n_2 y + n_3 z = d$.
- You should be able to find angles between lines and planes.
  - The angle between a line and a plane is $90° - \phi$, where ϕ is the angle between the direction of the line and the normal to the plane.
  - The angle between two planes is the angle between their normals.
- You should know that if a line intersects a plane, you can find the coordinates of the intersection point by substituting (x, y, z) from the equation of the line (in terms of λ) into the Cartesian equation of the plane.
- You should know two planes are either parallel or intersect along a line.
- You should know that three planes can intersect at a single point, along a line, or not at all. Their geometrical configuration can be determined by solving the system of three simultaneous equations.

Mixed Practice

1 The diagram shows a rectangle $ABCD$, with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of BC.



a Express $\overrightarrow{MD}$ in terms of **a** and **b**.

b N is the midpoint of DM. Express $\overrightarrow{AN}$ in terms of **a** and **b**.

c P is the point on the extension of the side BC such that $CP = CM$. Show that A, N and P lie on the same straight line.

2 Given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{d} = 4\mathbf{i} - \mathbf{j} + p\mathbf{k}$,

a find $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$

b find the value of p such that **d** is perpendicular to $\mathbf{a} \times \mathbf{b}$.

3 Find the values of x, with $0 < x < \frac{\pi}{2}$, such that the vectors $\begin{pmatrix} 3\sin x \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4\cos x \\ 1 \\ -2 \end{pmatrix}$ are perpendicular.

4 Points A and C have position vectors

$\mathbf{a} = 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 3\mathbf{k}$.

a Find $\overrightarrow{OA} \times \overrightarrow{OC}$.

b Find the coordinates of the point B such that $OABC$ is a parallelogram.

c Find the exact area of $OABC$.

5 In this question, distance is measured in metres and time in seconds. The base vectors **i**, **j** and **k** point east, north and up, respectively.

An aeroplane takes off from the ground. It moves with constant velocity $\mathbf{v} = (116\mathbf{i} + 52\mathbf{j} + 12\mathbf{k})$.

a Find the speed of the aeroplane.
b How long does it take for the aeroplane to reach the height of 1 km?

6 Points A, B and D have coordinates (1, 1, 7) (−1, 6, 3), and (3, 1, k), respectively. AD is perpendicular to AB.
a Write down, in terms of k, the vector $\overrightarrow{AD}$.
b Show that $k = 6$.
Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.
c Find the coordinates of C.
d Find the exact value of $\cos(A\hat{D}C)$.

7 **a** Find a vector equation of the line through the points $A(1, -3, 2)$ and $B(2, 2, 1)$.
b Find the acute angle between this line and the line l_2 with equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}.$$

c Find the value of k for which the point $B(7, 3, k)$ lies on l_2.
d Find the distance AC.

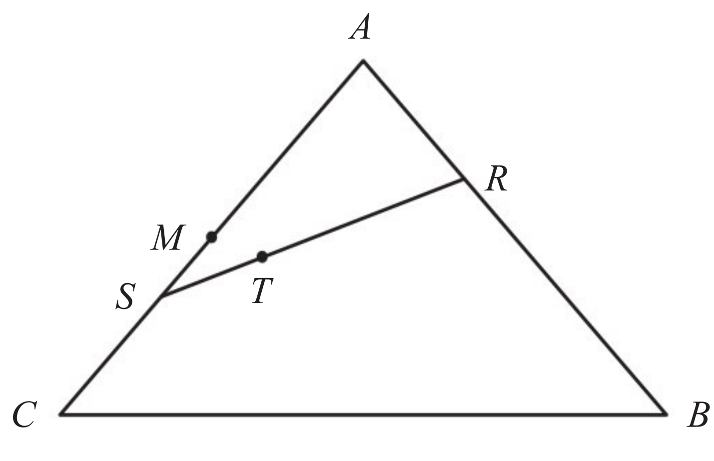
8 Line L_1 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and line L_2 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

a Find $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

b Find the coordinates of the point of intersection of the two lines.
c Write down a vector perpendicular to the plane containing the two lines.
d Hence find the Cartesian equation of the plane containing the two lines.

9 The vector $\mathbf{n} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is normal to a plane which passes through the point (3, −1, 2).
a Find an equation for the plane.
b Find a if the point $(a, 2a, a - 1)$ lies on the plane.

10 The position vectors of the points A, B and C are **a**, **b** and **c** respectively, relative to an origin O. The diagram shows the triangle ABC and points M, R, S and T.



M is the midpoint of $[AC]$.

R is a point on $[AB]$ such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$.

S is a point on $[AC]$ such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$.

T is a point on $[RS]$ such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$.

a **i** Express $\overrightarrow{AM}$ in terms of **a** and **c**.

ii Hence show that $\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}$.

b **i** Express $\overrightarrow{RA}$ in terms of **a** and **b**.

ii Show that $\overrightarrow{RT} = \frac{2}{9}\mathbf{a} - \frac{2}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}$.

c Prove that T lies on $[BM]$.

Mathematics HL November 2014 Paper 1 Q12

11 Find a vector of magnitude 3 in the same direction as $2\mathbf{i}-\mathbf{j}+\mathbf{k}$.

12 Let $\mathbf{a}=\begin{pmatrix}3\\0\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}-1\\5\\p\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}1\\4\\-3\end{pmatrix}$.

Find the value of p, given that $\mathbf{a}\times\mathbf{b}$ is parallel to $\mathbf{c}$.

13 Four points have coordinates $A(2, 4, 1)$, $B(k, 4, 2k)$, $C(k+4, 2k+4, 2k+2)$ and $D(6, 2k+4, 3)$.

- **a** Show that $ABCD$ is a parallelogram for all values of k.
- **b** When $k=1$, find the angles of the parallelogram.
- **c** Find the value of k for which $ABCD$ is a rectangle.

14 Show that $(2\mathbf{a}-\mathbf{b})\times(\mathbf{a}+3\mathbf{b}) = 7\mathbf{a}\times\mathbf{b}$.

15 Find the area of the triangle with vertices (2, 1, 1) (–1, 2, 2) and (0, 1, 5).

16 Given that $\mathbf{a}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=2q\mathbf{i}+\mathbf{j}+q\mathbf{k}$ find the values of scalars p and q such that $p\mathbf{a}+\mathbf{b}$ is parallel to vector $\mathbf{i}+\mathbf{j}+2\mathbf{k}$.

17 Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors and α the angle between them.

- **a** Express $|\mathbf{a}-\mathbf{b}|$ and $|\mathbf{a}+\mathbf{b}|$ in terms of $\cos\alpha$.
- **b** Hence find the value of α such that $|\mathbf{a}+\mathbf{b}| = 4|\mathbf{a}-\mathbf{b}|$.

18 **a** Determine whether the lines

$$\mathbf{r}=\begin{pmatrix}2\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}4\\-1\\2\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}-2\\0\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\-3\end{pmatrix} \text{ are parallel or perpendicular.}$$

b Determine whether the lines intersect.

19 **a** Calculate $\begin{pmatrix}-1\\0\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}$.

b Two lines have equations

$$l_1: \mathbf{r}=\begin{pmatrix}7\\-3\\2\end{pmatrix}+t\begin{pmatrix}-1\\0\\2\end{pmatrix}$$

$$l_2: \mathbf{r}=\begin{pmatrix}1\\1\\26\end{pmatrix}+s\begin{pmatrix}0\\1\\3\end{pmatrix}$$

- **i** Show that l_1 and l_2 intersect.
- **ii** Find the coordinates of the point of intersection, R.

c Plane Π contains lines l_1 and l_2. Find the Cartesian equation of Π.

20 Line l has Cartesian equation $\frac{2x-3}{2}=\frac{3-y}{4}=\frac{z}{5}$.

- **a** Write down a vector equation of l.
- **b** Point A lies on l such that OA is perpendicular to l. Find the coordinates of A.
- **c** Hence find the shortest distance of l from the origin.

21 Line l_1 has Cartesian equation $\frac{x-2}{4}=\frac{y+1}{-3}=\frac{z}{3}$. Line l_2 is parallel to l_1 and passes through point $A(0, -1, 2)$.

a Write down the vector equation of l_2.

b Find the coordinates of the point B on l_1 such that AB is perpendicular to l_1.

c Hence find, to three significant figures, the shortest distance between the two lines.

22 **a** Find the vector equation of the line L through point $A(-2, 4, 2)$ parallel to the vector $\mathbf{l}=\begin{pmatrix}1\\1\\0\end{pmatrix}$.

b Point B has coordinates $(2, 3, 3)$. Find the cosine of the angle between AB and the line L.

c Calculate the distance AB.

d Point C lies on L and BC is perpendicular to L. Find the exact distance AC.

23 **a** Show that the lines L_1: $\mathbf{r}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\3\\1\end{pmatrix}$ and L_2: $\frac{x-1}{4}=\frac{y+2}{3}=\frac{2z-1}{4}$ do not intersect.

b Calculate $\begin{pmatrix}-3\\3\\1\end{pmatrix}\times\begin{pmatrix}4\\3\\2\end{pmatrix}$.

c Hence find the Cartesian equation of the plane through the point $(3, 0, 1)$ which is parallel to both L_1 and L_2.

24 **a** Find a vector equation of the line with Cartesian equation $\frac{2x-1}{4}=\frac{y+2}{3}=\frac{4-3z}{6}$.

b Determine whether the line intersects the x-axis.

c Find the angle the line makes with the x-axis.

25 Point $A(3, 1, -4)$ lies on the line L which is perpendicular to plane Π: $3x-y-z=1$.

a Find the Cartesian equation of L.

b Find the intersection of the line L and plane Π.

c Point A is reflected in Π. Find the coordinates of the image of A.

d Point B has coordinates $(1, 1, 1)$. Show that B lies in Π.

e Find the exact distance between B and L.

26 The position vector of a particle at time t seconds is given by $\mathbf{r}=(4+3t)\,\mathbf{i}+(6-\mathrm{t})\,\mathbf{j}+(2t-7)\mathbf{k}$. The distance is measured in metres.

a Find the displacement of the particle from the starting point after 5 seconds.

b Find the speed of the particle.

c Determine whether the particle's path crosses the line connecting the points $(3, 0, 1)$ and $(1, 1, 5)$.

27 At time $t=0$ two aircraft have position vectors $5\mathbf{j}$ and $7\mathbf{k}$. The first moves with velocity $3\mathbf{i}-4\mathbf{j}+\mathbf{k}$ and the second with velocity $5\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

a Write down the position vector of the first aircraft at time t.

b Show that at time t the distance, d, between the two aircraft is given by $d^2=44t^2-88t+74$.

c Show that the two aircraft will not collide.

d Find the minimum distance between the two aircraft.

28 The plane with equation $3x-y+5z=30$ intersects the coordinate axes at points A, B and C. Find the area of the triangle ABC.

29 **a** Given that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$, show that $\mathbf{b} \times \mathbf{a} = 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$.

Two planes have equations $\mathbf{r} \cdot \mathbf{a} = 5$ and $\mathbf{r} \cdot \mathbf{b} = 12$.

b Show that the point (2, 2, 3) lies in both planes.

c Hence write down the Cartesian equation of the line of intersection of the two planes.

30 Four points have coordinates $A(7, 0, 1)$, $B(8, -1, 4)$, $C(9, 0, 2)$, $D(6, 5, 3)$.

a Show that $\overrightarrow{AD}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Write down the equation of the plane Π containing the points A, B and C in the form $\mathbf{r} \cdot \mathbf{n} = k$.

c Find the exact distance of point D from plane Π.

d Find the volume of the tetrahedron $ABCD$.

31 The planes $x - 2y + z = 15$ and $4x + y - z = k$ intersect along the line $45x = 9y + 180 = 5z + 125$. Find the value of k.

32 Two lines have equations

$$L_1: \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \text{ and } L_2: \mathbf{r} = \begin{pmatrix} 0 \\ 7 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$$

a Find the acute angle between the lines.

b The two lines intersect at the point X. Find the coordinates of X.

c Show that the point $Y(9, -7, 3)$ lies on L_1.

d Point Z lies on L_2 such that XY is perpendicular to YZ. Find the area of the triangle XYZ.

33 Consider the vectors $\boldsymbol{a} = \sin(2\alpha)\mathbf{i} - \cos(2\alpha)\mathbf{j} + \mathbf{k}$ and $\boldsymbol{b} = \cos\alpha\mathbf{i} - \sin\alpha\mathbf{j} - \mathbf{k}$, where $0 < \alpha < 2\pi$.

Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

a Express $\cos\theta$ in terms of α.

b Find the acute angle α for which the two vectors are perpendicular.

c For $\alpha = \frac{7\pi}{6}$, determine the vector product of $\mathbf{a}$ and $\mathbf{b}$ and comment on the geometrical significance of this result.

Mathematics HL November 2010 Paper 2 Q9

34 The equations of three planes, are given by

$$ax + 2y + z = 3$$

$$-x + (a+1)y + 3z = 1$$

$$-2x + y + (a+2)z = k$$

where $a \in \mathbb{R}$.

a Given that $a = 0$, show that the three planes intersect at a point.

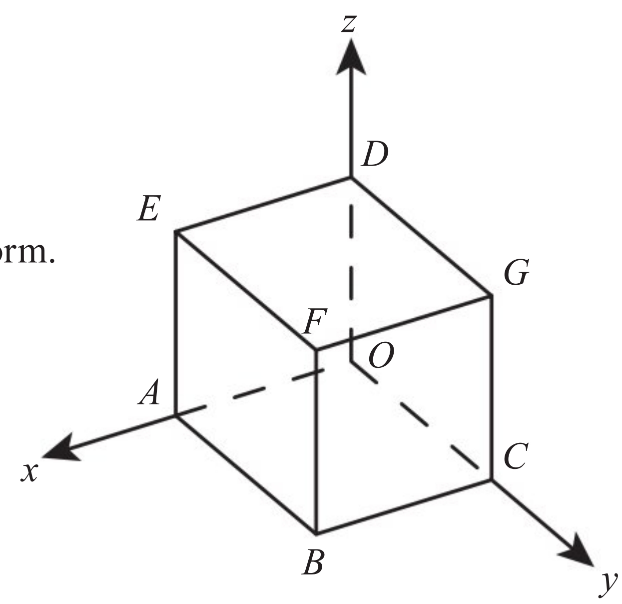
b Find the value of a such that the three planes do **not** meet at a point.

c Given a such that the three planes do **not** meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

Mathematics HL May 2011 Paper 2 TZ1 Q11

35 The diagram shows a cube *OABCDEFG*.



Let O be the origin, (OA) the x-axis, (OC) the y-axis and (OD) the z-axis. Let M, N and P be the midpoints of $[FG]$, $[DG]$ and $[CG]$, respectively.The coordinates of F are (2, 2, 2).

a Find the position vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$ and $\overrightarrow{OP}$ in component form.

b Find $\overrightarrow{MP} \times \overrightarrow{MN}$.

c Hence

i calculate the area of the triangle MNP

ii show that the line (AG) is perpendicular to the plane MNP

iii find the equation of the plane MNP.

d Determine the coordinates of the point where the line (AG) meets the plane MNP.

Mathematics HL November 2010 Paper 2 Q12

36 Two lines are given by $l_1: \mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

a l_1 and l_2 intersect at P. Find the coordinates of P.

b Show that the point $Q(5, 2, 5)$ lies on l_2.

c Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

d Find the area of the triangle PQM.

37 Points P and Q have position vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = (2+t)\,\mathbf{i} + (1-t)\,\mathbf{j} + (1+t)\,\mathbf{k}$. Find the value of t for which the distance PQ is the minimum possible and find this minimum distance.

38 The plane $3x + 2y - z = 2$ contains the line $x - 3 = \dfrac{2y+2}{5} = \dfrac{z-5}{k}$. Find k.

39 **a** Find the equation of the line l of intersection of the planes $\Pi_1: x - 3y + 5z = 12$ and $\Pi_2: 5x + y + z = 20$

b Find the Cartesian equation of the plane Π_3 which contains l and is perpendicular to Π_1.

40 Two lines with equations $l_1: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ intersect at point P.

a Show that $Q(8, 2, 6)$ lies on l_2.

b R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R.

c Find a vector equation of a line through P which bisects the angle QPR.

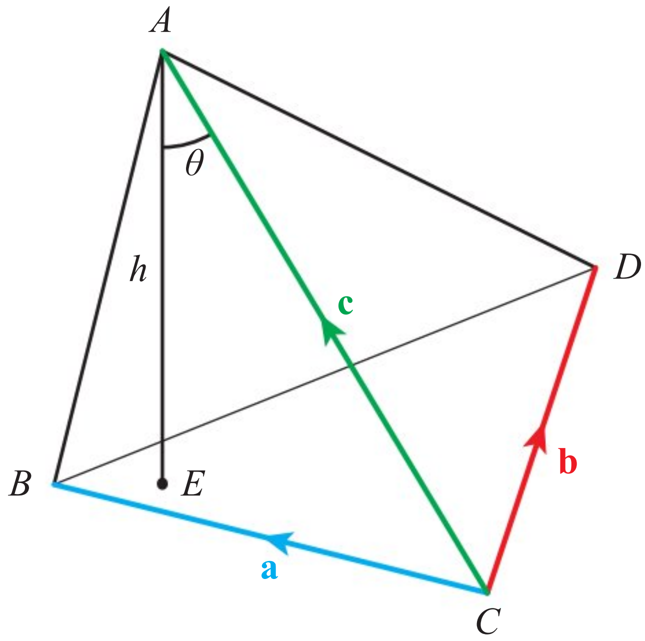
41 **a** Line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates (7, 2, 3).

Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

b Hence find the shortest distance from P to l.

c Q is a reflection of P in line l. Find the coordinates of Q.

42 Consider the tetrahedron shown in the diagram and define vectors $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CD}$ and $\mathbf{c} = \overrightarrow{CA}$.



a Write down an expression for the area of the base in terms of vectors **a** and **b** only.

b AE is the height of the tetrahedron, $|AE| = h$ and $\angle CAE = \theta$. Express h in terms of **c** and θ.

c Use the results of part **a** and part **b** to prove that the volume of the tetrahedron is given by $\left|\frac{1}{6}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right|$.

d Find the volume of the tetrahedron with vertices $A(0, 4, 0)$, $B(0, 6, 0)$, $C(1, 6, 1)$ and $D(3, -1, 2)$.

e Find the distance of the vertex A from the face BCD.

f Determine which of the vertices A and B is closer to its opposite face.

43 Two lines have equations l_1: $\mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 18 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -8 \end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

a Show that the lines do not intersect.

b Show that the point $P(1, 1, 2)$ lies on l_1.

c Show that there is a point Q on l_2 such that PQ is perpendicular to both lines, and find its coordinates.

d Find the equation of the plane Π which is parallel to both lines and passes half-way between them.

e The line l_3 is the reflection of l_2 in Π. Write down a vector equation for l_3.

44 Plane Π has equation $5x - 3y - z = 1$.

a Show that point $P(2, 1, 6)$ lies in Π.

b Point Q has coordinates $(7, -1, 2)$. Find the exact value of the sine of the angle between PQ and Π.

c Find the exact distance PQ.

d Hence find the exact distance of Q from Π.

45 Three planes have equations

$$x + 3y + (a-1)z = 1$$

$$2x + 2y + (a-2)z = 1$$

$$3x + y + (a-3)z = b.$$

a Show that, for all values of a, the three planes do not intersect at a single point.

b Find the value of b for which the intersection of the three planes is a straight line.

c For this value of b, find the value of a for which the intersection line is perpendicular to the line with parametric equations

$$x = 3 + \lambda,\ y = 1 - 3\lambda,\ y = 5\lambda - 3.$$

46 Plane Π has equation $\mathbf{r} \cdot \mathbf{n} = k$ and point P, outside Π, has position vector $\mathbf{p}$.

a Write down a vector equation of the line l through P which is perpendicular to Π.

b Line l intersects Π at Q. Show that $PQ = \left(\dfrac{k - \mathbf{p} \cdot \mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$.

c Hence show that the shortest distance from P to Π is given by $\dfrac{|\mathbf{p} \cdot \mathbf{n} - k|}{|\mathbf{n}|}$.

d Use the result from part **c** to find the shortest distance from the point with coordinates (4, –2, 8) to the plane with equation $3x + y - 4z = 22$.

47 Port A is defined to be the origin of a set of coordinate axes and port B is located at the point (70, 30) where distances are measured in kilometres. A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $\mathbf{r} = t\begin{pmatrix} 10 \\ 20 \end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

Mathematics HL May 2011 Paper 2 TZ1 Q10

48 **a** For non-zero vectors **a** and **b**, show that

i if $|\mathbf{a} - \mathbf{b}| = |\mathbf{a} + \mathbf{b}|$, then **a** and **b** are perpendicular

ii $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

b The points A, B and C have position vectors **a**, **b** and **c**.

i Show that the area of triangle ABC is $\dfrac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$.

ii Hence show that the shortest distance from B to AC is

$$\frac{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}{|\mathbf{c} - \mathbf{a}|}.$$

Mathematics HL November 2011 Paper 1 Q12

9 Probability

ESSENTIAL UNDERSTANDINGS

- The theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict occurrence of events.
- Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.
- Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical or observed.

In this chapter you will learn…

- how to reverse conditional probabilities (Bayes' theorem)
- how to find the variance of a discrete random variable
- how to represent continuous random variables and find associated probabilities
- how to find mean, median, mode and variance of a continuous random variable.

CONCEPTS

The following concepts will be addressed in this chapter:

- Probability methods such as Bayes' theorem can be applied to real-world **systems**, such as medical studies or economics, to inform decisions and to better understand outcomes.
- **Approximation** in data can approach the truth but may not always achieve it.
- **Changes** to variables can have predictable effects on observed **quantities**.

LEARNER PROFILE – Thinkers

How rich are you? How good are you at maths?? How fit are you? Can you estimate how you compare to other people in your school? In your country? In the world? See if you can find data to quantify where you might be on some of these scales. In the top 50%? The top 1%? Very few people have a good estimate of where they fit into society, and sometimes statistics are a great way of holding a mirror up to your life.

■ **Figure 9.1** How do we compare different distributions?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. A bag contains five red and eight blue balls. Two balls are selected at random. Using a tree diagram
   a. find the probability that the balls are the same colour
   b. given that the first ball is red, find the probability that the balls are the same colour.
2. The probability distribution of a random variable X is shown in the table.

x	1	2	3	4
$P(X = x)$	0.3	0.1	p	0.2

   a. Find the value of P.
   b. Find $E(X)$.
3. a. Evaluate $\int_0^{\pi} \sin x \, dx$.
   b. Find the value of k such that $\int_1^k \frac{1}{3x} \, dx = \frac{1}{2}$.

You are already familiar with a number of techniques for finding probabilities (tree diagrams, Venn diagrams, tables of outcomes) and with the concept of conditional probability, where the probability of an event changes depending on the outcome of a previous event. Bayes' theorem is an important tool in more advanced work with conditional probabilities and has many applications, for example in medical and legal trials.

Random variables can be used to model many situations where the outcome depends on chance. In this chapter you will learn how to work with both discrete and continuous variables, and how to calculate various measures of average and spread.

Starter Activity

Look at the pictures in Figure 9.1. Discuss which of the two archers is better. What measures could you use to compare them?

Now look at this problem:

What is the probability that a person is exactly 170 cm tall?

You met this formula in Chapter 19 of the Mathematics: analysis and approaches SL book.

Tip

Remember that $A \cap B$ means that both A and B occur.

9A Bayes' theorem

You have already met conditional probabilities, where the probability of an event occurring changes depending on the outcome of some other event. The conditional probability of A occurring given B has occurred is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' theorem is a way of reversing this formula to find the probability of B given A.

Proof 9.1

Show that $P(B \mid A) = \dfrac{P(B)P(A \mid B)}{P(A)}$.

Start by using the conditional probability formula for $P(B \mid A)$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

We want to introduce $P(A \mid B)$, so use $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

But $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

so

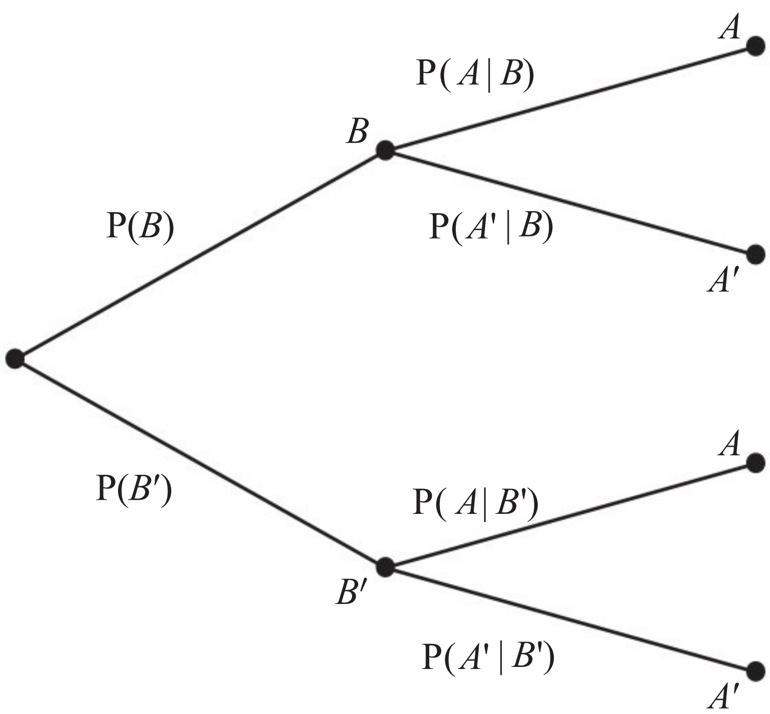
$$P(A \cap B) = P(B)\,P(A \mid B)$$

Substitute this into the first equation, noting that $P(A \cap B)$ and $P(B \cap A)$ are the same thing

Hence,

$$P(B \mid A) = \frac{P(B)P(A \mid B)}{P(A)}$$

To use the formula, you also need to know P(A). This can be found by using a tree diagram. Remember that the probabilities you know are conditional on B, so the first level of the tree diagram needs to show event B.



Thomas Bayes was an eighteenth-century English theologian and mathematician whose work was based on the idea that acquiring new evidence modifies probabilities.

KEY POINT 9.1

Bayes' theorem for two events:

$$P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B')P(A \mid B')}$$

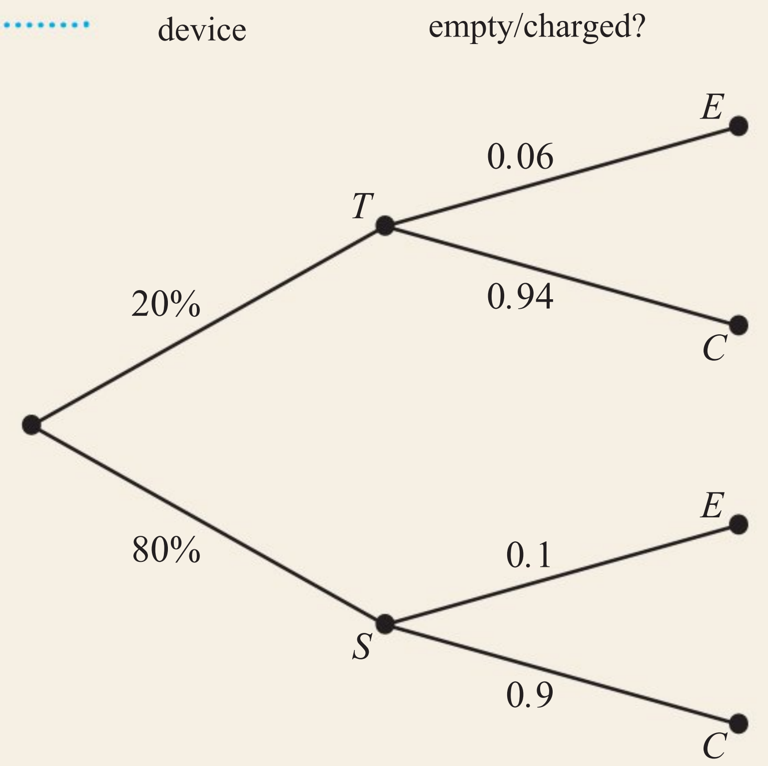
This formula looks rather complicated and difficult to remember. To use it in a question, you can simply draw a tree diagram to identify all the relevant probabilities.

WORKED EXAMPLE 9.1

All students in a class use a personal device to do some research. 20% of the students use a tablet and the rest of them use a smartphone. The probability that a tablet runs out of charge during a lesson is 0.06 and the probability that a smartphone runs out of charge is 0.1. Given that a student's device runs out of charge during a lesson, find the probability that this student is using a tablet.

Start by drawing a tree diagram to show conditional probabilities

The probability of running out of charge depends on the type of device, so the first level of the diagram is the type of device



The required probability is P(tablet | empty). This is a conditional probability

$$P(\text{tablet} \mid \text{empty}) = \frac{P(\text{tablet and empty})}{P(\text{empty})}$$

Use the tree diagram to find both probabilities needed to work out P(tablet | empty)

P(tablet and empty) is found by multiplying along the top branch

P(empty) is found by adding the probabilities of the two branches which end in empty

$$= \frac{0.2 \times 0.06}{0.2 \times 0.06 + 0.8 \times 0.1}$$

$$= 0.130$$

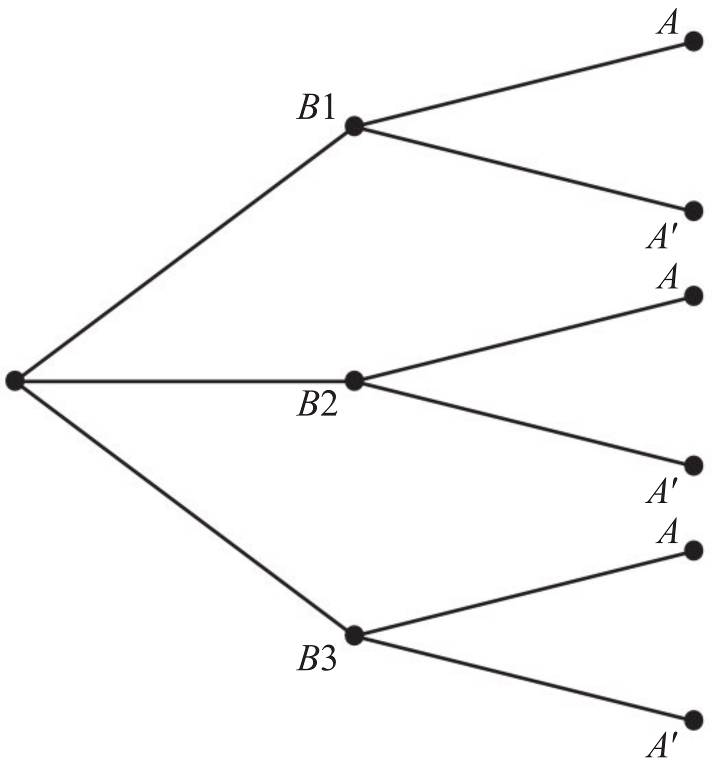
TOK Links

In Worked Example 9.1, we used the ideas from the proof of Bayes' theorem to solve a specific problem. Does this change your opinion about the usefulness of mathematical proof?

CONCEPTS – SYSTEMS

In most real-world **systems**, the probability of a certain outcome will depend on several factors. Bayes' theorem enables us to adjust our estimates of probability based on new data.

Bayes' theorem can be extended to the case where there are more options than just 'B occurs' and 'B does not occur'. In this course you only need to deal with examples that extend to three possible outcomes for the first event.



KEY POINT 9.2

Bayes' theorem for three events:

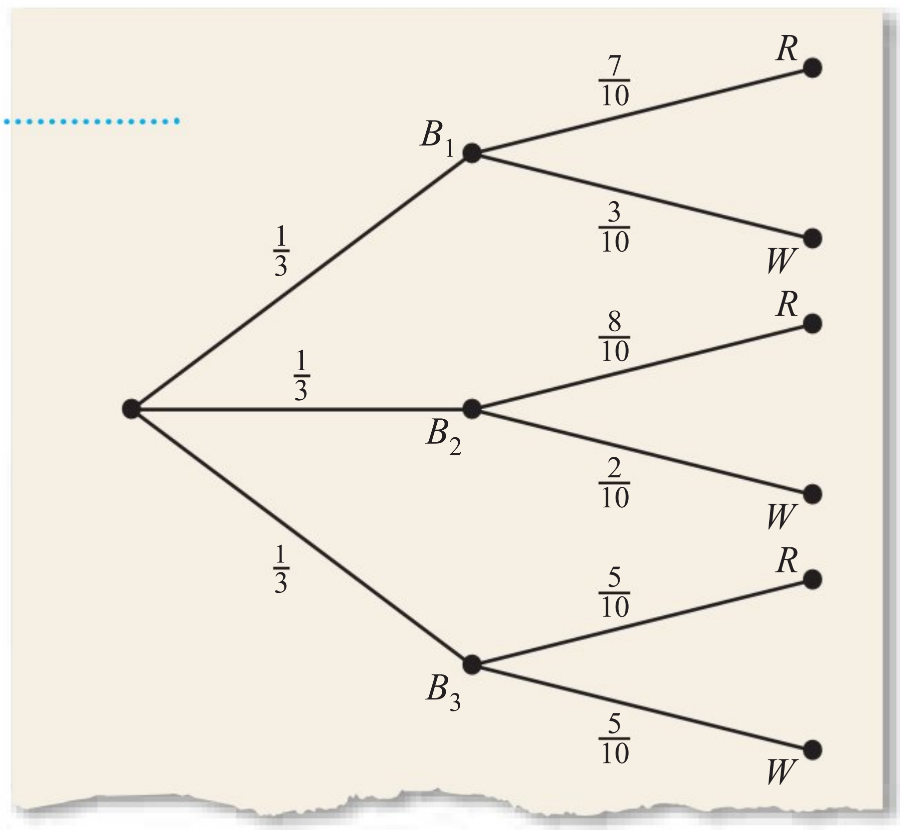
$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$$

As before, the best way to tackle a question is usually to draw a tree diagram.

WORKED EXAMPLE 9.2

Three bags, B_1, B_2 and B_3, contain red and white balls. Bag B_1 contains seven red and three white balls. Bag B_2 contains eight red and two white balls. Bag B_3 contains five red and five white balls. One of the bags is selected at random, and then a ball is selected from that bag. Given that the ball is red, find the probability that it came from bag B_1.

Draw a tree diagram to represent the situation. The probability of a red ball depends on the bag, so the first level of the diagram should show the bags



The required probability is $P(B_1 | R)$, so use the conditional probability formula. $P(B_1 | R) =$

$$P(B_1 | R) = \frac{P(B_1 \cap R)}{P(R)}$$

Use the tree diagram to find the two probabilities:

$P(B_1 \cap R) = P(B_1)\,P(R | B_1)$

$P(R) = P(B_1)\,P(R | B_1) + P(B_2)\,P(R | B_2) + P(B_3)\,P(R | B_3)$

$$= \frac{\frac{1}{3}\times\frac{7}{10}}{\left(\frac{1}{3}\times\frac{7}{10}\right)+\left(\frac{1}{3}\times\frac{8}{10}\right)+\left(\frac{1}{3}\times\frac{5}{10}\right)}$$

$$= \frac{7}{20}$$

TOOLKIT: Problem Solving

There is a famous puzzle, that we covered in Chapter 7 of Mathematics: analysis and approaches SL, called the Monty Hall problem in which a game show contestant has three doors to choose from. Behind one is a luxury car and behind the other two are goats. The contestant chooses one door, but before it is opened the host, Monty Hall, (who knows what is behind each door) opens a door to show a goat. He then gives the contestant the opportunity to stick with his original door or switch to the other unopened door. In a magazine column, mathematician Marilyn vos Savant suggested that switching was better, resulting in a huge number of letters including from leading mathematicians claiming that she was wrong.

a Simulate this situation using either a computer or just repeated experiments with balls under cups or playing cards. Does this suggest that Marilyn was correct or incorrect?

b Prove your assertion using Bayes' theorem.

TOK Links

By considering the story about Marilyn vos Savant in the box opposite, discuss the role of expert opinion in mathematics. What is the role of intuition?

Exercise 9A

For questions 1 to 3, use the method demonstrated in Worked Example 9.1, or the formula from Key Point 9.1, to find the required probability.

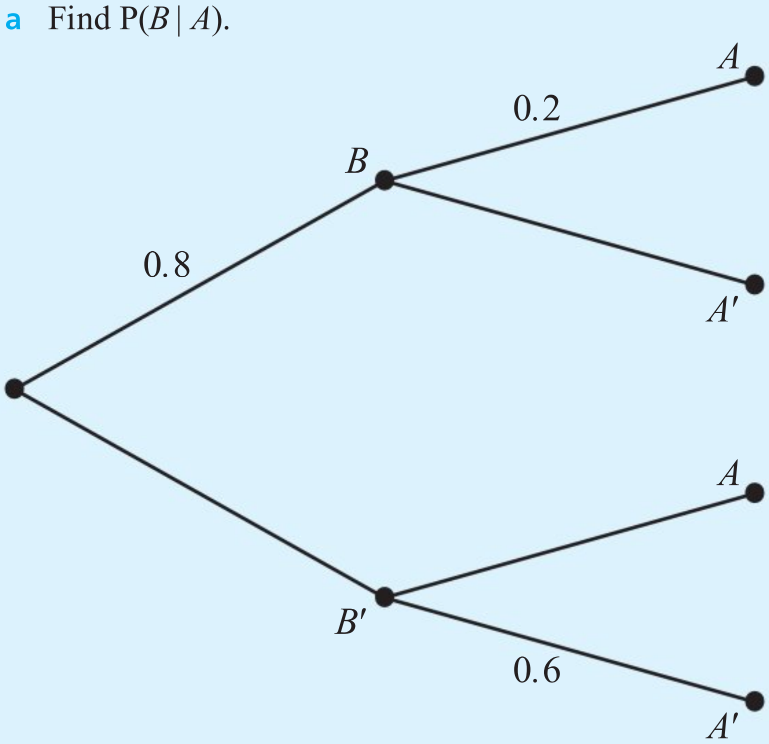
1 **a** $P(B) = 0.3$, $P(A | B) = 0.6$, $P(A | B') = 0.8$, find $P(B | A)$

b $P(B) = 0.7$, $P(A | B) = 0.9$, $P(A | B') = 0.2$, find $P(B | A)$

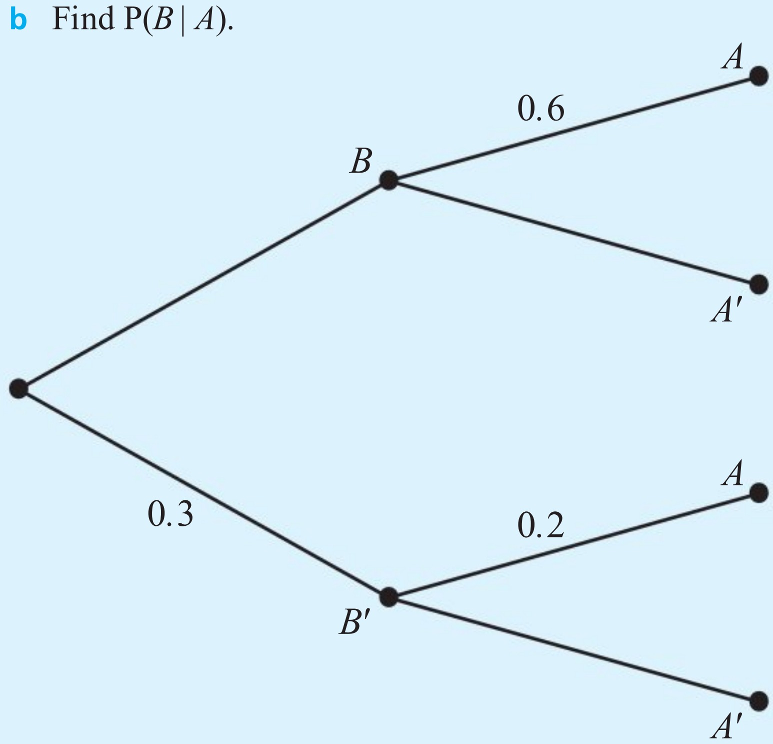
2 **a** $P(B) = 0.3$, $P(A | B) = 0.6$, $P(A | B') = 0.8$, find $P(B | A')$

b $P(B) = 0.7$, $P(A | B) = 0.9$, $P(A | B') = 0.2$, find $P(B | A')$

3 **a** Find $P(B | A)$.



b Find $P(B | A)$.



For questions 4 to 6, use the method demonstrated in Worked Example 9.2 to find the required probability.

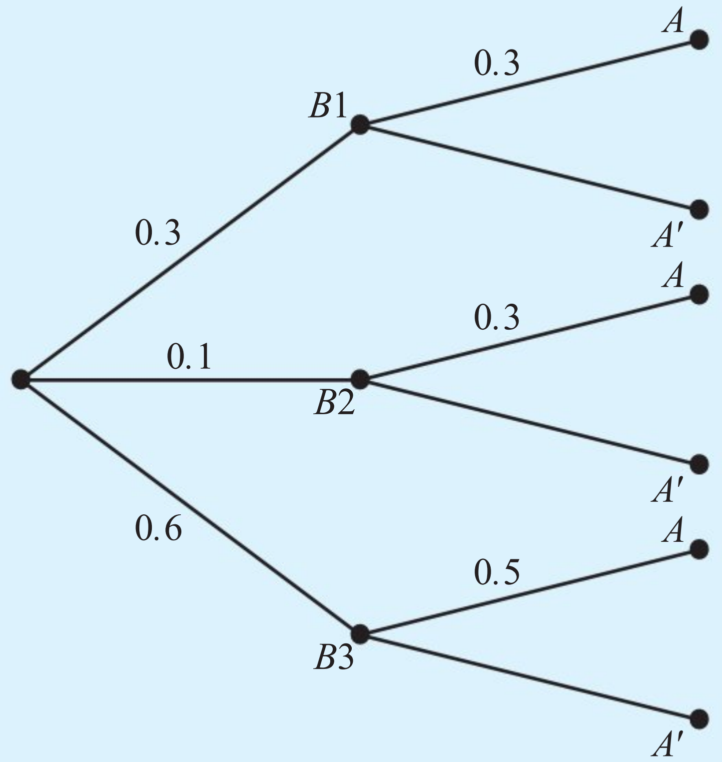
4 **a** $P(B_1) = 0.2$, $P(B_2) = 0.5$, $P(B_3) = 0.3$;
$P(A \mid B_1) = 0.3$, $P(A \mid B_2) = 0.7$, $P(A \mid B_3) = 0.1$.
Find $P(B_1 \mid A)$.

b $P(B_1) = 0.2$, $P(B_2) = 0.5$, $P(B_3) = 0.3$;
$P(A \mid B_1) = 0.4$, $P(A \mid B_2) = 0.2$, $P(A \mid B_3) = 0.8$.
Find $P(B_2 \mid A)$.

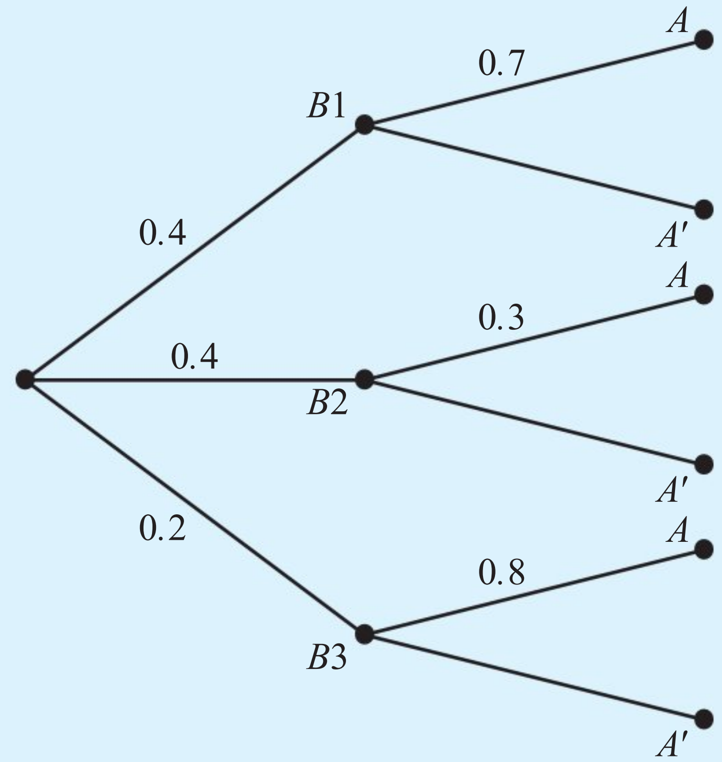
5 **a** $P(B_1) = 0.4$, $P(B_2) = 0.3$, $P(B_3) = 0.3$;
$P(A \mid B_1) = 0.8$, $P(A \mid B_2) = 0.5$, $P(A \mid B_3) = 0.3$.
Find $P(B_1 \mid A')$.

b $P(B_1) = 0.4$, $P(B_2) = 0.3$, $P(B_3) = 0.3$;
$P(A \mid B_1) = 0.2$, $P(A \mid B_2) = 0.9$, $P(A \mid B_3) = 0.7$.
Find $P(B_3 \mid A')$.

6 **a** Find $P(B_2 \mid A)$.



b Find $P(B_3 \mid A)$.



7 Every morning, I either take the bus or the train. The probability that I take the bus is 0.6. When I take the bus, the probability that I arrive to work on time is 0.7. If I take the train, the probability that I am on time is 0.9.

a Draw a tree diagram to represent this situation.

b Find the probability that I arrive on time.

c Given that I am on time on a particular morning, find the probability that I took the bus.

8 A box contains six small and eight large eggs. The probability that a small egg is broken is 0.1 and the probability that a large egg is broken is 0.2.

a Draw a tree diagram to represent this situation.

b Find the probability that a randomly selected egg is broken.

c Given that an egg is broken, find the probability that it is a small egg.

9 In a large population of cats, 20% are white and the rest are yellow. 40% of white cats have spots and 10% of yellow cats have spots.

a Find the probability that a randomly selected cat from the population has spots.

b Given that a cat has spots, find the probability that it is white.

10 A bag contains ten red and eight blue balls. Two balls are selected at random, without replacement.

a Find the probability that the two balls are different colours.

b Given that the first ball is red, find the probability that they are different colours.

c Given that the two balls are different colours, find the probability that the first one was red.

11 A bag contains a large number of tokens, 30% of which are green and the rest are orange. One token is selected at random and replaced. Then a second token is selected.

a Find the probability that the second token is green.

b Given that the second token is green, find the probability that the first one was also green.

12 Alessia has either chips or beans for lunch. The probability that she has chips is 0.7. If she has chips, the probability that she eats an ice-cream afterwards is 0.9. If she has beans, the probability that she eats an ice-cream is 0.8.

a Find the probability that Alessia eats an ice-cream after lunch.

b Alessia is seen eating an ice-cream. What is the probability that she had chips for lunch?

13 Asher and Elsa are doing their Mathematics homework. The probability that Elsa makes a mistake on any question is 0.07 and the probability that Asher makes a mistake is 0.1. Asher does 20 questions and Elsa does 18 questions. One question is selected at random.

a Find the probability that this question is correct.

b Given that the question is correct, find the probability that it was done by Elsa.

14 Each morning I either walk, cycle or drive to work. The probability that I walk is 0.2 and I am equally likely to cycle or drive. When I walk, the probability that I am late is 0.05. When I cycle, the probability that I am late is 0.1. When I drive, the probability that I am late is 0.2.

a Draw a tree diagram to represent this situation.

b Find the probability that I am late.

c Given that I am late, find the probability that I walked.

15 Each week I buy either strawberries, bananas or peaches, each with equal probability. The probability that the strawberries are ripe is 0.6, the probability that the bananas are ripe is 0.9 and the probability that the peaches are ripe is 0.5.

a Draw a tree diagram representing this situation.

b Find the probability that my fruit is ripe.

c Given that my fruit is ripe, find the probability that I bought strawberries.

16 The probability that I pass a test that I sit in the morning is 0.8. The probability that I pass a test that I sit in the afternoon is 0.6. 40% of all my tests take place in the morning. I passed my last test. What is the probability that the test was in the morning?

17 I have a box of twenty 10p coins and a box of fifteen 20p coins. My 10p coins are fair, but the 20p coins are biased so that the probability of coming up heads is $\frac{2}{3}$. I pick a box at random (with equal probability of picking either) and toss all the coins in it. Given that exactly ten coins come up heads, what is the probability that I selected the box with 10p coins?

18 A bag contains 10 four-sided dice (with sides numbered 1 to 4), 15 six-sided dice (with numbers 1 to 6) and 20 eight-sided dice (with sides numbered 1 to 8). All the dice are known to be fair. I select a dice and random and roll it. Given that the outcome is smaller than 6, find the probability that it was a four-sided dice.

19 A football team has three goal keepers, Ali, Bobby and Carter. The probability that they save a penalty is 0.3 for Ali, 0.2 for Bobby and 0.2 for Carter. They are each equally likely to be in goal when a penalty is taken. Given that a penalty is saved, find the probability that Ali was in goal.

20 A new test is developed for a rare disease. The test gives a positive result in 10% of patients who don't have the disease, and gives a negative result in 2% of patients who do have the disease. It is estimated that 0.3% of the population have this disease. Given that a patient receives a positive test results, what is the probability that they have the disease?

21 A test for a disease is known to be 90% accurate, so it gives positive result for 90% of patients who have the disease and negative result for 90% of patients who do not have the disease. It is found that, out of 200 individuals – that were chosen in a way that is representative of the population – who tested positive for the disease, 45 actually did not have it. Estimate the percentage of the population who have the disease.

22 In a game, a machine dispenses tokens which are either yellow or red. 30% of the tokens contained in the machine are yellow, and there are a large number of well-mixed tokens in the machine. When a lever is pushed, one of two slots is selected at random, with each slot being equally likely to be chosen. The machine dispenses either seven tokens from Slot A or ten tokens from Slot B. The lever is pushed once. Given that exactly three yellow tokens are dispensed, find the probability that they came from Slot A.

23 **a** A bag contains m black and n white balls. Two balls are selected at random, without replacement. Find an expression for the probability that both balls are the same colour.

b A yellow bag contains 5 black and 15 white balls. A blue bag contains 7 green and 8 red balls. A bag is selected at random, and two balls are randomly selected from that bag, without replacement. Given that the two balls are the same colour, find the probability that they came from the blue bag.

9B Variance of a discrete random variable

In Chapter 8 of the Mathematics: analysis and approaches SL book, you met discrete random variables and their probability distributions (possible values and their probabilities). You learnt that the expected value of a discrete random variable represents its average value, and is given by the formula $\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X = x)$.

Knowing the average value alone does not tell you much about the values the variable is likely to take. It is also useful to know how spread out those values are. One way to measure spread is the **variance**.

You met variance and standard deviation of data sets in Chapter 6 of the Mathematics: analysis and approaches SL book.

KEY POINT 9.3

The variance of a random variable X is:

- $\mathrm{Var}(X) = \sum_x x^2\ \mathrm{P}(X = x) - (\mathrm{E}(X))^2$
- The standard deviation is $\sqrt{\mathrm{Var}(X)}$.

Tip

This formula can also be written as $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$.

WORKED EXAMPLE 9.3

A discrete random variable X has the probability distribution given in this table.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.3	0.4	0.1

Find $\mathrm{Var}(X)$.

You first need to find $\mathrm{E}(X)$

$$\mathrm{E}(X) = \sum xp = 0 + 0.3 + 0.8 + 0.3 = 1.4$$

Then find $\mathrm{E}(X^2)$

$$\mathrm{E}(X^2) = \sum x^2 p = 0 \times 0.2 + 1 \times 0.3 + 4 \times 0.4 + 9 \times 0.1 = 2.8$$

Use the formula for the variance

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 2.8 - 1.4^2 = 0.84$$

Linear transformations of a random variable

If you add the same constant to every possible value of a random variable, its expected value will increase by the same constant, but the variance will remain unchanged. If you multiply all the value by a constant, both the expected value and the variance will change.

KEY POINT 9.4

- $E(aX + b) = aE(X) + b$
- $Var(aX + b) = a^2 Var(X)$

You learnt in Chapter 6 of the Mathematics: analysis and approaches SL book that the same rules apply to constant changes to data sets.

CONCEPTS – QUANTITIES AND CHANGE

The results from Key Point 9.4 are used when rescaling **quantities** or **changing** units. In such situations it is important to know what changes and what remains the same.

WORKED EXAMPLE 9.4

A random variable X has expected value 12.5 and variance 4.8. A random variable Y is given by $Y = 3X - 2$. Find the expected value and the variance of Y.

Use $E(aX + b) = aE(X) + b$

$$E(Y) = E(3X - 2) = 3E(X) - 2 = 3 \times 12.5 - 2 = 35.5$$

Use $Var(aX + b) = a^2 Var(X)$

$$Var(Y) = Var(3X - 2) = 3^2 \times 4.8 = 43.2$$

Be the Examiner 9.1

A random variable X has $E(X) = 5$ and $Var(X) = 3.5$. Find the variance of the random variable $Y = 10 - 4X$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$Var(10 - 4X) = 10 - 4(3.5)$ $= -5$	$Var(10 - 4X) = -4 \times (3.5)$ $= -14$	$Var(10 - 4X) = 16 \times (3.5)$ $= 56$

Tip

Variance can never be negative.

TOOLKIT: Modelling

Consider a random variable X which has $P(X = 1) = P(X = 0) = 0.5$.

a Find $E(X)$ and $Var(X)$.

Two independent observations of X are made and their mean found. This is called $\bar{X}$.

b What values can $\bar{X}$ take? Write down the probability distribution of $\bar{X}$.

c Find $E(\bar{X})$ and $Var(\bar{X})$. How does this compare to your answers in part **a**?

d Repeat parts **b** and **c** for a mean of three independent observations of X.

e What does this tell you about the design of scientific experiments?

Exercise 9B

For questions 1 to 4, use the method demonstrated in Worked Example 9.3 to find the variance of the random variable with the given probability distribution.

1 a

x	0	1	2	3
$P(X = x)$	0.1	0.2	0.3	0.4

b

x	0	1	2	3
$P(X = x)$	0.4	0.3	0.2	0.1

2 a

x	−2	−1	1	2
$P(X = x)$	0.25	0.25	0.25	0.25

b

x	−5	−2	2	5
$P(X = x)$	0.25	0.25	0.25	0.25

3 a $P(X = x) = \frac{x}{10}$ for $x = 1, 2, 3, 4$

b $P(X = x) = \frac{x+1}{10}$ for $x = 0, 1, 2, 3$

4 a $P(X = x) = \frac{x^2}{14}$ for $x = 1, 2, 3$

b $P(X = x) = \frac{6}{11x}$ for $x = 1, 2, 3$

For questions 5 to 11, to you are given the mean and variance of a random variable X. Find the mean and variance of the related random variable Y.

5 a $E(X) = 3$, $\text{Var}(X) = 7$, $Y = 2X + 5$

b $E(X) = 11$, $\text{Var}(X) = 5$, $Y = 4X + 3$

6 a $E(X) = 3$, $\text{Var}(X) = 7$, $Y = 2X - 5$

b $E(X) = 11$, $\text{Var}(X) = 5$, $Y = 4X - 3$

7 a $E(X) = 9$, $\text{Var}(X) = 2$, $Y = \frac{1}{3}X + 1$

b $E(X) = 10$, $\text{Var}(X) = 15$, $Y = \frac{1}{5}X + 2$

8 a $E(X) = 10.5$, $\text{Var}(X) = 3.7$, $Y = -3X + 1$

b $E(X) = 7.5$, $\text{Var}(X) = 1.8$, $Y = -10X + 20$

9 a $E(X) = 0.6$, $\text{Var}(X) = 0.2$, $Y = 5 - X$

b $E(X) = 1.2$, $\text{Var}(X) = 0.3$, $Y = 4 - X$

10 a $E(X) = 10$, $\text{Var}(X) = 2$, $Y = 3(X - 2)$

b $E(X) = 20$, $\text{Var}(X) = 3$, $Y = 2(X - 5)$

11 a $E(X) = 14$, $\text{Var}(X) = 6$, $Y = \frac{X-2}{6}$

b $E(X) = 18$, $\text{Var}(X) = 10$, $Y = \frac{X-3}{5}$

12 The table shows the probability distribution of a random variable X.

x	0	1	2	3	4
$P(X = x)$	k	0.2	0.1	0.1	0.3

Find

a the value of k

b $E(X)$

c $\text{Var}(X)$.

13 Random variable X has the probability distribution shown in the table.

x	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	k

Find

a the value of k

b the standard deviation of X.

14 The table shows the probability distribution of a random variable X.

x	1	3	5	7
$P(X = x)$	0.2	0.3	0.2	0.3

a Find $E(X)$ and $\text{Var}(X)$.

b The random variable Y is given by $Y = 3X + 1$. Find $E(Y)$ and $\text{Var}(Y)$.

15 The random variable W has the probability distribution given by $P(W = w) = \frac{k}{w}$ for $W = 1, 2, 4, 8$.

- **a** Find the value of k.
- **b** Show that $\text{Var}(W) = 3.45$ to three significant figures.
- **c** The random variable V is given by $V = 2W - 2$. Find $\text{Var}(V)$.

16 The probability distribution of a random variable X is given by $P(X = x) = \frac{2x - 1}{16}$ for $x = 1, 2, 3, 4$.

- **a** Find $E(X)$ and show that $\text{Var}(X) = \frac{55}{64}$.
- **b** Find the mean and the variance for the random variable $Y = 10X + 3$.

17 Random variable X has the following probability distribution.

x	2	3	5	8
$P(X = x)$	0.2	0.3	0.4	0.1

- **a** Find $E(X)$ and $\text{Var}(X)$.

Random variable Y is defined as $Y = 3X + 1$.

- **b** Write down the probability distribution of Y.
- **c** Verify that $\text{Var}(Y) = 9\text{Var}(X)$.

18 The probability distribution of a random variable V is given by $P(V = v) = \frac{v}{20}$ for $v = 2, 4, 6, 8$.

- **a** Find the mean and variance of V.

Random variable W is such that $V + W = 5$.

- **b** Construct a probability distribution table for W.
- **c** Verify that $\text{Var}(W) = \text{Var}(V)$.

19 The random variable A has mean 3.8 and variance 1.2. The random variable B is such that $A + 2B = 10$. Find the mean and the variance of B.

20 The random variable U has mean 25 and variance 16. The random variable V satisfies $2U + 5V = 20$. Find the mean and variance of V.

21 The probability distribution of a random variable X is given in this table.

x	3	103	203	303	403
$P(X = x)$	0.1	0.2	0.2	0.3	0.2

The random variable Y is defined by $Y = \frac{X - 3}{100}$.

- **a** Write down the probability distribution of Y.
- **b** Find the mean and the variance of Y.
- **c** Hence find the mean and the variance of X.

22 A fair coin is flipped three times. X is the number of tails.

- **a** Write down the probability distribution of X.
- **b** Write down $E(X)$.
- **c** Show that the standard deviation of X is $\frac{\sqrt{3}}{2}$.

> You learnt in Section 8B of the Mathematics: analysis and approaches SL book that the variance of a binomial distribution is $np(1 - p)$.

23 A fair six-sided dice has its sides numbered 1, 1, 2, 2, 2, 3. The dice is rolled twice and the scores are added. Find the mean and standard deviation of the total.

24 Random variable W has the expected value 2. The probability distribution of W is shown in the table.

w	1	2	3	4
$P(W = w)$	0.4	0.3	a	b

a Show that $3a + 4b = 1$.

b Write down another equation for a and b.

c Find the variance of W.

25 The random variable X has the probability distribution shown in the table.

x	0	1	2	3
$P(X = x)$	0.1	p	q	0.2

Given that $E(X) = 1.5$, find $Var(X)$.

26 a A fair six-sided dice is rolled 30 times and X denotes the number of sixes. Write down the mean and variance of X.

In a game, a player rolls a fair six-sided dice 30 times and receives 10 cents each time he rolls a six. He is charged c cents to play the game. Let T denote a player's total profit.

b Find, in terms of c, the mean and variance of T.

c Find the value of c so that the game is fair.

27 A fair six-sided dice has sides numbered 1, 2, 5, 8, 10 and c, where $c > 10$. The dice is rolled once and the score is the random variable S. Given that the variance of S is $\frac{185}{9}$, find the value of c.

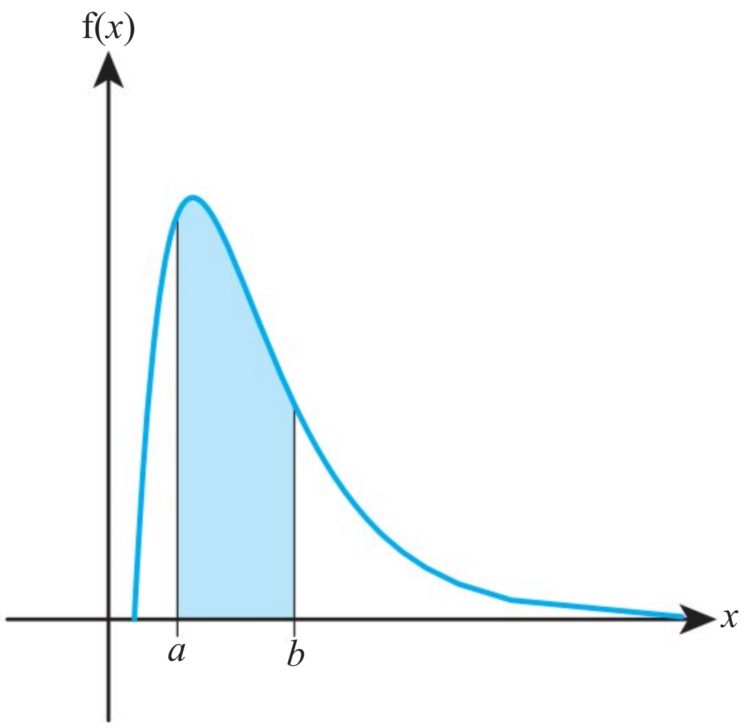
28 The probability distribution of a random variable X is given in the table.

x	0	1	2	3
$P(X = x)$	p	q	0.2	0.2

Given that $Var(X) = 0.85$, find $E(X)$.

9C Continuous random variables

A **continuous random variable** can take any real value in a given interval (which may be finite or infinite). It is impossible to list all the values and their probabilities. Instead, the distribution of a continuous random variable is described by a **probability density function**. The probability of the variable taking a value in a certain interval in given by the area under the graph of this function.



KEY POINT 9.5

For a continuous random variable X with probability density function $f(x)$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Tip

The probability that a continuous random variable takes any specific value is zero. It only makes sense to talk about the probability of it taking a value in an interval. This also means that, in the formula in Key Point 9.5, it does not matter whether you use $<$ or $\leqslant$.

Just like with discrete variables, the total probability needs to be equal to 1. Also, the probability can never be negative.

KEY POINT 9.6

If $f(x)$ is a probability density function, then

$$f(x) \geqslant 0 \text{ for all } x \text{ and } \int_{-\infty}^{\infty} f(x)\,dx = 1$$

In Chapters 8 and 19 of the Mathematics: analysis and approaches SL book you met the normal distribution, which is an example of a continuous distribution. This distribution can take all real values. The probability density function for a normal distribution cannot be integrated exactly (its equation is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma}(x-\mu)^2}$), so you need to use your GDC to find the probabilities.

Tip

In practice, a random variable often (but not always!) only takes values in a finite interval. In that case, the limits of the integral in Key Point 9.6 become the end points of that interval.

WORKED EXAMPLE 9.5

A continuous random variable X has probability density function

$$f(x) = \begin{cases} kx(5-x) & \text{for } 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k. **b** Find $P(1 < X < 3)$.

The total area under the graph of the probability density function must be 1. In this case, the only possible values of X are between 0 and 5

a $\int_0^5 kx(5-x)\,dx = 1$

k is a constant, so it can be taken out of the integral

$k\int_0^5 x(5-x)\,dx = 1$

Evaluate the integral using your GDC

$\frac{125}{6}k = 1$

$k = \frac{6}{125}$

The probability is given by the area under the graph

b $P(1 < X < 3) = \int_1^3 \frac{6}{125}x(5-x)\,dx$

Evaluate the integral using your GDC

$= 0.544$

Piecewise defined functions

A probability density function can have a different equation on different parts of its domain.

WORKED EXAMPLE 9.6

The probability density function of a continuous random variable X is given by

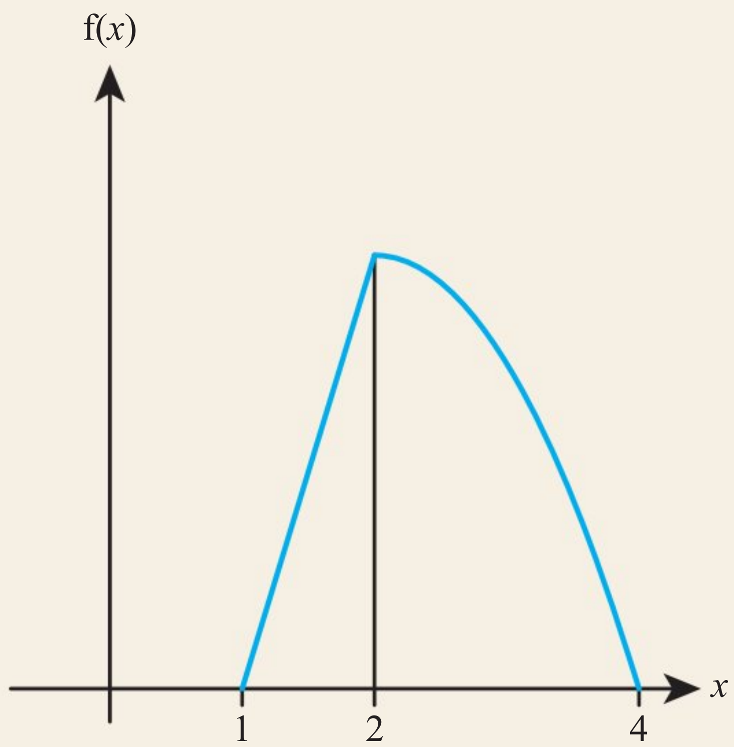
$$f(x) = \begin{cases} \frac{6}{11}(x-1) & \text{for } 1 \leqslant x \leqslant 2 \\ \frac{3}{22}x(4-x) & \text{for } 2 < x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $P(1.5 < X < 3)$.

You can use your GDC to draw the graph. Remember to only show the relevant parts

a



The probability is the area under the graph. Make sure you use the correct limits: use the first equation up to $x = 2$ and then the second equation from 2 to 3

b $P(1.5 < X < 3) = \int_{1.5}^{2} \frac{6}{11}(x-1)\,dx + \int_{2}^{3} \frac{3}{22}x(4-x)\,dx$

Use your GDC to evaluate each integral

$= \frac{9}{44} + \frac{1}{2}$

$= 0.705$ (3 s.f.)

CONCEPTS – APPROXIMATION

Continuous random variables can be used to model quantities such as length, time or mass. However, continuous quantities cannot be measured exactly, so the predictions from those models can only be **approximations** of actual values.

Mode and median of a continuous random variable

KEY POINT 9.7

The mode of a continuous random variable is the value with the largest probability density.

The mode could be the value of x for which $f'(x) = 0$, but it could also occur at an end-point. Just like discrete random variables, there can be more than one mode.

WORKED EXAMPLE 9.7

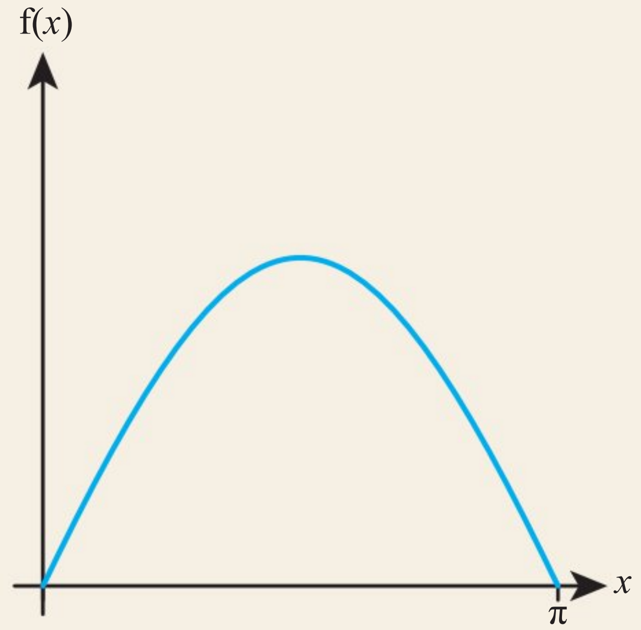
Sketch each probability density function and hence find the mode(s).

a $f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \leqslant x \leqslant \pi \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

Remember that x is in radians

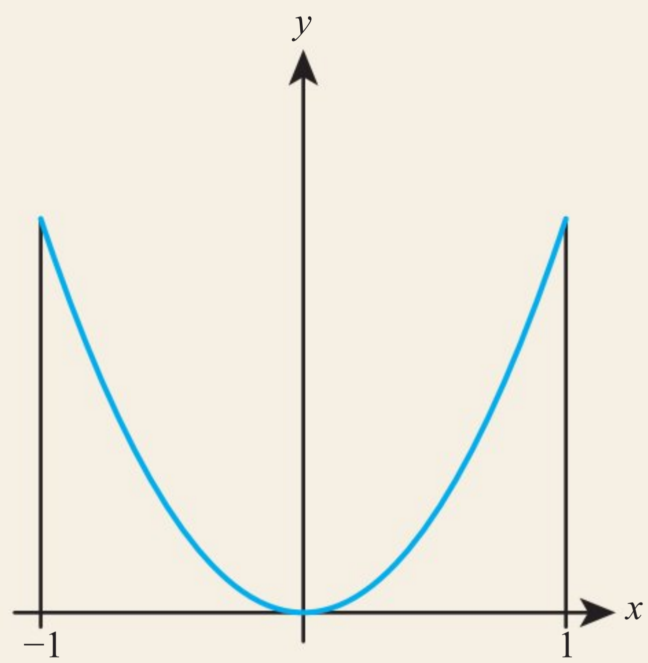
a



The mode corresponds to the maximum point on the graph. In this case you don't need to find $f'(x)$

The mode is $x = \frac{\pi}{2}$.

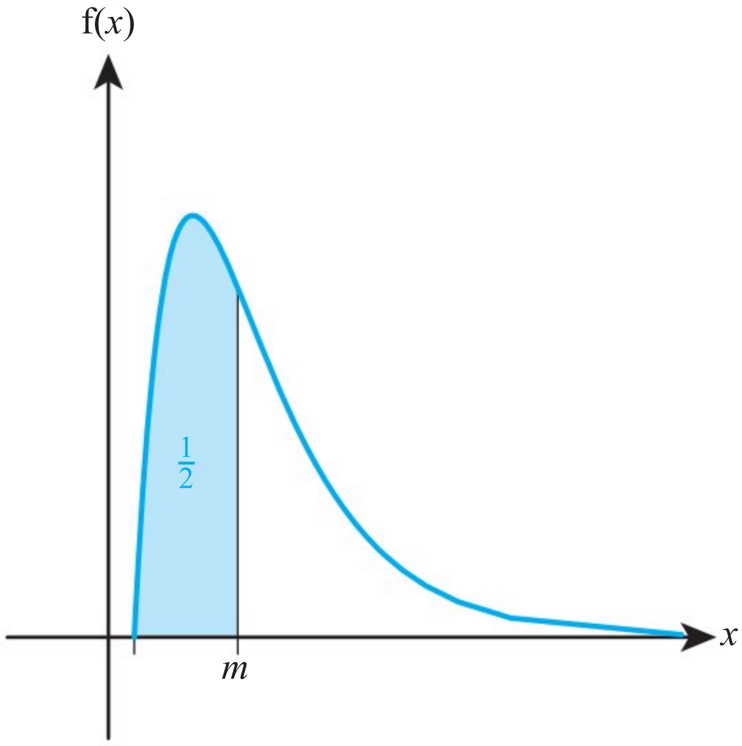
b



In this case, the largest value of $f(x)$ is at the end-points – there are two modes

The modes are $x = -1$ and $x = 1$.

The median is the value of x which splits the probability distribution in half. In other words, the probability on either side of the median is $\frac{1}{2}$.



KEY POINT 9.8

If m is the median of a continuous random variable X, then

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$

WORKED EXAMPLE 9.8

Find the exact value of the median of a random variable with the probability density function

$$f(x) = \begin{cases} \frac{1}{2}(x-1) & \text{for } 1 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

In this case, the lower limit of the integral is 1, so the median satisfies $\int_1^m f(x)\,dx = \frac{1}{2}$

$$\int_1^m \frac{1}{2}(x-1)\,dx = \frac{1}{2}$$

Integrate and apply the limits to get an equation for m

$$\left[\frac{1}{4}(x-1)^2\right]_1^m = \frac{1}{2}$$

$$\frac{1}{4}(m-1)^2 - \frac{1}{4}(0)^2 = \frac{1}{2}$$

$$(m-1)^2 = 2$$

Remember ± when taking square root of both sides

$$m-1 = \pm\sqrt{2}$$

$$m = 1 \pm \sqrt{2}$$

The median must be between the smallest and largest possible values of x

But $1 \leqslant x \leqslant 3$, so $m = 1 + \sqrt{2}$.

For a piecewise defined probability density function, you need to check in which part the median lies.

WORKED EXAMPLE 9.9

A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{4}x & \text{for } 0 \leqslant x \leqslant 1 \\ \frac{3}{4}e^{1-x} & \text{for } 1 \leqslant x \leqslant \ln(6e) \\ 0 & \text{otherwise} \end{cases}$$

Show that the median of X is $\ln\left(\frac{6e}{5}\right)$.

Check whether the median is between 0 and 1 by finding the area under this part of the graph

$$\int_0^1 \frac{3}{4}x\,dx = \frac{3}{8} < \frac{1}{2}$$

So, the median is greater than 1.

The area between 0 and 1 is $\frac{3}{8}$, so you need $\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$ to make the area $\frac{1}{2}$

$$\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$$

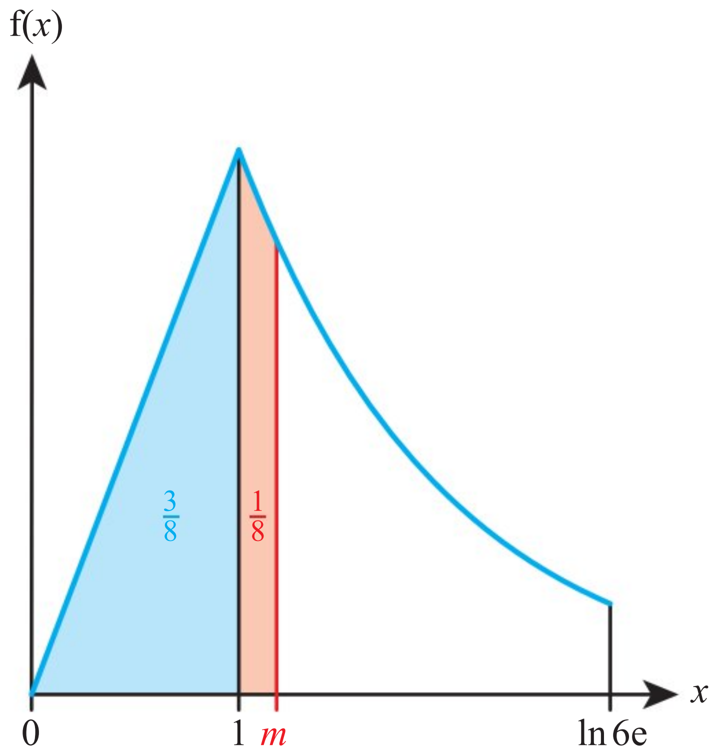
So,

$$\int_1^m \frac{3}{4}e^{1-x}\,dx = \frac{1}{8}$$

$$\frac{3}{4}\left[-e^{1-x}\right]_1^m = \frac{1}{8}$$

$$-e^{1-m} + e^0 = \frac{1}{8} \times \frac{4}{3} = \frac{1}{6}$$

$$e^{1-m} = e^0 - \frac{1}{6} = \frac{5}{6}$$



Take logs of both sides

$$1 - m = \ln\left(\frac{5}{6}\right)$$

$$m = 1 - \ln\left(\frac{5}{6}\right)$$

Use rules of logs and the fact that 1 = ln e

$$= \ln\left(e \div \frac{5}{6}\right)$$

$$= \ln\left(\frac{6e}{5}\right)$$

Be the Examiner 9.2

Find the median of a random variable with the probability density function

$$f(x) = \begin{cases} 0.1x & \text{for } 0 \leqslant x < 2 \\ 0.25 - 0.025x & \text{for } 2 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\int_0^m 0.1x\ dx = \frac{1}{2}$ $0.05m^2 = \frac{1}{2}$ $m^2 = 10$ $m > 0$ so $m = \sqrt{10}$	$\int_m^{10} 0.25 - 0.025x\ dx = \frac{1}{2}$ $\left[0.25x - \frac{0.025x^2}{2}\right]_m^{10} = \frac{1}{2}$ $2.5 - 0.25m + 0.0125m^2 = \frac{1}{2}$ $m = 10 \pm 2\sqrt{2}$ $0 < m < 10$ so $m = 10 - 2\sqrt{2}$	$\int_0^2 0.1x\ dx = 0.2 < 0.5$ $\int_2^m 0.25 - 0.025x\ dx = \frac{1}{2} - 0.2$ $\left[0.25x - \frac{0.025x^2}{2}\right]_2^m = 0.3$ $0.25m - 0.0125m^2 - 0.45 = 0.3$ $m = 10 \pm 2\sqrt{2}$ from GDC

Mean and variance of a continuous random variable

KEY POINT 9.9

For a continuous random variable X with probability density function $f(X)$

- $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$
- $Var(X) = E(X^2) - [E(X)]^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - [E(X)]^2$

WORKED EXAMPLE 9.10

Find the exact values of the mean and the variance of a random variable with the probability density function

$$f(x) = \begin{cases} \frac{1}{x \ln 3} & \text{for } 1 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

Use $E(X) = \int_{-\infty}^{\infty} x\ f(x)\, dx$

$$E(X) = \int_1^3 \frac{1}{\ln 3} dx$$

$$= \left[\frac{x}{\ln 3}\right]_1^3$$

$$= \frac{2}{\ln 3}$$

Find $E(X^2)$ first

$$E(X^2) = \int_1^3 x^2 f(x)\,dx$$
$$= \int_1^3 \frac{x}{\ln 3}\,dx$$
$$= \left[\frac{x^2}{2\ln 3}\right]_1^3$$
$$= \frac{4}{\ln 3}$$

Use $Var(X) = E(X^2) - [E(X)]^2$

$$Var(X) = \frac{4}{\ln 3} - \left(\frac{2}{\ln 3}\right)^2$$
$$= \frac{4(\ln 3 - 1)}{(\ln 3)^2}$$

For a piecewise defined probability density function, you need to split the integration over the separate parts of the domain.

WORKED EXAMPLE 9.11

Find the mean and variance of a random variable with the probability density function

$$f(x) = \begin{cases} \frac{3}{5}\ln x & \text{for } 1 \leqslant x < e \\ \frac{3}{5}\left(1-(x-e)^2\right) & \text{for } e \leqslant x \leqslant e+1 \\ 0 & \text{otherwise} \end{cases}$$

$E(X) = \int x\,f(x)\,dx$

Use the relevant expression for $f(x)$ on each part of the domain

$$E(X) = \int_1^e \frac{3}{5}x\ln x\,dx + \int_e^{e+1} \frac{3}{5}x\left(1-(x-e)^2\right)dx$$
$$= 2.4956\ldots$$

Find $E(X^2)$ first

$$E(X^2) = \int_1^e \frac{3}{5}x^2\ln x\,dx + \int_e^{e+1} \frac{3}{5}x^2\left(1-(x-e)^2\right)dx$$
$$= 6.5958\ldots$$

Use $Var(X) = E(X^2) - [E(X)]^2$

$$Var(X) = 6.5958 - 2.4956^2$$
$$= 0.378 \text{ (3 s.f.)}$$

Exercise 9C

For questions 1 to 3, you are given the probability density function for a random variable X. Use the method demonstrated in Worked Example 9.5 to find the value of k and the required probability.

1 **a** $f(x) = \begin{cases} kx^3 & \text{for } 2 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$

Find $P(2 < X < 2.5)$.

b $f(x) = \begin{cases} k\sqrt{x} & \text{for } 1 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$

Find $P(1 < X < 3)$.

2 **a** $f(x) = \begin{cases} \frac{x}{10} + k & \text{for } 0 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$

Find $P(1 \leqslant X \leqslant 2)$.

b $f(x) = \begin{cases} \frac{x^2}{4} + k & \text{for } -1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

Find $P(0 \leqslant X \leqslant 1)$.

3 **a** $f(x) = \begin{cases} x^3 & \text{for } 0 < x < k \\ 0 & \text{otherwise} \end{cases}$

Find $P\left(X < \frac{k}{2}\right)$.

b $f(x) = \begin{cases} x^2 & \text{for } k < x < 2k \\ 0 & \text{otherwise} \end{cases}$

Find $P\left(X < \frac{3k}{2}\right)$.

For questions 4 to 6, use the method demonstrated in Worked Example 9.6 to sketch the probability density function and find the required probability.

4 **a** $f(x) = \begin{cases} \frac{x}{25} & \text{for } 0 \leqslant x < 5 \\ \frac{1}{5} & \text{for } 5 \leqslant x < 7.5 \\ 0 & \text{otherwise} \end{cases}$

Find $P(2 < X < 6)$.

b $f(x) = \begin{cases} \frac{x}{16} & \text{for } 0 \leqslant x < 4 \\ \frac{1}{4} & \text{for } 4 \leqslant x < 6 \\ 0 & \text{otherwise} \end{cases}$

Find $P(3 < X < 5)$.

5 **a** $f(x) = \begin{cases} \frac{2}{3}e^{x} & \text{for } x < 0 \\ \frac{2}{3}e^{-2x} & \text{for } x \geqslant 0 \end{cases}$

Find $P(-\ln 2 < X < \ln 2)$.

b $f(x) = \begin{cases} \frac{2}{3}e^{2x} & \text{for } x \leqslant 0 \\ \frac{2}{3}e^{-x} & \text{for } x > 0 \end{cases}$

Find $P(-1 < X < 2)$.

6 **a** $f(x) = \begin{cases} 0.8\sin x & 0 \leqslant x < \frac{\pi}{2} \\ 0.8\cos 4x & \frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{8} \\ 0 & \text{otherwise} \end{cases}$

Find $P(1.2 < X \leqslant 1.8)$.

b $f(x) = \begin{cases} 1.094\sin 2x & 0 \leqslant x < \frac{\pi}{4} \\ 1.547\cos x & \frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

Find $P(0.4 < X \leqslant 1.2)$.

For questions 7 to 9, use the method demonstrated in Worked Example 9.7 to find the mode(s) of a random variable with the given probability density function.

7 **a** $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{x}{32} & 0 < x < 8 \\ 0 & \text{otherwise} \end{cases}$

8 **a** $f(x) = \begin{cases} \frac{3}{64}(4x^2 - x^3) & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} 768(x^2 - 4x^3) & 0 < x < 0.25 \\ 0 & \text{otherwise} \end{cases}$

9 **a** $f(x) = \begin{cases} \frac{4}{3\pi}\cos^2 x & 0 \leqslant x < \frac{3\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{1}{\pi}\sin^2 x & 0 \leqslant x < 2\pi \\ 0 & \text{otherwise} \end{cases}$

For questions 10 to 12, use the method demonstrated in Worked Example 9.8 to find the median of a random variable with the given probability density function.

10 **a** $f(x) = \begin{cases} \frac{x}{32} & 0 < x < 8 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{x}{16} & 2 < x < 6 \\ 0 & \text{otherwise} \end{cases}$

11 **a** $f(y) = \begin{cases} 3e^{-3y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

b $f(y) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$

12 **a** $f(x) = \begin{cases} \frac{2}{\ln 2}\tan x & 0 < x < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{1}{x} & 1 < x < e \\ 0 & \text{otherwise} \end{cases}$

For questions 13 to 15, use the method demonstrated in Worked Example 9.9 to find the median of a random variable with the given piecewise-defined probability density function.

13 **a** $f(x) = \begin{cases} 0.8\sin x & 0 \leqslant x < \frac{\pi}{2} \\ 0.8\cos 4x & \frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{8} \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} 1.094\sin 2x & 0 \leqslant x < \frac{\pi}{4} \\ 1.547\cos x & \frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

14 **a** $f(x) = \begin{cases} \frac{3}{32}x^2 & \text{for } 0 \leqslant x \leqslant 2 \\ \frac{3}{32}(6 - x) & \text{for } 2 < x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{2}{63}x^2 & \text{for } 0 \leqslant x \leqslant 3 \\ \frac{2}{35}(8 - x) & \text{for } 3 < x \leqslant 8 \\ 0 & \text{otherwise} \end{cases}$

15 **a** $f(x) = \begin{cases} \frac{x}{16} & \text{for } 0 \leqslant x < 4 \\ \frac{1}{4} & \text{for } 4 \leqslant x < 6 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{x}{25} & \text{for } 0 \leqslant x < 5 \\ \frac{1}{5} & \text{for } 5 \leqslant x < 7.5 \\ 0 & \text{otherwise} \end{cases}$

For questions 16 to 18, use the method demonstrated in Worked Example 9.10 to find the mean and variance of a random variable with the given probability density function.

16 **a** $f(x) = \begin{cases} \frac{x}{32} & \text{for } 0 < x < 8 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{x}{16} & \text{for } 2 < x < 6 \\ 0 & \text{otherwise} \end{cases}$

17 **a** $f(x) = \begin{cases} \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} e^x & \text{for } 0 < x < \ln 2 \\ 0 & \text{otherwise} \end{cases}$

18 **a** $f(x) = \begin{cases} \frac{3}{64}(4x^2 - x^3) & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} 768(x^2 - 4x^3) & 0 < x < 0.25 \\ 0 & \text{otherwise} \end{cases}$

For questions 19 to 21, use the method demonstrated in Worked Example 9.11 to find the mean and variance of a random variable with the given piecewise-defined probability density function.

19 **a** $f(x) = \begin{cases} \frac{x}{25} & \text{for } 0 \leqslant x < 5 \\ \frac{1}{5} & \text{for } 5 \leqslant x < 7.5 \\ 0 & \text{otherwise} \end{cases}$

b $f(x) = \begin{cases} \frac{x}{16} & \text{for } 0 \leqslant x < 4 \\ \frac{1}{4} & \text{for } 4 \leqslant x < 6 \\ 0 & \text{otherwise} \end{cases}$

20 **a** $f(x)=\begin{cases}\dfrac{x}{25} & \text{for } 0\leqslant x<5\\ \dfrac{10-x}{25} & \text{for } 5\leqslant x<10\\ 0 & \text{otherwise}\end{cases}$

b $f(x)=\begin{cases}\dfrac{x}{16} & \text{for } 0\leqslant x<4\\ \dfrac{8-x}{16} & \text{for } 4\leqslant x<8\\ 0 & \text{otherwise}\end{cases}$

21 **a** $f(x)=\begin{cases}0.8\sin x & 0\leqslant x<\dfrac{\pi}{2}\\ 0.8\cos 4x & \dfrac{\pi}{2}\leqslant x\leqslant\dfrac{5\pi}{8}\\ 0 & \text{otherwise}\end{cases}$

b $f(x)=\begin{cases}1.094\sin 2x & 0\leqslant x<\dfrac{\pi}{4}\\ 1.547\cos x & \dfrac{\pi}{4}\leqslant x\leqslant\dfrac{\pi}{2}\\ 0 & \text{otherwise}\end{cases}$

22 Random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{\ln 2}\tan x & \text{for } 0<x<\dfrac{\pi}{4}\\ 0 & \text{otherwise}\end{cases}$$

Find

a $P\left(\dfrac{\pi}{16}\leqslant X\leqslant\dfrac{\pi}{8}\right)$

b $P\left(X>\dfrac{3\pi}{16}\right)$.

23 Random variable Y has the probability density function given by $f(y)=\dfrac{3}{4}y(2-y)$ for $0\leqslant y\leqslant 2$. Find

a $P(0.5<Y<1)$

b $P\left(Y>\dfrac{2}{3}\right)$.

24 Random variable X has the probability density function:

$$f(x)=\begin{cases}ke^{-x} & 3<x<8\\ 0 & \text{otherwise}\end{cases}$$

Find

a the value of k

b $P(X>5)$

c $E(X)$.

25 The probability density function of a random variable X is given by:

$$f(x)=\begin{cases}kx^2 & 0<x<3\\ 0 & \text{otherwise}\end{cases}$$

a Find the value of k.

b Find the expected value of X.

26 Find the median of a random variable with the probability density function:

$$f(x)=\begin{cases}2-2x & \text{for } 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

27 Random variable T has the probability density function $f(t)=\dfrac{t}{2}$ for $0\leqslant t\leqslant 2$.

a Find $P(X>1)$.

b Find the median of T.

28 Random variable Y has the probability density function:

$$f(y) = \begin{cases} \frac{1}{2\pi}\sin\sqrt{y} & \text{for } 0 < y < \pi^2 \\ 0 & \text{otherwise} \end{cases}$$

Find

- **a** the mean of Y
- **b** the standard deviation of Y
- **c** the median of Y
- **d** the mode of Y.

29 For a random variable with the probability density function:

$$f(x) = \begin{cases} \frac{1}{2}\cos x & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

- **a** Write down the expected value.
- **b** Calculate the variance.

30 X is a random variable with the probability density function $f(x) = \frac{e}{2e-5}x^2e^{-x}$ on the domain $0 \leqslant x \leqslant 1$. Find

- **a** $P(X > 0.5)$
- **b** the mode of X
- **c** $E(X)$.

31 Random variable Y has the probability density function:

$$f(y) = \begin{cases} \frac{1}{ky} & 1 \leqslant y \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

Find

- **a** the value of k
- **b** the median of Y.

32 Random variable X takes values between 0 and 1 and has the probability density function $f(x) = 4x(1 - x^2)$.

- **a** Find the mode of X.
- **b** Find the expected value of X.

33 A teacher finds that the time students take to complete a certain homework task can be modelled by the continuous random variable T minutes with the probability density function:

$$f(t) = \frac{3}{40000}(30 - t)(t - 10)^2 \text{ for } 10 \leqslant t \leqslant 30$$

- **a** Find the mean time taken to complete the task.
- **b** In a class of 30 students, how many can be expected to take more than 25 minutes to complete the task?

34 The time I have to wait for the bus in the morning can be modelled by a random variable with the probability density function

$$f(x) = \frac{e^4}{5(e^4 - 1)}e^{-\frac{x}{5}} \text{ for } 0 < x < 20$$

where x is measured in minutes.

- **a** Find the probability that I wait
  - **i** between 5 and 10 minutes
  - **ii** less than 10 minutes.
- **b** How long should I expect to wait for the bus on average?

35 A function is defined by

$$f(x)=\begin{cases}\dfrac{2x}{k^2} & \text{for } 0 \leqslant x \leqslant k\\ 0 & \text{otherwise}\end{cases}$$

a Show that f(*x*) is a valid probability density function for all $k > 0$.

X is a random variable with probability density function f(*x*).

b Find the median of *X*.

c Find, in terms of *k*, the value of *c* such that $P(X \geqslant c) = 0.19$.

36 The random variable *T* has the probability density function

$$f(x)=\begin{cases}\dfrac{1}{1+t} & \text{for } k < t < k+1\\ 0 & \text{otherwise}\end{cases}$$

a Find the value of *k*.

b Find the variance of *T*.

37 The probability density function of random variable *X* is given by $f(x)=\dfrac{x^2}{9}$ for $0 \leqslant x \leqslant 3$.

a Find the value of *a* such that $P(X > a) = 0.05$.

b Find the interquartile range of *X*.

38 Find the interquartile range of a random variable with the probability density function $f(x)=\dfrac{1}{x}$ for $1 \leqslant x \leqslant e$.

39 A continuous random variable *X* has the probability density function

$$f(x)=\begin{cases}kx & 0 \leqslant x < 10\\ k(20-x) & 10 \leqslant x \leqslant 20\\ 0 & \text{otherwise}\end{cases}$$

a Find the value of *k*.

b Find $P(X > 15)$.

c Find the value of *a* such that $P(X \leqslant a) = 0.9$.

d Find Var(*X*).

40 The continuous uniform distribution on [*a*, *b*] has the probability density function given by

$$f(x)=\begin{cases}k & \text{for } a \leqslant x \leqslant b\\ 0 & \text{otherwise}\end{cases}$$

a Sketch the graph of $y = f(x)$.

Random variable *X* follows the continuous uniform distribution on [*a*, *b*].

b Write down E(*X*).

c Show that $\text{Var}(X)=\dfrac{(b-a)^2}{12}$.

41 The continuous random variable *X* has probability density function

$$f(x)=\begin{cases}\dfrac{3}{20}(4x^2-x^3) & 0 < x < 2\\ 0 & \text{otherwise}\end{cases}$$

a Sketch the probability density function.

b Find the mode of *X*.

c Find the mean of *X*.

42 Y is a continuous random variable with probability density function

$$f(y) = \begin{cases} ay^2, & -k < y < k \\ 0 & \text{otherwise} \end{cases}$$

a Show that $a = \dfrac{3}{2k^3}$.

b Given that $\text{Var}(Y) = 5$ find the exact value of k.

43 Random variable X takes values between -3 and 3 has the probability density function $f(x) = k(9 - x^2)$. Find the probability that X takes a value within two standard deviations of the mean.

44 For the random variable Y with the probability density function

$$f(y) = \begin{cases} \dfrac{1}{27}y^2 & \text{for } 0 \leqslant y < 3 \\ \dfrac{1}{27}y(6-y) & \text{for } 3 \leqslant y \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$$

Find

a $P(Y < 5)$

b the median of Y

c the expected value of Y.

45 Random variable T has the probability density function

$$f(t) = \begin{cases} \dfrac{1}{125}t(10-t) & \text{for } 0 \leqslant t < 5 \\ \dfrac{1}{125}(t-10)^2 & \text{for } 5 \leqslant t \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the probability density function.

b Find $P(T < 5)$.

c Show that the median of T satisfies $2m^3 - 30m^2 + 375 = 0$.

46 For a continuous random variable T with the probability density function

$$f(t) = \begin{cases} kt & \text{for } 0 \leqslant t < 1 \\ k\sin\left(\dfrac{\pi t}{2}\right) & \text{for } 1 \leqslant t \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find the exact value of k.

b Sketch the probability density function.

c Find the median of T.

Although you can calculate probabilities for a normal distribution exactly, you will be able to find its mean and variance when you learn more about integration in Chapter 10. See Mixed Practice 10, question 56.

Checklist

- You should know that Bayes' theorem reverses the conditional probability formula:
  - $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B)\mathrm{P}(A \mid B')}$
- If $\mathrm{P}(B_1) + \mathrm{P}(B_2) + \mathrm{P}(B_3) = 1$, then
  - $\mathrm{P}(B_1 \mid A) = \dfrac{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$
- You should know that the variance of a discrete random variable X is
  - $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \sum_x x^2\ \mathrm{P}(X = x) - (\mathrm{E}(X))^2$

    and the standard deviation is $\sqrt{\mathrm{Var}(X)}$.
- You should know that for a random variable X and constants a and b,
  - $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- You should know that the distribution of a continuous random variable is described by a probability density function, which satisfies:
  - $\mathrm{f}(x) \geqslant 0$ for all x
  - $\int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = 1$
  - $\mathrm{P}(a < X < b) = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$
- You should know that the mode of a continuous random variable is the value of x with the maximum value of $\mathrm{f}(x)$.
- You should know that the median of a continuous random variable satisfies:
  - $\int_{-\infty}^{m} \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{2}$
- You should know that the mean and variance of a continuous random variable are given by:
  - $\mathrm{E}(X) = \int_{-\infty}^{\infty} x\ \mathrm{f}(x)\,\mathrm{d}x$
  - $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \int_{-\infty}^{\infty} x^2\ \mathrm{f}(x)\,\mathrm{d}x - [\mathrm{E}(X)]^2$

Mixed Practice

1 The probability distribution of a random variable X is given in the following table.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.1	0.2	0.3	0.2	0.2

a Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b The random variable Y is given by $Y = 2 - 3X$. Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

2 Random variable V has the probability distribution given in the table.

v	2	3	5	7
$\mathrm{P}(V = v)$	0.4	p	$2p$	$3p$

a Find the value of p.

b Find the mean and standard deviation of V.

c Find the mean and standard deviation of the random variable $W = 10 - V$.

3 A continuous random variable X has the probability density function given by

$$\mathrm{f}(x) = \begin{cases} kx(6-x) & \text{for } 0 \leqslant x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$$

a Show that $k = \frac{1}{36}$.

b Find $\mathrm{P}(X > 2)$.

c Find $\mathrm{E}(X)$.

4 A probability density function is given by

$$\mathrm{f}(x) = \begin{cases} \frac{3}{64}x^2(4-x), & 0 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of $y = \mathrm{f}(x)$.

b Find the mode of a random variable X with the probability density function $\mathrm{f}(x)$.

c Show that the median of X satisfies $3m^4 - 16m^3 + 128 = 0$. Hence find the median of X.

5 A continuous random variable Y has the probability density function

$$\mathrm{f}(y) = \begin{cases} \frac{1}{\pi} y \sin y & \text{for } 0 \leqslant y \leqslant \pi \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

b Random variable Z is given by $Z = 4Y - 1$. Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

6 Events A and B satisfy

$\mathrm{P}(B) = 0.4$, $\mathrm{P}(A \mid B) = 0.6$ and $\mathrm{P}(A \mid B') = 0.2$

Use Bayes' theorem to find $\mathrm{P}(B \mid A)$.

7 The random variable X has the probability distribution given by

$\mathrm{P}(X = x) = \frac{3x - 1}{26}$ for $x = 1, 2, 3, 4$.

a Show this probability distribution in a table.

b Find the exact value of $\mathrm{E}(X)$.

c Show that $\mathrm{Var}(X) = 0.92$ to two significant figures.

d Find $\mathrm{Var}(20 - 5X)$ correct to two significant figures.

8 Three fair coins are tossed and H is the number of heads.

a Construct a table showing the probability distribution of H.

b Find $E(H)$ and show that $Var(H) = 0.75$.

In a game, a player tosses three fair coins and gets 3 counters for every head. It costs 5 counters to play the game. The random variable W represents the amount of money won or lost.

c Decide whether the game is fair.

d Find $Var(W)$.

9 The distribution of a random variable X is given in the table.

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.2	p	q

Given that $E(X) = 3.3$, find $Var(X)$.

10 The table shows the probability distribution of a discrete random variable Y.

y	1	3	5	7
$P(Y = y)$	a	b	b	a

a Write down the expected value of Y.

b Given that the variance of Y is 4.2, find the values of a and b.

11 A continuous random variable X takes values between 1 and 3 and its probability density function is given by $f(x) = ax + b$. Given that $E(X) = 2.08$, find the values of a and b.

12 Every weekend, Theo takes a trip either to Spain or to Sweden. The probability that he goes to Spain is 0.3. There is a probability of 0.2 that it rains in Spain and a probability of 0.6 that it rains in Sweden.

a Find the probability that Theo has a rainy trip.

b Given that Theo had a rainy trip, find the probability that he went to Sweden.

13 Each day the school canteen serves either cheese sandwiches or a salad. When there are cheese sandwiches, the probability that Emma eats in the canteen is 0.4. When there is salad, the probability that she eats in the canteen in 0.7.

a Emma eats in the canteen 52% of the time. Find the probability that the canteen serves cheese sandwiches.

b Given that Emma eats in the canteen on a particular day, find the probability that there is salad.

14 Three mutually exclusive events, B_1, B_2 and B_3, satisfy

$P(B_1) = 0.2$, $P(B_2) = 0.3$, $P(B_3) = 0.5$

The event A satisfies

$P(A \mid B_1) = 0.6$, $P(A \mid B_2) = 0.8$, $P(A \mid B_3) = 0.2$

Find $P(B_1 \mid A)$.

15 A continuous random variable X takes values between 0 and 3 and has the probability density function

$$f(x) = \frac{10}{81} x^2(x - 3)^2 \text{ for } 0 \leqslant x \leqslant 3$$

a Find $P(X > 2.5)$.

b Find the standard deviation of X.

The random variable X is used to model the mass, in kilograms, of blocks of wood. In a game, a player selects a block of wood at random and receives the amount of money (in dollars) equal to its mass.

c The game is played 80 times. How many players should expect to win more than \$2.50?

d What should the charge for playing the game be to make it a fair game?

16 A discrete random variable X follows the binomial distribution $\mathrm{B}\left(200, \frac{1}{4}\right)$. Find the probability that a randomly chosen value of X is more than one standard deviation from the mean.

17 A botanist models the length of leaves of a certain plant, X cm by the probability distribution

$$f(x) = \frac{3}{4000}(x - 10)(30 - x) \text{ for } 10 \leqslant x \leqslant 30$$

a Find the mean and variance of the distribution.
b Find the probability that a randomly chosen leaf is more than 25 cm long.
A student uses the normal distribution with the same mean and variance to model the length of the leaves.
c Using the student's model, find the probability that a randomly chosen leaf is more then 25 cm long.
d In a sample of 300 leaves, 41 are more than 25 cm long. Whose model seems to give a better prediction?

18 The lifetime, in years, of a certain brand of lightbulb can be modelled by the probability density function

$$f(x) = \begin{cases} kxe^{-x} & \text{for } 0 \leqslant x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k correct to 4 significant figures.
b Find the mean lifetime of lightbulbs, giving your answer to the nearest month.
c Find the probability that a randomly chosen lightbulb fails in the first year.
d I have five such lightbulbs in my living room. Assuming that they fail independently of each other, find the probability that
  i all of them are still working after six months
  ii at least one of them is still working after two years.

19 A bag contains seven blue, five red and eight yellow balls. One ball is selected at random and not replaced. Then a second ball is selected. Given that the second ball is blue, find the probability that the first ball was also blue.

20 The masses of cats, M kg, can be modelled by the probability density function

$$f(m) = \begin{cases} km\sin^2\left(\frac{\pi m}{10}\right) & \text{for } 0 \leqslant m \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k.
b Sketch the probability density function.
c Find the probability that a randomly selected cat has a mass of more than 6 kg.
d Find the mean and standard deviation of M.
e A student decides to model the masses using a normal distribution with the same mean and standard deviation as M. He uses this normal distribution to estimate the probability that a cat weighs has a mass of more than 6 kg. Find the percentage error in his estimate. Give your answer to the nearest integer.

21 A scientist investigates a particular genetic mutation in flies. He finds that 90% of flies without the mutation and 10% of flies with the mutation live longer than three days. It is known that 3% of the fly population have this mutation. Given that a randomly selected fly died within the first three days, find the probability that it did not have the mutation.

22 Jenny goes to school by bus every day. When it is not raining, the probability that the bus is late is $\frac{3}{20}$. When it is raining, the probability that the bus is late is $\frac{7}{20}$. The probability that it rains on a particular day is $\frac{9}{20}$. On one particular day the bus is late. Find the probability that it is not raining on that day.

Mathematics HL November 2010 Paper 1 Q4

23 **a** Mobile phone batteries are produced by two machines. Machine A produces 60% of the daily output and machine B produces 40% . It is found by testing that on average 2% of batteries produced by machine A are faulty and 1% of batteries produced by machine B are faulty.

i Draw a tree diagram clearly showing the respective probabilities.

ii A battery is selected at random. Find the probability that it is faulty.

iii A battery is selected at random and found to be faulty. Find the probability that it was produced by machine A.

b In a pack of seven transistors, three are found to be defective. Three transistors are selected from the pack at random without replacement. The discrete random variable X represents the number of defective transistors selected.

i Find $P(X = 2)$.

ii Copy and complete the following table.

x	0	1	2	3
$P(X = x)$				

iii Determine $E(X)$.

Mathematics HL May 2014 Paper 1 TZ2 Q11

24 A continuous random variable T has the probability density function

$$f(t) = \begin{cases} \frac{1}{12}(8t - t^3) & \text{for } 0 \leqslant t \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a Evaluate the expected value of T.

b Find the mode of T.

c Find the exact value of the median of T.

25 The lifetime of batteries, T hundred hours, can be modelled by a random variable with the probability density function

$$f(t) = \frac{4}{27}(t - 3)^2(6 - t) \text{ for } 3 \leqslant t \leqslant 6$$

a Find the probability that a randomly chosen battery lasts

i more than 500 hours

ii between 500 and 550 hours.

A toy needs three such batteries. Alessia put in three new batteries.

b Find the probability that the toy will work for more than 500 hours.

c Given that the toy has been working for 500 hours, find the probability that it will continue to work for another 50 hours.

26 A farmer keeps three breeds of chicken. The weights of chickens of each breed are normally distributed, with the means and standard deviations shown in the table. The table also shows the percentage of chickens of each breed.

Breed	Percentage	Mean (kg)	Standard deviation (kg)
A	20	1.5	0.3
B	45	1.2	0.2
C	35	1.9	0.5

The farmer selects a chicken at random and finds that it weighs more than 1.8 kg. What is the probability that the chicken is breed B?

27 A continuous random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ ae^{-ax} & x \geqslant 0 \end{cases}$$

It is known that $P(X < 1) = 1 - \frac{1}{\sqrt{2}}$.

a Show that $a = \frac{1}{2}\ln 2$.

b Find the median of X.

c Calculate the probability that $X < 3$ given that $X > 1$.

Mathematics HL May 2010 Paper 1 TZ1 Q12

28 The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{\sin x}{4} & 0 \leqslant x \leqslant \pi \\ a(x - \pi) & \pi \leqslant x \leqslant 2\pi \\ 0 & 2\pi < x \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $P(X \leqslant \pi)$.

c Show that $a = \frac{1}{\pi^2}$.

d Write down the median of X.

e Calculate the mean of X.

f Calculate the variance of X.

g Find $P\left(\frac{\pi}{2} \leqslant X \leqslant \frac{3\pi}{2}\right)$.

h Given that $\frac{\pi}{2} \leqslant X \leqslant \frac{3\pi}{2}$ find the probability that $\pi \leqslant X \leqslant 2\pi$.

Mathematics HL May 2015 Paper 2 TZ1 Q11

10 Further calculus

ESSENTIAL UNDERSTANDINGS

- Calculus describes rates of change between two variables and the accumulation of limiting areas.
- Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

In this chapter you will learn…

- about the theoretical foundations of calculus: limits, continuity and differentiability
- how to carry out differentiation from first principles
- about higher derivatives
- how to evaluate limits using L'Hôpital's rule
- how to find gradients of functions defined implicitly
- how to find related rates of change
- how to find maximum and minimum values of functions in more complex situations
- how to differentiate and integrate more functions, including use of partial fractions
- about the techniques of integration by substitution and integration by parts
- how to calculate the area between a curve and the y-axis, and volumes of revolution.

LEARNER PROFILE – Open-minded

How useful is academic education?
Is the mathematics chosen to be assessed in this course appropriate and beneficial?

CONCEPTS

The following concepts will be addressed in this chapter:

- The derivative may be represented physically as a rate of **change** and geometrically as the gradient or slope function.
- Areas under curves can be **approximated** by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Mathematical **modelling** can provide effective solutions to real-life problems in optimization by maximizing or minimizing a **quantity**, such as cost or profit.
- Some functions may be continuous everywhere but not differentiable everywhere.
- Limits describe the output of a function as the input approaches a certain value and can represent convergence and divergence.
- Examining limits of a function at a point can help determine continuity and differentiability at a point.

Figure 10.1 How could you measure the volume of these objects?

PRIOR KNOWLEDGE

Before you start this chapter, you should already be able to complete the following:

1 Differentiate

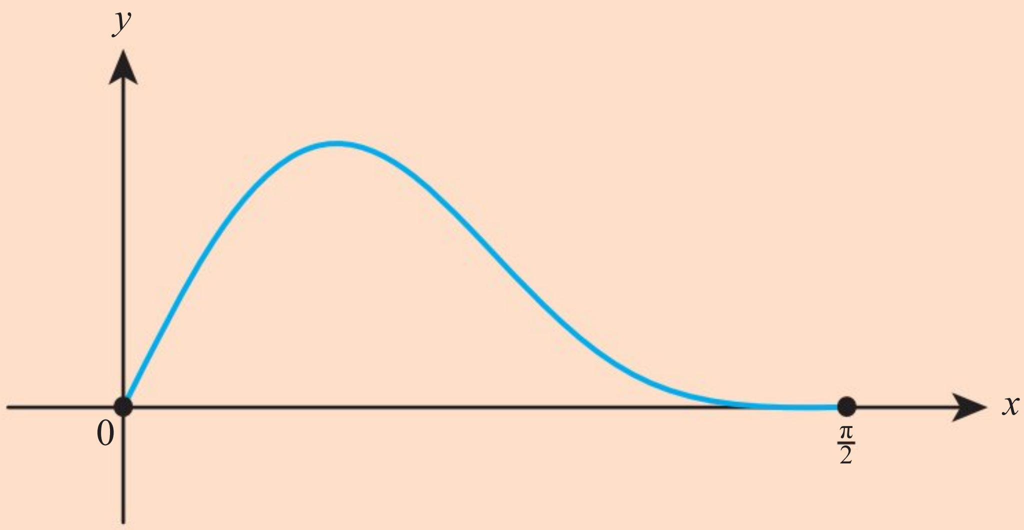
a $(2x+1)^7$ **b** $x^3 \ln x$ **c** $\frac{\sin x}{e^x}$.

2 Given that $y = x^2e^{2x}$, find $\frac{d^2y}{dx^2}$.

3 Find the coordinates of the local maximum point on the graph of $y = x^3 - 12x + 2$.

4 Find $\int \frac{3}{2x} + 5\sqrt{x}\,dx$.

5 Calculate the area bounded by the x-axis and the part of the curve with equation $y = \sin x \cos^4 x$ shown in the diagram.



6 Express $3 + \frac{5}{\cos x} - 2\tan^2 x$ in terms of $\sec x$.

7 Express $\frac{2x+1}{x^2+x-2}$ in partial fractions.

Calculus provides us with powerful tools to study properties of functions, such as their gradients, their maximum and minimum points, and the areas their graphs enclose. The main tools of calculus are differentiation and integration, and in this chapter you will extend those techniques to a wider variety of functions. You will also look in more detail at the concept of limits, which underpin rigorous foundations of calculus.

Starter Activity

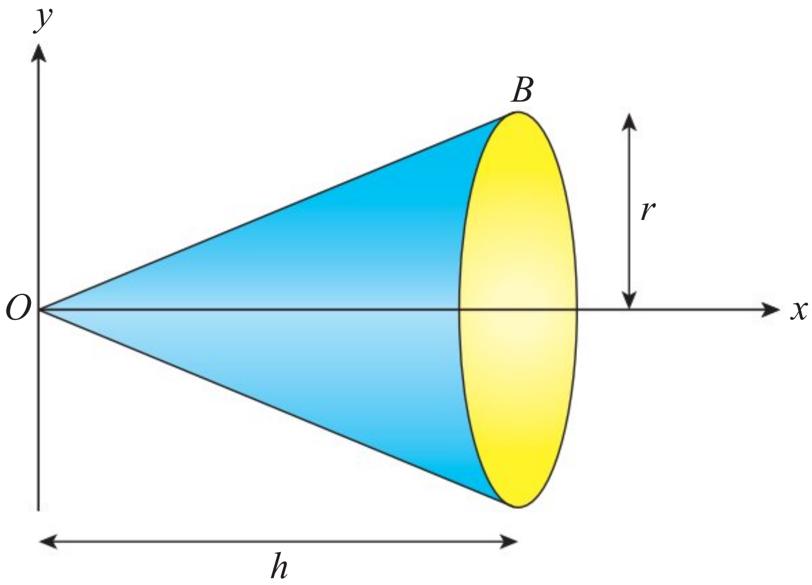
Look at the solid shapes shown in Figure 10.1. For each solid

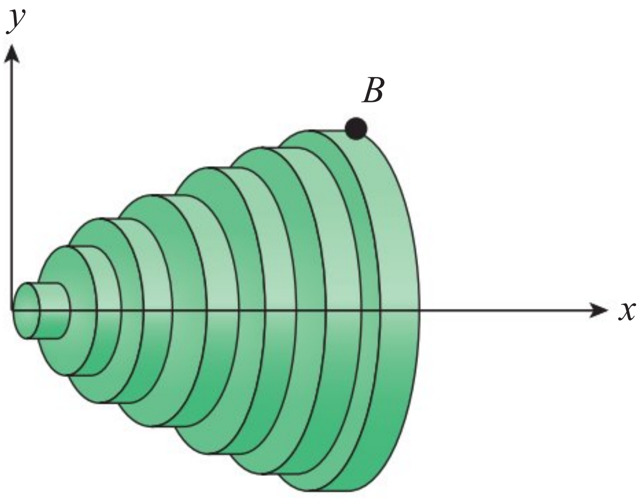
a draw a curve that represents its outline

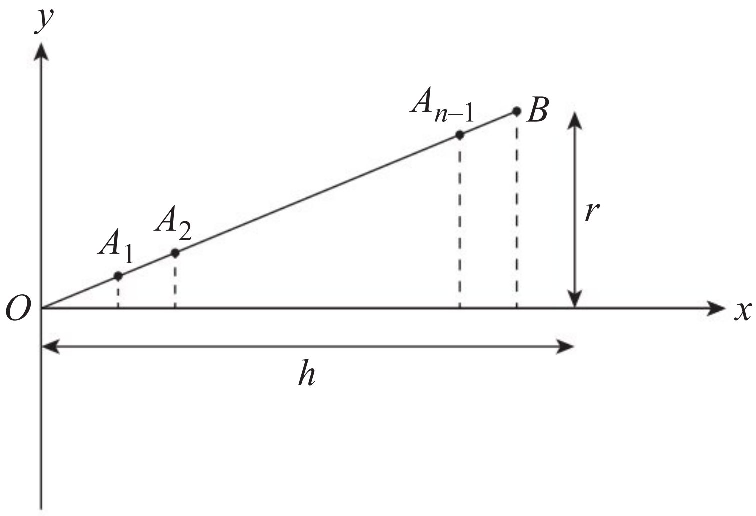
b draw three different cross-sections (perpendicular to the axis of symmetry).

Now look at this problem:

A cone with base radius r and height h is placed so that its vertex is at the origin and the line of symmetry lies along the x-axis. The volume of the cone can be found approximately by splitting the cone into lots of vertical slices, each one approximately a cylinder.







1 Write down the equation of the line OB.

2 When there are n cylinders, write down the lengths of each one. Hence, write down the coordinates of the points marked A_1, A_2 and A_{n-1}.

3 Explain why the sum of the volumes of all the slices can be written as $\frac{\pi r^2 h}{n^3}\sum_{k=1}^{n} k^2$.

4 Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

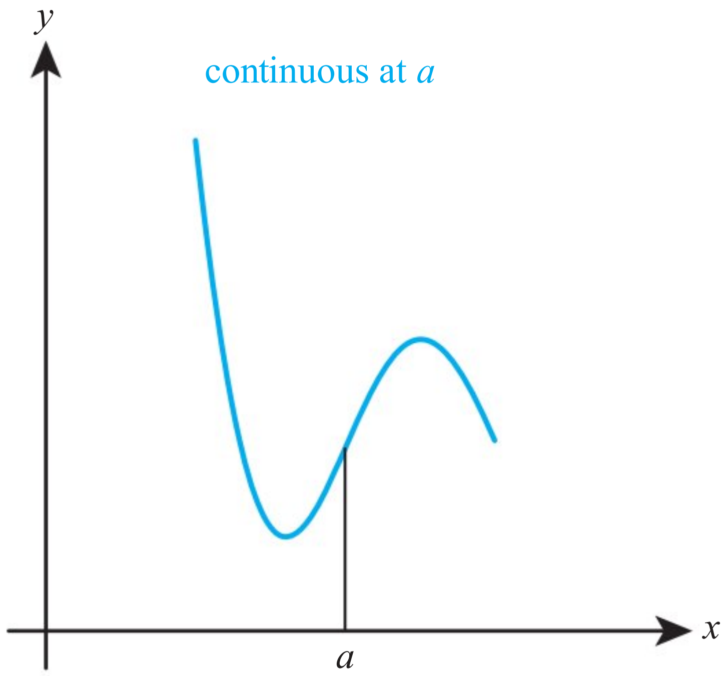
5 Hence, show that the sum from question 3 can be written as $\pi r^2 h\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)$.

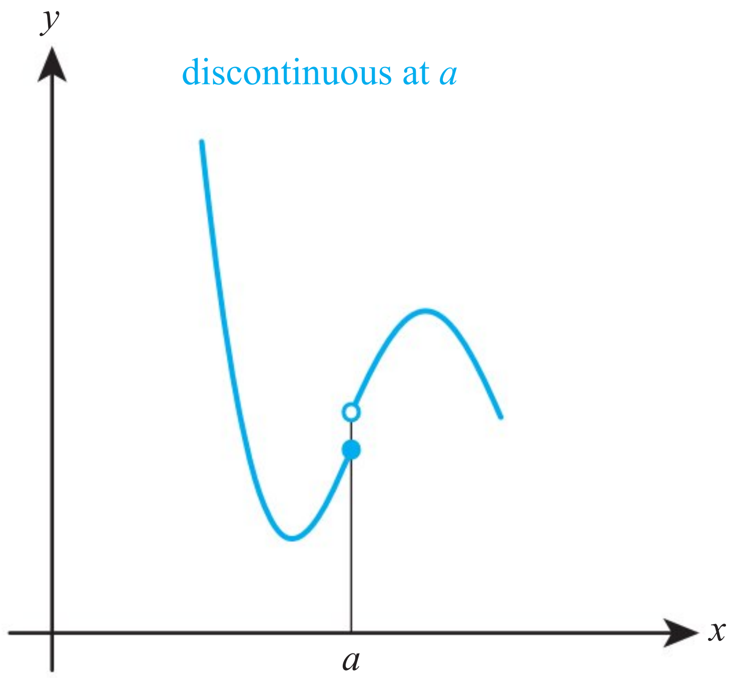
6 What happens to this expression as n gets larger? You should find the familiar formula for the volume of the cone.

10A Fundamentals of calculus

Informal understanding of continuity and differentiability

Not all functions have graphs which can be drawn without lifting your pencil off the paper. For example, $y = \frac{1}{x}$ has two separate branches. If it can be drawn without picking your pencil up off the paper, it is called a **continuous function**. This leads to the idea of continuity at a point, which basically means that the limit of function from one side of the point equals the limit of the function from the other side – they connect together.





WORKED EXAMPLE 10.1

a Find the value of a that makes the function below continuous.

$$f(x) = \begin{cases} x^2 & x \leqslant 3 \\ x + a & x > 3 \end{cases}$$

b Sketch $y = f(x)$.

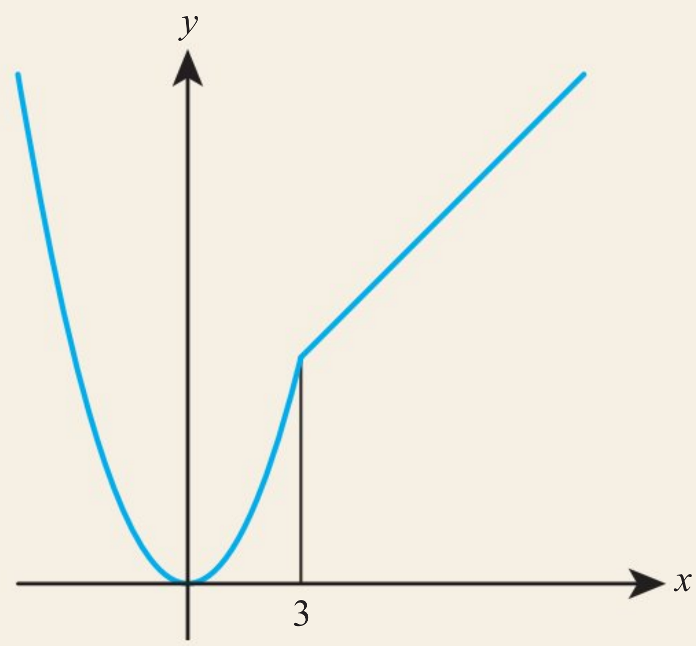
Evaluate f(x) at the boundary on the two different branches. These must be equal

a $f(3) = 9 = 3 + a$

Therefore, $a = 6$

You should recognize the basic shape of each branch

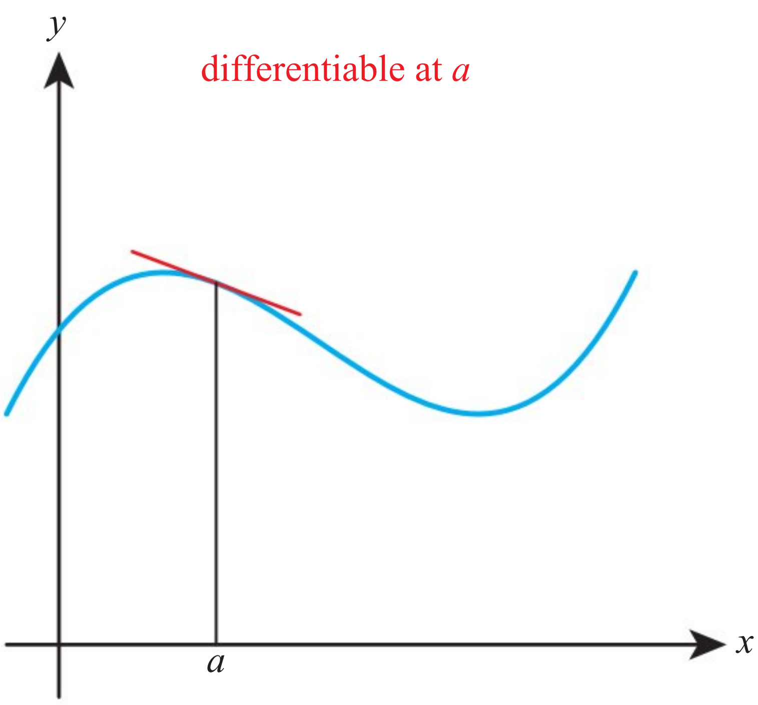
b

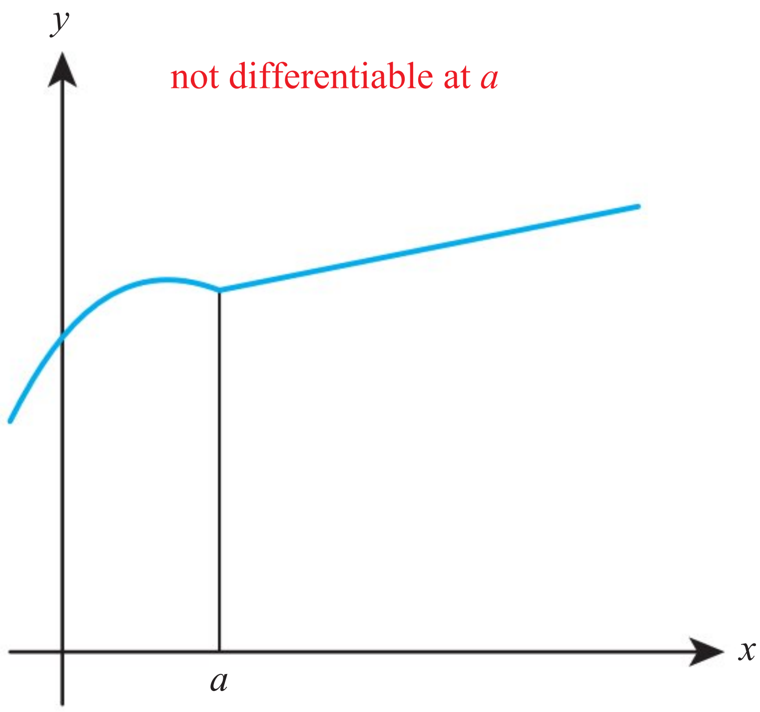


You are the Researcher

There is a much more complicated looking (but more rigorous) definition of continuity called the $\epsilon - \delta$ definition. You might wonder why a more rigorous definition is required. You might want to look at the Dirichlet function – which is 1 for all rational numbers and 0 for all irrational numbers – or the Thomae function – which is continuous at all irrational numbers but discontinuous at all rational numbers – to understand why we sometimes need a more abstract definition.

There is also the concept of differentiability at a point. Informally, this means that at that point a unique tangent can be drawn:





In most circumstances, this definition implies that the derivative must be continuous – the gradient just before $x = a$ matches the gradient just after it.

WORKED EXAMPLE 10.2

a Find the value of a and b that makes the function below continuous and differentiable at $x = 1$.

$$f(x) = \begin{cases} \frac{1}{x} & x \leqslant 1 \\ ax + b & x > 1 \end{cases}$$

b Sketch $y = f(x)$.

Evaluate f(x) at the boundary on the two different branches. These must be equal

a $f(1) = 1 = a + b$

Find an expression for f′(x)

$$f'(x) = \begin{cases} \frac{-1}{x^2} & x < 1 \\ a & x > 1 \end{cases}$$

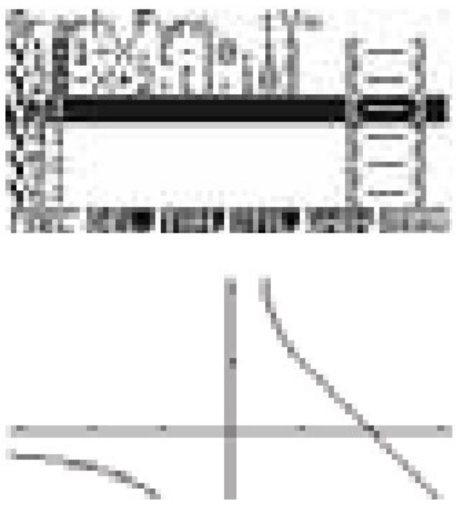
Evaluate f′(x) at the boundary on the two different branches. These must be equal

$f'(1) = -1 = a$

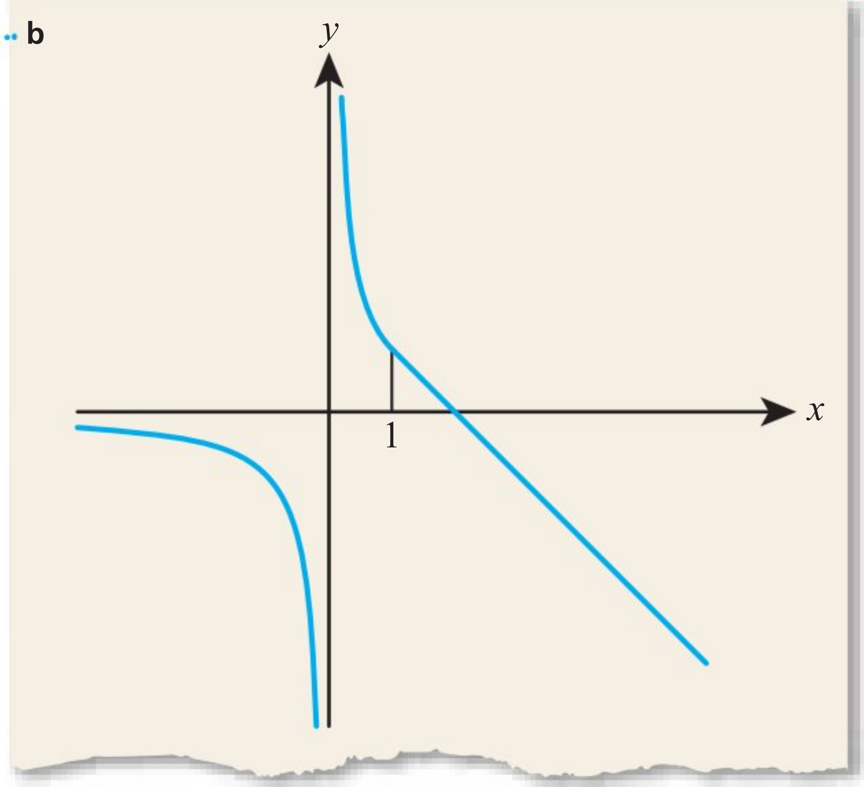
Solve these equations simultaneously

$a = -1, b = 2$

Use your calculator to help:



b



Tip

If a curve has a vertical tangent, for example, $y = \sqrt[3]{x}$ at $x = 0$, then it is said to not be differentiable at that point.

You are the Researcher

You might wonder when the method shown above breaks down. The function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ is an example of a function which is differentiable everywhere, but does not have a continuous derivative. You might want to research what this means and how it is justified.

Understanding of limits

It is often useful to consider the limit of an expression, particularly as it tends to infinity. This is written as $\lim\limits_{x\to\infty} f(x)$. There are various techniques for dealing with limits, but one common strategy when looking at fractions is to divide top and bottom by the same thing that turns a ratio of two infinite expressions into a ratio of two finite expressions.

TOK Links

The normal rules of arithmetic do not apply to infinity – for example, $\frac{\infty}{\infty}$ is not necessarily 1. This is an area of mathematics where intuition is a very dangerous Way of Knowing.

WORKED EXAMPLE 10.3

Determine the limit of $\frac{e^x - 1}{2e^x + 1}$ as x tends to infinity.

Letting x tend to infinity in the given expression gives $\frac{\infty}{\infty}$ which is undetermined. We first need to divide top and bottom by e^x

$$\frac{e^x - 1}{2e^x + 1} = \frac{1 - e^{-x}}{2 + e^{-x}}$$

With the new form, letting x tend to infinity results in finite values

As $x \to \infty$, this expression tends to

$$\frac{1+0}{2+0} = \frac{1}{2}$$

Tip

Not every function is either convergent or divergent as x tends to infinity. For example, $\sin x$ is neither convergent nor divergent as x tends to infinity.

A function which gets arbitrarily large as x tends towards a value is said to be divergent at that value. For example, $\lim\limits_{x\to 0} \frac{1}{x}$ is divergent. If the function gets closer and closer to a value as x tends towards a value, then the function is convergent at that value. For example, $\lim\limits_{x\to 4} 3x = 12$.

Tip

If you draw the graphs of $y=\frac{x^2}{1+3x}$ and $y=\frac{1}{3}x$ on your GDC, you can see that $y=\frac{1}{3}x$ is an oblique asymptote to $y=\frac{x^2}{1+3x}$.

WORKED EXAMPLE 10.4

Determine whether the expression $\frac{x^2}{1+3x}$ is convergent or divergent as $x \to \infty$.

Letting x tend to infinity in the given expression gives $\frac{\infty}{\infty}$ which is undetermined. We first need to divide top and bottom by x

$$\frac{x^2}{1+3x}=\frac{x}{\frac{1}{x}+3}$$

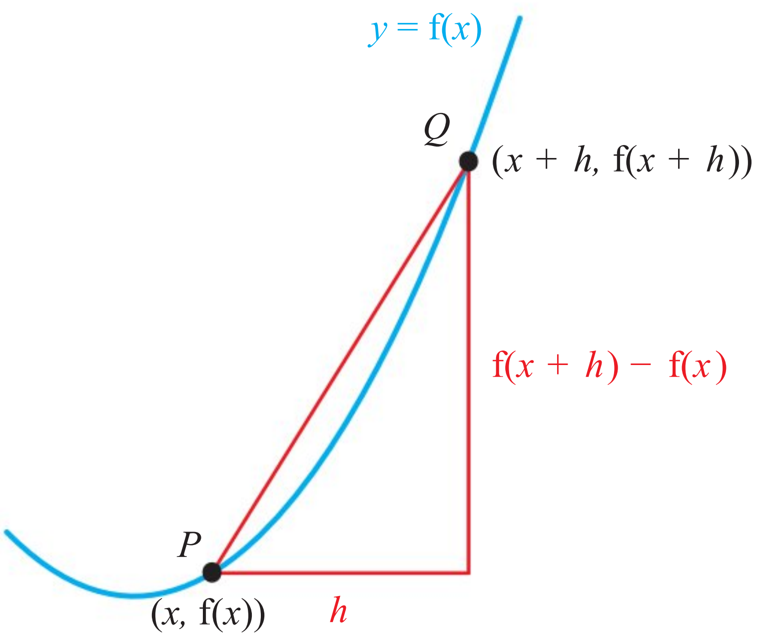
As $x \to \infty$, this expression tends to

$$\frac{x}{0+3}=\frac{1}{3}x$$

This grows without limit as $x \to \infty$, so the expression diverges.

Differentiation from first principles

You saw in Worked Example 9.2 in Mathematics: analysis and approaches SL that the gradient of a chord from a point gets closer and closer to the gradient of the tangent at that point as the chord gets smaller. We can generalize this argument to any point on a curve to find an expression for the gradient of the tangent at any point. This is called differentiation from first principles.



The gradient of the chord from P when the horizontal distance between points P and Q is h is given by

- $\frac{f(x+h)-f(x)}{h}$.

As h tends towards zero we find the gradient of the tangent at P. Since P is a general point, this gives us the derivative of the function.

KEY POINT 10.1

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The main trick that comes up with applying this formula is that if you just set $h = 0$ you find that both the numerator and the denominator equal zero. The expression $\frac{0}{0}$ is undetermined – it might take a value, but it is not obvious what that value is. So before setting $h = 0$ normally some algebra has to be done to remove this issue.

WORKED EXAMPLE 10.5

Prove from first principles that if $f(x) = x^2$ then $f'(x) = 2x$.

Write the expression for differentiation from first principles with $f(x) = x^2$. Putting in $h = 0$ currently makes the expression $\frac{0}{0}$ which is undetermined…

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

… so some algebra is necessary first

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} 2x + h$$

At this point, we can safely set $h = 0$ without any division by zero occurring

$$= 2x$$

Sometimes, the algebra needs to be a little more imaginative.

WORKED EXAMPLE 10.6

If $y = \frac{1}{x}$, prove that $\frac{dy}{dx} = -\frac{1}{x^2}$.

Write the expression for differentiation from first principles with $f(x) = \frac{1}{x}$. Putting in $h = 0$ currently makes the expression $\frac{0}{0}$ which is undetermined…

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

… so some algebra is necessary first. Starting with the writing the numerator as a single fraction

$$= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{-h}{x(x+h)}\right)}{h}$$

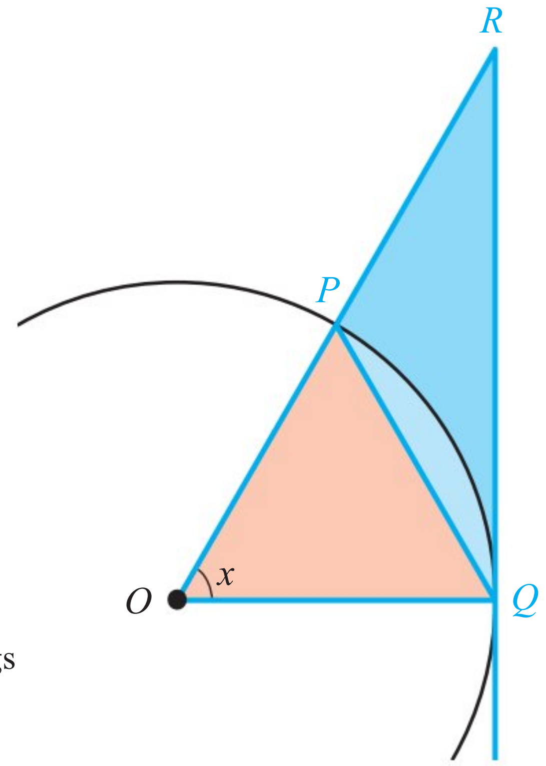
$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$

Now h can be set to zero

$$= -\frac{1}{x^2}$$

TOOLKIT: Proof

Why are radians so important when differentiating trigonometric functions?
The answer lies in the small angle approximation, which states that if x is small and in radians, $\sin x \approx x$. To prove this, consider the following diagram showing a circle of radius 1. A sector OPQ is drawn with an angle x at the centre of the circle, O.
A tangent is drawn at Q, which meets the line OP at R. Considering the right-angled triangle OQR, you can see that $|QR| = \tan x$.



Then we can compare areas:

Triangle OPQ < Sector OPQ < Triangle OQR

Therefore:

$$\frac{1}{2} \times 1^2 \times \sin x < \frac{1}{2} \times 1^2 \times x < \frac{1}{2} \times 1 \times \tan x$$

Notice that the formula for the area of a sector is the only point in this proof that uses the fact that x is in radians.

Simplifying:

$$\sin x < x < \frac{\sin x}{\cos x}$$

If x is small, $\cos x \approx 1$, so in this limit:

$\sin x < x < \sin x$

We see that x is sandwiched between two things which both tend towards $\sin x$, so x must also tend towards $\sin x$.

This fact can then be combined with the compound angle formula to find the derivative of $\sin x$. If $f(x) = \sin x$, then using the formula for differentiation from first principles gives:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \end{aligned}$$

Since h is small in the limit, we can use $\cos h \approx 1$ and $\sin h \approx h$ therefore:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin x + h\cos x - \sin x}{h} \\ &= \lim_{h \to 0} \frac{h\cos x}{h} \\ &= \lim_{h \to 0} \cos x \end{aligned}$$

At this point, we can take the limit of h tending towards zero without it having any effect, so $f'(x) = \cos x$.

Higher derivatives

You have already met the first derivative, $\frac{dy}{dx}$ or $f'(x)$ and the second derivative, $\frac{d^2y}{dx^2}$ or $f''(x)$. We can continue differentiating functions. The nth derivative is given the notation $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

WORKED EXAMPLE 10.7

Find $\frac{d^3y}{dx^3}$ if $y = x^4$.

Work through derivatives starting with the first derivative

$$\frac{dy}{dx} = 4x^3$$

$$\frac{d^2y}{dx^2} = 12x^2$$

$$\frac{d^3y}{dx^3} = 24x$$

WORKED EXAMPLE 10.8

If $f(x) = xe^x$, prove that $f^{(n)}(x) = (x+n)e^x$ for $n \in \mathbb{Z}^+$.

Proving something about positive integers suggest induction

Using the product rule,

$$f'(x) = xe^x + 1e^x$$
$$= (x+1)e^x$$

So, the statement holds for $n = 1$

Assume that the statement is true for $n = k$

$$f^{(k)}(x) = (x+k)e^x$$

Link the statement for $n+k+1$ to the statement for $n = k$

Then

$$f^{(k+1)}(x) = \frac{d}{dx}(f^{(k)}(x))$$

Using the assumption:

$$f^{(k+1)}(x) = \frac{d}{dx}((x+k)e^x)$$

This expression can be manipulated using the product rule

Using the product rule with $u = x + k$ and $v = e^x$,

then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$, therefore

$$f^{(k+1)}(x) = (x+k)e^x + 1e^x$$
$$= (x+(k+1))e^x$$

The final line is the statement with n replaced with $k + 1$

Therefore, the statement would also hold for $n = k + 1$

Summarize the logic of what you have found

The statement holds for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$. This implies that the statement is true for all positive integers.

You are the Researcher

You know that in kinematics, the velocity and the acceleration are the first and second derivatives of the displacement. Find out about the meaning and uses of higher derivatives – they also have exciting names.

Exercise 10A

For questions 1 to 6, use the method demonstrated in Worked Example 10.1 to find the values of a which make the following functions continuous at all points.

1 a $f(x) = \begin{cases} -x & x \leqslant 1 \\ x + a & x > 1 \end{cases}$

b $f(x) = \begin{cases} 3x & x \leqslant 2 \\ x + a & x > 2 \end{cases}$

2 a $f(x) = \begin{cases} x + 1 & x \leqslant 2 \\ ax & x > 2 \end{cases}$

b $f(x) = \begin{cases} x + 1 & x \leqslant 2 \\ ax + 2 & x > 2 \end{cases}$

3 a $f(x) = \begin{cases} x^2 & x \leqslant 3 \\ x + a & x > 3 \end{cases}$

b $f(x) = \begin{cases} x^2 - 4 & x \leqslant 2 \\ x + a & x > 2 \end{cases}$

4 a $f(x) = \begin{cases} x + 6 & x \leqslant 2 \\ x^a & x > 2 \end{cases}$

b $f(x) = \begin{cases} 16x + 16 & x \leqslant 1 \\ (x+1)^a & x > 1 \end{cases}$

5 a $f(x) = \begin{cases} e^{ax} & x \leqslant 2 \\ \dfrac{1}{x} & x > 2 \end{cases}$

b $f(x) = \begin{cases} e^{ax} & x \leqslant 1 \\ x + 3 & x > 1 \end{cases}$

6 a $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ \ln(ax) & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^2 & x \leqslant 2 \\ \ln(ax) & x > 2 \end{cases}$

For questions 7 to 12, use the method demonstrated in Worked Example 10.2 to find the values of a and b which make the following functions continuous and differentiable at all points.

7 a $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ ax + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^3 & x \leqslant 1 \\ ax + b & x > 1 \end{cases}$

8 a $f(x) = \begin{cases} x^2 + a & x \leqslant 2 \\ bx + 1 & x > 2 \end{cases}$

b $f(x) = \begin{cases} x^2 + a & x \leqslant 3 \\ bx + 2 & x > 3 \end{cases}$

9 a $f(x) = \begin{cases} x^a & x \leqslant 1 \\ bx - 1 & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^a & x \leqslant 1 \\ bx - 2 & x > 1 \end{cases}$

10 a $f(x) = \begin{cases} e^x & x \leqslant 0 \\ ax + b & x > 0 \end{cases}$

b $f(x) = \begin{cases} e^x & x \leqslant 2 \\ ax + b & x > 2 \end{cases}$

11 a $f(x) = \begin{cases} e^{-x} & x \leqslant 1 \\ \dfrac{a}{x} + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} e^{-x} & x \leqslant 2 \\ \dfrac{a}{x} + b & x > 2 \end{cases}$

12 a $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ a\ln x + b & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ a\sqrt{x} + b & x > 1 \end{cases}$

For questions 13 to 18, use the method demonstrated in Worked Example 10.3 to find the limit of the given expression as x tends to infinity.

13 a $\dfrac{3x^2 + x}{x^2 - x}$ **b** $\dfrac{4x^2 + 1}{2x^2 + 5x}$

14 a $\dfrac{x}{x + 4}$ **b** $\dfrac{3x}{1 + 2x}$

15 a $\dfrac{x}{x^2 + 4}$ **b** $\dfrac{2x}{x^2 + 3x}$

16 a $\dfrac{e^x - 1}{2e^x + 1}$ **b** $\dfrac{e^x + 2}{e^x + 3}$

17 a $\dfrac{e^{2x} + e^x}{2e^{2x} + 3e^x}$ **b** $\dfrac{4e^{3x} + 3e^x}{e^x - e^{3x}}$

18 a $\dfrac{4^x - 3^x}{4^x + 3^x}$ **b** $\dfrac{10^x + 2^x}{4^x - 10^x}$

For questions 19 to 22, use the method demonstrated in Worked Example 10.4 to determine if the given expression is convergent or divergent as $x \to \infty$.

19 a $\dfrac{x^3}{x + 3x^2}$ **b** $\dfrac{x^2 + 1}{x + 3}$

20 a $\dfrac{1 + x}{1 + x^2}$ **b** $\dfrac{2 + 3x}{5x + x^2}$

21 a $\dfrac{e^x + e^{2x}}{1 + 2e^x}$ **b** $\dfrac{1 - e^{2x}}{e^x - 5}$

22 a $\dfrac{e^x}{e^x + 1}$ **b** $\dfrac{e^x + 2}{e^x + 1}$

For questions 23 to 26, use the method demonstrated in Worked Example 10.5 to differentiate the given expression from first principles.

23 a $2x$ **b** 8

24 a x^3 **b** x^4

25 a $x^2 + 3$ **b** $2x + 1$

26 a $x^2 + 2x$ **b** $x^3 + x$

For questions 27 to 30, use the method demonstrated in Worked Example 10.7 to find the third derivative of each function with respect to x.

27 a x^5
b x^6

28 a $x^4 + 2x^3$
b $3x^5 - 2x^4$

29 a $\ln x$
b e^{2x}

30 a $\sin x$
b $\cos x$

31 Given that $f(x) = e^{5x}$, use induction to prove that $f^n(x) = 5^n e^{5x}$.

32 a Find the limit of the function $f(x) = \frac{3e^x}{1+2e^x}$ as x tends towards:

i infinity
ii minus infinity
iii zero.

b Use the quotient rule to show that $f'(x) = \frac{3e^x}{(1+2e^x)^2}$. Hence explain why f(x) is an increasing function.

c Hence sketch $y = f(x)$.

33 The function

$$f(x) = \begin{cases} e^{ax} + x & x \leqslant 2 \\ 2e^{ax} & x > 2 \end{cases}$$

is continuous at $x = 2$. Find the value of a.

34 a Determine if the function $f(x) = a^x + \frac{3x}{2}$, where $a > 0$, is convergent or divergent as x tends to minus infinity.

b The function

$$f(x) = \begin{cases} a^x + \frac{3x}{2} & x \leqslant 2 \\ 4a^{x-1} & x > 2 \end{cases}$$

is continuous at $x = 2$. Find the possible values of a.

c In each case found in part b, sketch f(x).

35 The function

$$f(x) = \begin{cases} ax^2 + x & x \leqslant 2 \\ bx + 2 & x > 2 \end{cases}$$

is both continuous and differentiable at every point. Find the values of a and b.

36 Prove that if $y = xe^{2x}$, then $\frac{d^n y}{dx^n} = (n2^{n-1} + 2^n x)e^{2x}$.

37 a Prove from first principles that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

b Prove by induction that if $f(x) = \frac{1}{1-x}$, then $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$.

38 If $f(x) = \sin x$, prove that $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$.

39 Prove from first principles that if $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

40 Prove from first principles that if $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

41 If $f(x) = x \sin x$, prove that $f^{(2n)}(x) = (-1)^n(x \sin x - 2n - \cos x)$.

42 Prove by induction that if $f(x) = \ln x$, then

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}.$$

43 Prove that $\sqrt{x+4} - \sqrt{x}$ tends towards zero as x tends towards infinity.

10B L'Hôpital's rule

Evaluating limits using L'Hôpital's rule

When studying differentiation from first principles in Section 10A, you met the issue of zero divided by zero being undetermined. Sometimes algebraic manipulation can be applied to these fractions to remove this issue, which was the approach taken in Worked Example 10.6. However, there is another very important tool you can use in this situation called **L'Hôpital's rule**.

KEY POINT 10.2

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof 10.1

Prove L'Hôpital's rule.

First of all, we are going to change the limit of $x \to a$ into $x = a + h$ with h tending toward zero

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)}$$

Since f(a) and g(a) are both zero, we can insert them into the numerator and denominator. You might wonder why we are doing this – we are trying to link to the definition of f′(x) and g′(x) from first principles (Key Point 10.1)

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{g(a+h) - g(a)}$$

We then need to divide top and bottom by h to make it look even more like differentiation from first principles

$$= \lim_{h \to 0} \frac{\left(\frac{f(a+h) - f(a)}{h}\right)}{\left(\frac{g(a+h) - g(a)}{h}\right)}$$

In the limit as $h \to 0$ the top line becomes f′(a) and the bottom line becomes g′(a)

$$= \lim_{h \to 0} \frac{f'(a)}{g'(a)}$$

We can then turn the limit back into one involving x

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Notice that in the second to last line of the proof we are using the intuitive idea that the limit of a ratio is the ratio of the limits. It turns out this is true, but it is quite tricky to prove without more rigorous analytic tools.

WORKED EXAMPLE 10.9

Find $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$.

The limit of both top and bottom of the fraction is zero, so we can use L'Hôpital's rule.

Differentiate top and bottom of the fraction

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} \frac{\cos \theta}{1}$$

As θ gets very small, $\cos \theta$ tends towards 1

$$= \frac{1}{1} = 1$$

In the next chapter you will explore in more detail the idea of approximating functions by polynomials.

CONCEPTS – APPROXIMATION

Limits can be used to **approximate** the value of a function near a point where its expression is undetermined. For example, from Worked Example 10.9 it follows that $\frac{\sin\theta}{\theta} \approx 1$ when θ is close to 0. This in turn confirms the small angle approximation, which you proved in the last section: that, for small values of θ, $\sin\theta = \theta$. This idea of approximating a function by a simpler one is very important in advanced mathematics, enabling us to approximate solutions to many integral and differential equations which cannot be solved analytically.

If f(x) and g(x) both tend to infinity, then the functions $\frac{1}{f(x)}$ and $\frac{1}{g(x)}$ both tend to zero and so satisfy the conditions in Key Point 10.2.

Since $\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} = \frac{f(x)}{g(x)}$, this means that L'Hôpital's rule can also be applied to the indeterminate form $\frac{\infty}{\infty}$.

KEY POINT 10.3

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

WORKED EXAMPLE 10.10

Find $\lim_{x\to\infty} \frac{e^x}{x}$.

The limit of both top and bottom of the fraction is infinity, so we can use L'Hôpital's rule.

Differentiate top and bottom of the fraction

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1}$$

$$= \lim_{x\to\infty} e^x$$

This tends to infinity, so the expression is divergent at infinity.

Tip

In advanced mathematics, you will very frequently use the fact that exponential functions grow faster than any polynomial.

Sometimes it is not obvious that L'Hôpital's rule can be used. Some algebra is needed first to put the expression into the required form.

WORKED EXAMPLE 10.11

Find $\lim_{x\to 0} x\ln x$.

We need to turn this into a fraction where top and bottom are either both zero or both infinite. We can use the fact that $x = \frac{1}{1/x}$	$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x}$
Notice that for the purpose of L'Hôpital's rule we do not care about whether the value is positive or negative infinity	Now top and bottom of the fraction tend towards an infinite quantity.
Differentiate top and bottom of the fraction	$\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}$
Simplify the fractions	$= \lim_{x\to 0} (-x)$
You are now free to take the limit	$= 0$

Be the Examiner 10.1

Evaluate $\lim_{x\to 0} \frac{\cos x}{x}$.

Which is the correct solution? Identify the mistakes in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\lim_{x\to 0} \frac{\cos x}{x} = \lim_{x\to 0} \frac{-\sin x}{1}$ $= -\sin(0)$ $= 0$	$\lim_{x\to 0} \frac{\cos x}{x} = \lim_{x\to 0} \frac{1}{x}$ Which is undefined.	$\lim_{x\to 0} \frac{\cos x}{x} = \lim_{x\to 0} \frac{x\cos x}{x^2}$ $= \lim_{x\to 0} \frac{\cos x - x\sin x}{2x}$ $= \frac{1}{2}\lim_{x\to 0} \frac{\cos x}{x} - \lim_{x\to 0} \frac{\sin x}{2}$ Therefore $\frac{1}{2}\lim_{x\to 0} \frac{\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{2} = 0$ So $\lim_{x\to 0} \frac{\cos x}{x} = 0$

Be the Examiner 10.2

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x^3+x-2}$.

Which is the correct solution? Identify the mistakes in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\lim_{x\to 1} \frac{x^2-1}{x^3+x-2} = \lim_{x\to 1} \frac{2x}{3x^2+1}$ $= \lim_{x\to 1} \frac{2}{6x}$ $= \frac{1}{3}$	Numerator is dominated by x^2 Denominator is dominated by x^3 Therefore expression is approximately $\frac{x^2}{x^3}$ which tends towards 1.	$\lim_{x\to 1} \frac{x^2-1}{x^3+x-2} = \lim_{x\to 1} \frac{2x}{3x^2+1}$ $= \frac{2}{4} = \frac{1}{2}$

Repeated use of L'Hôpital's rule

Sometimes the result of using L'Hôpital's rule is still undetermined. If this is the case, you may have to use L'Hôpital's rule again.

WORKED EXAMPLE 10.12

Find the limit of $\frac{\cos x - 1}{x^2}$ as x tends to 0.

The limit of both top and bottom of the fraction is zero, so we can use L'Hôpital's rule.

Differentiate top and bottom of the fraction: $\lim_{x\to 0}\frac{\cos x - 1}{x^2} = \lim_{x\to 0}\frac{-\sin x}{2x}$

This is still zero divided by zero, so use L'Hôpital's rule again: $\lim_{x\to 0}\frac{-\cos x}{2}$

As $x \to 0$, $\cos x \to 1$: $= -\frac{1}{2}$

Exercise 10B

For questions 1 to 4, use the method demonstrated in Worked Example 10.9 to find the following limits.

1 a $\lim_{x\to 0}\frac{12x}{3x}$ b $\lim_{x\to 0}\frac{8x}{4x}$

2 a $\lim_{x\to 0}\frac{x^2 - 2x}{x}$ b $\lim_{x\to 0}\frac{x^2 + 5x}{2x}$

3 a $\lim_{x\to 0}\frac{e^x - 1}{\sin x}$ b $\lim_{x\to 0}\frac{e^{-x} - e^x}{x}$

4 a $\lim_{x\to 1}\frac{\ln x}{x - 1}$ b $\lim_{x\to 1}\frac{\sqrt{x} - 1}{x - 1}$

For questions 5 to 8, use the method demonstrated in Worked Example 10.10 to find the following limits.

5 a $\lim_{x\to\infty}\frac{10x + 3}{5x + 1}$ b $\lim_{x\to\infty}\frac{12x + 1}{3x}$

6 a $\lim_{x\to\infty}\frac{\ln x}{x}$ b $\lim_{x\to\infty}\frac{x}{e^x}$

7 a $\lim_{x\to\infty}\frac{e^x}{3x + 2}$ b $\lim_{x\to\infty}\frac{e^x}{x + 2}$

8 a $\lim_{x\to\infty}\frac{e^x}{\ln x}$ b $\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}$

9 Use L'Hôpital's rule to find $\lim_{x\to 0}\frac{e^x - 1}{x}$.

10 Use L'Hôpital's rule twice to find $\lim_{x\to\infty}\frac{x^2}{e^x}$.

11 Find $\lim_{x\to 0}\frac{1 - \cos(x^2)}{x^4}$.

12 Find the following limits.

a $\lim_{x\to 0}\frac{x - \cos x}{x + \cos x}$ b $\lim_{x\to 0}\frac{x - \sin x}{x + \sin x}$

13 Find the following limits.

a $\lim_{x\to 0}\frac{\sin x}{x^2}$ b $\lim_{x\to 2}\frac{\sin(x - 2)}{x^2 - 4}$

14 Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\cos^2(5x)}{\cos^2 x}$.

15 Evaluate $\lim_{x\to 1}\frac{(\ln x)^2}{x^2 - 2x + 1}$.

16 Evaluate $\lim_{x\to 1}\frac{(\ln x)^2}{x^3 + x^2 - 5x + 3}$.

17 a Find $\lim_{x\to 0}\frac{\sin x}{x}$. b Find $\lim_{x\to\infty}\frac{\sin x}{x}$. c Hence sketch $y = \frac{\sin x}{x}$.

18 Show that the function

$$f(x) = \begin{cases} \frac{\sin 3x}{\sin x} & x \leqslant 0 \\ \frac{e^{3x} - 1}{x} & x > 0 \end{cases}$$

is continuous at $x = 0$.

19 Find $\lim_{x\to -\infty} xe^x$.

20 Find $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

21 Find $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$.

22 Use L'Hôpital's rule to prove that

$$\lim_{x\to\infty}\frac{e^x + e^{-x}}{e^x - e^{-x}} = 1.$$

10C Implicit differentiation

Most of the functions you have differentiated so far have been expressed in the form $y = f(x)$.

You might not have realized that when we say to 'differentiate' this type of expression we are really differentiating both sides with respect to x:

$$\frac{d}{dx}(y) = \frac{d}{dx}(f(x))$$

The left-hand side is just written as $\frac{dy}{dx}$.

However, not all relationships can be expressed in the form $y = f(x)$ – for example, $x^2 + y^2 = 4$. Both sides of this relationship can still be differentiated with respect to x, but there is an issue when we try to differentiate y^2 with respect to x. We can get around this by using the chain rule:

$$\frac{d}{dx}(y)^2 = \frac{d}{dy}(y)^2 \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$$

In general:

KEY POINT 10.4

$$\frac{d}{dx}(g(y)) = g'(y) \times \frac{dy}{dx}$$

WORKED EXAMPLE 10.13

If $x^4 + y^4 = 17$, find the gradient of the curve at (1, 2).

The gradient of the curve is $\frac{dy}{dx}$. To find $\frac{dy}{dx}$ we need to differentiate both sides with respect to x

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(17)$$

Differentiation can be applied separately to each term

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = \frac{d}{dx}(17)$$

We can now do the differentiation, using Key Point 10.4

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

We can then rearrange to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

Substitute in for x and y

$$\frac{dy}{dx} = -\frac{1}{8}$$

The same idea can be applied in more complicated situations, combining with the product, quotient and chain rules.

WORKED EXAMPLE 10.14

If $e^{3y} + x^2y^3 = x$, find $\frac{dy}{dx}$ in terms of x and y.

To find $\frac{dy}{dx}$ we need to differentiate both sides with respect to x

$$\frac{d}{dx}(e^{3y} + x^2y^3) = \frac{d}{dx}(x)$$

Differentiation can be applied separately to each term

$$\frac{d}{dx}(e^{3y}) + \frac{d}{dx}(x^2y^3) = \frac{d}{dx}(x)$$

The second term is a product, so the product rule is required

$$\frac{d}{dx}(e^{3y}) + \frac{d}{dx}(x^2)y^3 + x^2\frac{d}{dx}(y^3) = \frac{d}{dx}(x)$$

We can now do the differentiation, using Key Point 10.4

$$3e^{3y}\frac{dy}{dx} + 2xy^3 + x^2 \times 3y^2\frac{dy}{dx} = 1$$

Move everything not involving $\frac{dy}{dx}$ onto the right hand side and factorize the rest

$$\frac{dy}{dx}(3e^{3y} + 3x^2y^2) = 1 - 2xy^3$$

$$\text{So } \frac{dy}{dx} = \frac{1 - 2xy^3}{3e^{3y} + 3x^2y^2}$$

You are the Researcher

Implicit equations can be used to describe many curves whose equations cannot be written in the form $y = f(x)$. Use technology to explore implicit curves; for example, plot the curve with equation $x \sin x = y \sin y$. Find out about famous curves, such as the asteroid and the cardioid.

Exercise 10C

For questions 1 to 4, use the method demonstrated in Worked Example 10.13 to find the gradient of the curve at the given point.

1 a $x^2 + y^2 = 25$ at $(3, -4)$

b $x^3 + y^2 = 9$ at $(2, 1)$

2 a $e^x + e^y = 2e$ at $(1, 1)$

b $e^x + e^{2y} = e + 1$ at $(1, 0)$

3 a $\ln x + \ln y = 4$ at (e, e^3)

b $\ln(x) + \ln(y^2) = 2$ at $(1, e)$

4 a $x^2 + y^2 = x + y + 2$ at $(2, 1)$

b $x^3 + y^3 = x + 4y$ at $(1, 2)$

For questions 5 to 9, use the method demonstrated in Worked Example 10.14 to find $\frac{dy}{dx}$ in terms of x and y.

5 a $xy^2 + yx^2 = 3$

b $xy + x^2y^2 = 5$

6 a $\frac{x}{y} + \frac{y}{x} = x$

b $\frac{x}{x+y} = y^2 - 3$

7 a $\ln x + \ln y = \ln(x + y) - 1$

b $\ln(x + 2y) = x^2 - y$

8 a $e^{x+y} = x + \frac{1}{y}$

b $e^{(y^2)} + e^{(x^2)} = 3$

9 a $\sin(xy) = x + y$

b $\cos(x + y) = y$

10 A curve is given by the implicit equation $3x^2 + y^3 = 11$.

a Show that the point $(1, 2)$ lies on the curve.

b Find the gradient to the curve at this point.

c Find the equation of the normal to the curve at this point.

11 **a** Find the coordinates of the point where the curve with equation $\ln y = \sin x$ crosses the y-axis.

b Find the equation of the tangent to the curve at that point.

12 A curve has equation $e^x + \ln y = 0$.

a The point $A\left(a, \frac{1}{\sqrt{e}}\right)$ lies on the curve. Find the exact value of a.

b Find the gradient of the curve at A.

13 Find the equation of the tangent to the curve $x^2 - 2x - y^3 + y = 3$ at the point (3, 1).

14 Find the tangent to the curve $\frac{x+y}{x-y} = 2y$ at the point (3, 1).

15 A curve has equation $\sin(x + y) = \sqrt{2}\cos(x - y)$.

a Show that the point $\left(\frac{13\pi}{24}, \frac{5\pi}{24}\right)$ lies on the curve.

b Find the gradient of the curve at this point, giving your answer in the form $a + b\sqrt{3}$.

16 Find the equations of the two possible tangents to the curve $x^2 + 3xy + y^2 = 1$ when $x = 0$.

17 **a** Find the y intercepts of the curve $y^3 - y - x = 0$.

b Find the gradients of the tangents at each of these points.

18 **a** Find the x intercepts of the curve $e^y - y = x^2$.

b Find the equations of the tangents at each of these points.

19 **a** Find the possible values of y with an x coordinate of 1 in the equation $x^2 - 5xy + y^2 = 1$.

b Find the equations of the tangents at each of these points.

20 Find $\frac{dy}{dx}$ if $e^y - x\sin y = \ln y$.

21 Given that $x\sin x = y\sin y$, find an expression for $\frac{dy}{dx}$ in terms of x and y.

22 If $x^2 + y^2 = 9$, find an expression for $\frac{d^2y}{dx^2}$ in terms of y.

23 Find the coordinates of the stationary points on the curve $x^2 + 4xy + 2y^2 + 1 = 0$.

24 Find the coordinates of the turning points on the curve $y^3 - 3xy^2 + x^3 = 8$.

25 **a** Sketch the curve $y^2 = x^3$.

b Show that the equation of the tangent to the curve at the point (4, 8) is $y = 3x - 4$.

c Find the coordinates of the point where this tangent meets the curve again.

10D Related rates of change

There are many situations where we are given one rate and want to link it to another rate. One of the most common methods to do this is to use implicit differentiation to link the two rates.

WORKED EXAMPLE 10.15

If $y = x^2$ find $\frac{dy}{dt}$ when $x = 4$ and $\frac{dx}{dt} = 12$.

Use implicit differentiation to find $\frac{dy}{dt}$: $\frac{dy}{dt} = 2x \times \frac{dx}{dt}$

Substitute in values for x and $\frac{dx}{dt}$: $= 8 \times 12 = 96$

Frequently the link between the variables comes from a geometric context.

WORKED EXAMPLE 10.16

A rectangle has width x and height y. When $x = 2$ cm and $y = 4$ cm, these values are changing according to $\frac{dx}{dt} = 3\,\text{cm s}^{-1}$ and $\frac{dy}{dt} = -1\,\text{cm s}^{-1}$. What is the rate of change of the area at this time?

Define a variable for the area — Let the area be $A\,\text{cm}^2$.

$A = xy$

Differentiate both sides with respect to t — $\frac{dA}{dt} = \frac{d}{dt}(xy)$

This requires the product rule — $= x\frac{dy}{dt} + y\frac{dx}{dt}$

Substitute in the given values — $= 2 \times -1 + 4 \times 3$

$= 10$

So, the area is increasing at a rate of $10\,\text{cm}^2\,\text{s}^{-1}$.

CONCEPTS – QUANTITY, CHANGE

Worked Example 10.16 illustrates that the rate of **change** can depend on the shape's current size. For example, if the radius of the circle increases at a constant rate then the area will increase faster as the circle gets larger; when a balloon is inflated at a constant rate (so that the rate of increase of volume is constant) the rate of change of the radius will decrease with the size of the balloon. Implicit differentiation is an important tool for **quantifying** the relationship between the different rates.

Exercise 10D

For questions 1 to 4, use the method demonstrated in Worked Example 10.15 to find the required rate of change.

1 **a** If $y = x^3$, find $\frac{dy}{dt}$ when $x = 1$ and $\frac{dx}{dt} = -1$.

b If $y = x^2 + x$, find $\frac{dy}{dt}$ when $x = 0$ and $\frac{dx}{dt} = 4$.

2 **a** If $A = e^{2z}$, find the rate of increase of A when $z = 0$ and $\frac{dz}{dt} = 6$.

b If $p = e^q$, find the rate of increase of p when $q = 1$ and $\frac{dq}{dt} = 7$.

3 **a** If $a = \frac{1}{b}$, find the rate of increase of a when $b = 2$ and b is increasing at a rate of 3 per second.

b If $a = \frac{1}{b^2}$, find the rate of increase of a when $b = 1$ and b is increasing at a rate of 2 per second.

4 **a** If $y = \ln x$, find the rate of increase of y when $x = 2$ and x is decreasing at a rate of 4 per hour.

b If $y = \ln(x + 1)$, find the rate of increase of y when $x = 1$ and x is decreasing at a rate of 3 per hour.

5 Given that $A = x^2 + y^2$, find $\frac{dA}{dt}$ when $x = 3, y = 4, \frac{dx}{dt} = 1, \frac{dy}{dt} = -1$.

6 Given that $B = x^3 + y^3$, find $\frac{dB}{dt}$ when $x = 1, y = 2, \frac{dx}{dt} = 1, \frac{dy}{dt} = -2$.

7 Given that $C = \frac{x}{y}$, find $\frac{dC}{dt}$ when $x = 3, y = 4, \frac{dx}{dt} = 3, \frac{dy}{dt} = -1$.

8 The sides of a square are increasing at a rate of $2\,\text{cm s}^{-1}$. Find the rate of increase of the area when the area is $25\,\text{cm}^2$.

9 Circular mould is spreading on a leaf. When the radius is 3 mm the rate of increase is 1.2 mm per day. What is the rate of increase of the area?

10 The volume of a spherical balloon is increasing at a rate of $200\,\text{cm}^3$ per second. Find the rate of increase of the radius when the volume is $100\,\text{cm}^3$.

11 An x by y rectangle is expanding with $\frac{dx}{dt} = 4\,\text{cm s}^{-1}$ and $\frac{dy}{dt} = -2\,\text{cm s}^{-1}$. When $x = 3$ cm and $y = 4$ cm, find

- **a** the rate of increase of the rectangle's area
- **b** the rate of increase of the length of the diagonal.

12 An inverted cone is being filled with water at a constant rate of $5\,\text{cm}^3\,\text{s}^{-1}$. The surface of the water is always horizontal as it is being filled. The largest diameter of the cone is 10 cm and its height is 30 cm. If the volume of water in the cone is V at time t, and h is the height of the water above the vertex of the cone,

- **a** show that $V = \frac{\pi h^3}{108}$
- **b** find the rate that the height is increasing when $h = 18$ cm.

13 A circular stain of radius r cm and area $A\,\text{cm}^2$ is increasing in size. At a certain time, the rate of increase of the radius is $1.8\,\text{cm s}^{-1}$ and the rate of increase of the area is $86.5\,\text{cm}^2\,\text{s}^{-1}$. Find the radius of the stain at this point.

14 A sportsman throws a ball. When it is 2 m above the sportsman and 4 m away horizontally it is moving with purely horizontally with a speed of $3\,\text{m s}^{-1}$. Find the rate at which the ball is moving away from the sportsman.

15 The density of a reactive substance is given by its mass divided by its volume. When the density is $5\,\text{g cm}^3$ the mass is decreasing at a rate of $2\,\text{g s}^{-1}$ and the volume is decreasing at a rate of $1\,\text{cm}^3\,\text{s}^{-1}$. Determine, with justification, whether the density is increasing or decreasing.

16 A ladder of length 3 m is sliding down a vertical wall. The foot of the ladder is on horizontal ground. When the point of contact with the wall is 2 m above the horizontal that point is moving down at a rate of $0.1\,\text{m s}^{-1}$. At what speed is the foot of the ladder moving away from the wall, assuming that the ladder always stays in contact with both the wall and the ground?

10E Optimization

You saw in Chapter 20 of Mathematics: analysis and approaches SL that you can use calculus to find optimum solutions – maximum or minimum points. At higher level you will use the same technique, but in situations where the function you have to maximize is not so obvious.

Tip

Note that because L is only ever a positive value, the minimum and maximum of L will occur for the same values as the minimum and maximum of L^2. Dispensing with the need to take the square root can make this sort of problem much easier!

WORKED EXAMPLE 10.17

Find the point(s) closest to (0, 1) on the curve $y = x^2$.

First define a variable to quantify the thing you want to optimize. Then write it in terms of the other quantities in the question

Let L be the distance between the point (x, y) on the curve and (0, 1). Then,

$$L = \sqrt{x^2 + (y-1)^2}$$

We need there to be only one variable in the expression, so we can use the constraint that $y = x^2$ to eliminate y

$$= \sqrt{x^2 + (x^2-1)^2}$$

Multiply out the brackets to make the later calculus easier

$$= \sqrt{x^4 - x^2 + 1}$$

Use the chain rule to find $\frac{dL}{dx}$

$$\frac{dL}{dx} = \frac{4x^3 - 2x}{2\sqrt{x^4 - x^2 + 1}}$$

At a maximum or minimum point, $\frac{dL}{dx} = 0$. This happens when the numerator equals zero

At the minimum,

$$4x^3 - 2x = 0$$

This can be solved by factorizing

$2x(2x^2 - 1) = 0$

So, either $x = 0$ or $x^2 = \frac{1}{2}$

It is not clear which of these solutions provides a minimum. Since it is a continuous function we can just check the value of L for each possibility

When $x = 0$, $L = 1$

When $x^2 = \frac{1}{2}$, $L = \frac{\sqrt{3}}{2}$

Since $\frac{\sqrt{3}}{2} < 1$, $x^2 = \frac{1}{2}$ corresponds to the minimum

Therefore, the minimum occurs when $x^2 = \frac{1}{2}$,

corresponding to the, points $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

CONCEPTS – MODELLING

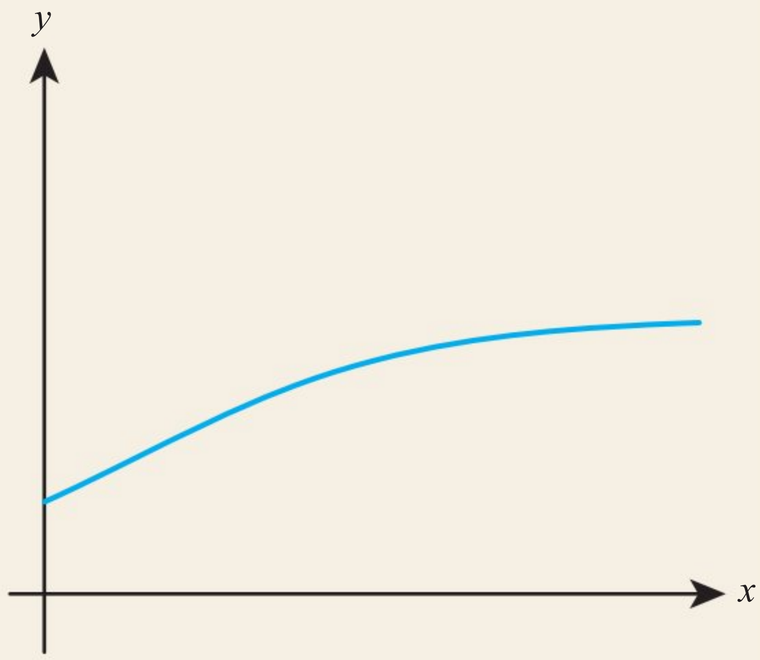
Optimization is often applied in real-life situations to equations **modelling** physical quantities. Can you think of any situation where the problem like the one in Worked Example 10.17 would be relevant?

Sometimes students immediately jump to differentiation when asked to do optimization. However the maximum or minimum value of a function does not always occur when the gradient is zero. It can also occur at the end point of the domain of the function. Often sketching a graph is very helpful, especially if you have your GDC available.

WORKED EXAMPLE 10.18

Find the minimum value of $\frac{2}{1 + 2e^{-x}}$ for $x \geqslant 0$.

Sketching the graph on the GDC gives:



The minimum occurs at $x = 0$. This can be found either substituting in or by using the trace function on the calculator

The maximum value is $\frac{2}{1 + 2e^0} = \frac{2}{3}$.

TOOLKIT: Problem Solving

How might you have found the answer to Worked Example 10.18 without a calculator? One useful approach is to show that the function is always increasing; therefore, its minimum value must be at the lowest x value. To do this, try differentiating the given function, simplifying your answer. What is it about the answer which quickly tells you that the function is always increasing?

Exercise 10E

For questions 1 to 3, use the method demonstrated in Worked Example 10.18 to find the minimum value of each of the following functions.

1 a $x^2 + x, x \geqslant 0$
  b $x^3, x \geqslant 2$

2 a $3 - x^2, 0 \leqslant x \leqslant 3$
  b $1 - 2^x, -1 \leqslant x \leqslant 3$

3 a $xe^x, -1 \leqslant x \leqslant 4$
  b $xe^x, -4 \leqslant x \leqslant -2$

4 Find the maximum and minimum value of f(x) if $f(x) = x^2 - x, 0 \leqslant x \leqslant 2$.

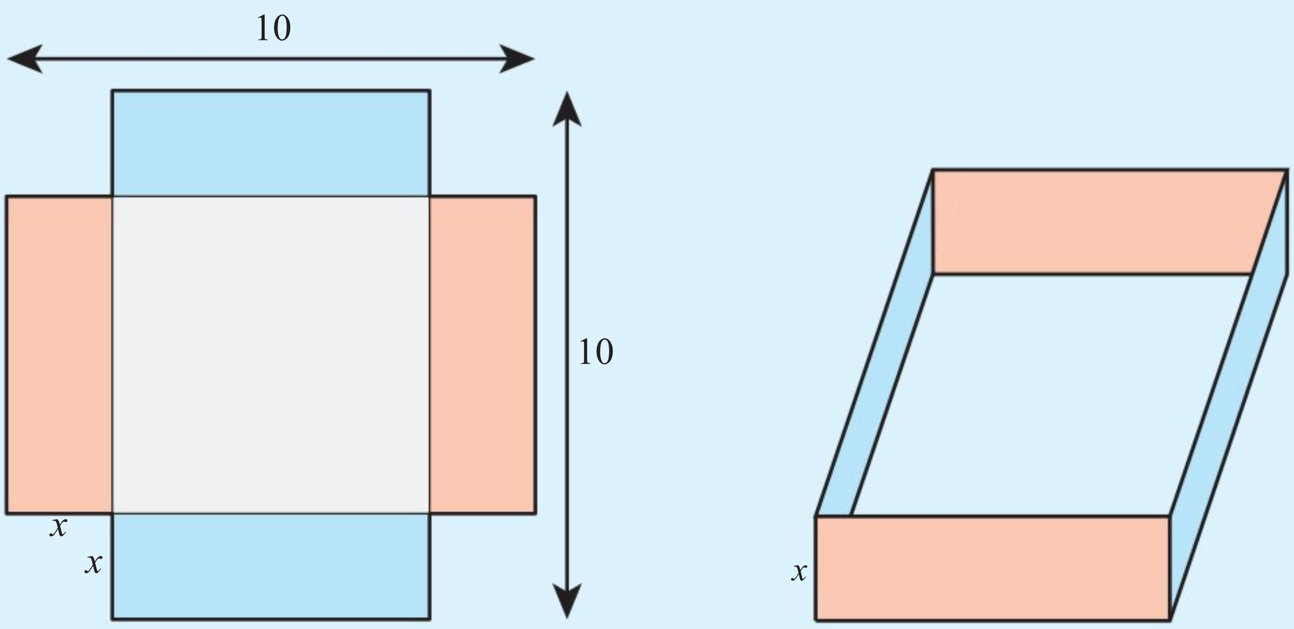
5 Find the maximum and minimum value of xe^{-x}, $0.5 < x < 2$.

6 Find the largest possible value of xy given that $x + 2y = 6$.

7 Find the smallest possible value of $x^3 + y^3$ given that $x + y = 1$.

8 Find the smallest possible value of $x + y$ given that $xy = 5$ and both x and y are positive.

9 A square sheet of card of side 10 cm has four squares of side x cm cut from the corners. The sides are then folded to make a small open box.



a Find an expression for the volume of the box in cm^3 in terms of x, including an appropriate domain.

b Find the value of x for which the volume is the maximum possible.

c What is the maximum possible volume?

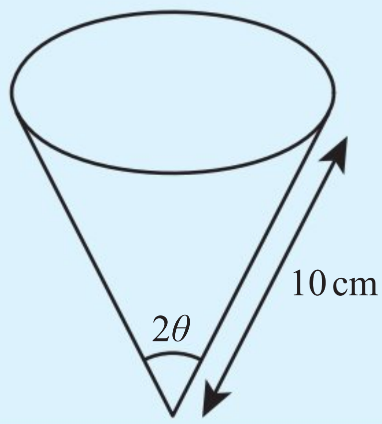
d What is the minimum possible volume?

10 A square based cuboid has a volume of 64 cm^3. Find the minimum possible surface area.

11 As shown in the diagram the apex angle of a cone is 2θ and the slope length is 10 cm. The rate of increase of θ is 0.01 radians per second. The cone starts with $\theta = \frac{\pi}{6}$.

a Find the initial rate of change of the volume of the cone.

b How long does it take the cone to reach its maximum volume?



12 One side of a rectangle lies on the x-axis and two corners lie on the curve $y = \sin x$, $0 \leqslant x \leqslant \pi$. Find the largest possible area of the rectangle.

13 Find the closest distance from the point (1, 2) to the curve $y = x^3$.

14 A piece of wire is bent to form an isosceles triangle. Prove that the largest possible area is formed when the triangle is equilateral.

15 Two corridors meet at a right angle. Find the longest ladder that would fit horizontally around the corner if

- a both corridors are 1 m wide
- b one corridor is 1 m wide and the other corridor is 8 m wide.

You may assume for your calculations that the ladder has negligible width.

You are the Researcher

Question 14 of Exercise 10E is an example of the 'isoperimetric problem': out of all shapes with a fixed perimeter, which has the largest area? Similar questions can be asked about 3D shapes, for example, what is the minimum surface area for a solid with a fixed volume? Question 10 is an example of this type, which have applications in the design of packaging.

You are the Researcher

Question 15 of Exercise 10E is a specific example of something called the 'Ladder problem'. It has a very beautiful general solution. The more realistic generalization of this problem is called the 'Moving Sofa problem' and it is much harder.

10F Calculus applied to more functions

More derivatives

There are several functions you have met which you can now differentiate. These are summarized below.

KEY POINT 10.5

f(x)	f′(x)
$\tan x$	$\sec^2 x$
$\sec x$	$\tan x \sec x$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$
$\cot x$	$-\operatorname{cosec}^2 x$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x \ln a}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

Proof 10.2

Prove that if $y = \arcsin x$, then $\frac{dy}{dx} = \sqrt{1 - x^2}$.

You know how to differentiate sin, so rewrite the equation by taking the inverse function of both sides

$$\sin y = x$$

Differentiate both sides with respect to x – this is implicit differentiation, so remember to use the chain rule for the left-hand side

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

You need an expression for $\frac{dy}{dx}$ in terms of x

The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

For all of these values, $\cos y$ is non-negative.

You can use $\sin^2\theta + \cos^2\theta = 1$ to relate $\cos y$ to $\sin y$, but you need to make sure you select the correct sign

We can therefore rearrange $\cos^2 y + \sin^2 y = 1$ to get

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2}$$

Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

Proof 10.3

Prove that if $y = a^x$ with $a > 0$, then $\frac{dy}{dx} = a^x \ln a$.

You know how to differentiate logarithms, so take ln of both sides

$$\ln y = \ln a^x$$

$$= x \ln a$$

Use implicit differentiation: $\frac{d}{dy}(\ln y) = \frac{1}{y}$

$\ln a$ is just a constant multiplying x

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

$$\frac{dy}{dx} = y \ln a$$

Remember that $y = a^x$

$$= a^x \ln a$$

TOOLKIT: Problem Solving

This is one method for proving this. It could also have been done using the quotient rule or the chain rule. Are there any advantages or disadvantages to each method?

Proof 10.4

Prove that if $y = \sec x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x \tan x$.

You know how to differentiate $\cos x$, so write $\sec x$ as $\frac{1}{\cos x}$: $\frac{1}{y} = \cos x$

Use implicit differentiation: $-\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \sin x$

Remember that $y = \sec x = \frac{1}{\cos x}$: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos^2 x}\sin x$

You want $\sec x$ in the answer, so separate $\frac{1}{\cos x}$: $= \frac{1}{\cos x}\frac{\sin x}{\cos x}$

$= \sec x \tan x$

WORKED EXAMPLE 10.19

Find the gradient of the graph of $y = x + 2\cot x$ at $x = \frac{\pi}{4}$.

Use the result from the table: $\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\mathrm{cosec}^2 x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2\mathrm{cosec}^2 x$

You need to know that $\mathrm{cosec}\, x = \frac{1}{\sin x}$, and the exact value of $\sin\left(\frac{\pi}{4}\right)$

At $x = \frac{\pi}{4}$, $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ so $\mathrm{cosec}\left(\frac{\pi}{4}\right) = \sqrt{2}$

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2 \times (\sqrt{2})^2 = -3$.

More integrals

All of the derivatives given in the table in Key Point 10.5 above can be reversed.

KEY POINT 10.6

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\sec^2 x$	$\tan x + c$
$\tan x \sec x$	$\sec x + c$
$\cot x\, \mathrm{cosec}\, x$	$-\mathrm{cosec}\, x + c$
$\mathrm{cosec}^2 x$	$-\cot x + c$
$a^x \ln a$	$a^x + a$
$\frac{1}{x \ln a}$	$\log_a x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\frac{1}{1+x^2}$	$\arctan x + c$

TOOLKIT: Problem Solving

Can you explain why we did not include the reverse of the arccos x result in the table of derivatives above?

WORKED EXAMPLE 10.20

Find $\int \frac{1+2\sin x}{\cos^2 x}\,dx$.

Split the fraction up into two parts, writing the second part in a way which can be neatened up…

$$\int \frac{1+2\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} + 2\frac{\sin x}{\cos x}\frac{1}{\cos x}\right)dx$$

Then rewrite these trig expressions into forms given in the table of integrals

$$= \int (\sec^2 x + 2\tan x \sec x)dx$$

We can split the integrals into two and bring out the constant factor of 2 from the second integral

$$= \int \sec^2 x\,dx + 2\int \tan x \sec x\,dx$$

Then apply the results from the table. Notice that the sum of two constants can be combined into one constant

$$= \tan x + \sec x + c$$

The composite of these functions with a linear function

As seen in Key Point 21.6 of Mathematics: analysis and approaches SL,
if $\int f(x)dx = F(x)$ then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$.
This can be combined with the results in Key Point 10.6 to integrate more functions.

WORKED EXAMPLE 10.21

Find $\int 6\sec^2(3x+1)dx$.

This is a composition of a standard result and a linear function. In this situation, we can integrate the outer function as normal and just divide by the coefficient of x and simplify the result – no further working is required

$$\frac{6}{3}\tan(3x+1) + c = 2\tan(3x+1) + c$$

A few standard forms come up so frequently that it is useful to learn them specifically.

KEY POINT 10.7

- $\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx = \arcsin\left(\frac{x+b}{a}\right) + c$
- $\int \frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\arctan\left(\frac{x+b}{a}\right) + c$

Proof 10.5

Use the fact that $\int \frac{1}{1+x^2}\,dx = \arctan x + c$ to prove that $\int \frac{1}{a^2+(x+b)^2}\,dx = \frac{1}{a}\arctan\left(\frac{x+b}{a}\right) + c$.

Pull out a factor of $\frac{1}{a^2}$

$$\int \frac{1}{a^2+(x+b)^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x+b}{a}\right)^2}\,dx$$

This is a composition of the standard integral and a linear function. We can therefore integrate as normal, dividing by the coefficient of x – in this case $\frac{1}{a}$

$$= \frac{\left(\frac{1}{a^2}\right)}{\left(\frac{1}{a}\right)}\arctan\left(\frac{x+b}{a}\right) + c$$

Simplify the fractions

$$= \frac{1}{a}\arctan\left(\frac{x+b}{a}\right) + c$$

We could also have proved this by differentiating $\arctan\left(\frac{x+b}{a}\right)$.

Conventionally, when using this formula we take a to be positive, but in the proof this fact was not used. What would happen if a were negative?

In order to apply the results from Key Point 10.7 you may need to complete the square first.

WORKED EXAMPLE 10.22

Find $\int \frac{1}{x^2+2x+10}\,dx$.

Complete the square for the denominator

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx$$

This is now in the standard form from Key Point 10.7

$$= \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + c$$

Partial fractions

If an integral is a fraction with a quadratic denominator that factorizes, one useful tool is to apply partial fractions before integrating.

WORKED EXAMPLE 10.23

Find $\int \frac{1}{x^2-5x+6}\,dx$.

Since there is no obvious other method for integrating, it is a good idea to factorize the denominator of the integrand and use partial fractions

$$\frac{1}{x^2-5x+6} \equiv \frac{1}{(x-2)(x-3)}$$

Use a trial function with unknown numerators

$$\equiv \frac{A}{x-2} + \frac{B}{x-3}$$

Multiply through by the common denominator $(x-2)(x-3)$

Therefore,

$1 \equiv A(x-3) + B(x-2)$

Use x-values which make one of the terms zero

When $x = 3$, we find $B = 1$

When $x = 2$, we find $A = -1$

So,

$$\frac{1}{x^2-5x+6} \equiv -\frac{1}{x-2} + \frac{1}{x-3}$$

You can now split the integral into two separate integrals

Then,

$$\int \frac{1}{x^2-5x+6}\,dx = -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}\,dx$$

Remember the modulus signs when integrating $\ln x$

$$= -\ln|x-2| + \ln|x-3| + c$$

Exercise 10F

For questions 1 to 5, use the technique demonstrated in Worked Example 10.19, together with the results from the table in Key Point 10.5, to find the gradient of the function at the given point.

1 a $y = x + 3\tan x$ at $x = \frac{\pi}{4}$

b $y = 2x - \cot x$ at $x = \frac{\pi}{3}$

2 a $y = 3\sec x + 4\operatorname{cosec} x$ at $x = \frac{\pi}{6}$

b $y = 2\operatorname{cosec} x - 5\sec x$ at $x = \frac{\pi}{4}$

3 a $y = 3^{4x}$ at $x = 1$

b $y = 2^{-3x}$ at $x = 2$

4 a $y = \log_3(x+2)$ at $x = 0$

b $y = \log_5(x-3)$ at $x = 4$

5 a $y = \arcsin(2x) + \arccos(3x)$ at $x = \frac{1}{6}$

b $y = 5x - 3\arctan 2x$ at $x = 1$

For questions 6 to 10, use the technique demonstrated in Worked Example 10.20, and the results from the table in Key Point 10.6, to find the following integrals.

6 a $\int \sec x(2\sec x + 3\tan x)\,dx$

b $\int \operatorname{cosec} x(5\operatorname{cosec} x + 2\cot x)\,dx$

7 a $\int \frac{2+3\sin x}{\cos^2 x}\,dx$

b $\int \frac{3-\cos x}{\sin^2 x}\,dx$

8 a $\int 2^x\,dx$

b $\int 3^x\,dx$

9 a $\int \frac{3}{\sqrt{1-x^2}}\,dx$

b $\int \frac{4}{1+x^2}\,dx$

10 a $\int \frac{2+\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$

b $\int \frac{2\sqrt{1-x^2}-3}{\sqrt{1-x^2}}\,dx$

For questions 11 to 15, use the technique demonstrated in Worked Example 10.21 to find the following integrals.

11 a $\int 8\operatorname{cosec}^2(2x-1)\,dx$

b $\int 9\sec^2(3x+1)\,dx$

12 a $\int 5\operatorname{cosec}^2\left(\frac{x}{3}\right)dx$

b $\int 3\sec^2\left(\frac{x}{4}\right)dx$

13 a $\int 3^{2x}\,dx$

b $\int 2^{5x}\,dx$

14 a $\int \frac{1}{1+4x^2}\,dx$

b $\int \frac{1}{1+16x^2}\,dx$

15 a $\int \frac{3}{\sqrt{1-9x^2}}\,dx$

b $\int \frac{10}{\sqrt{1-25x^2}}\,dx$

For questions 16 to 20, use the technique demonstrated in Worked Example 10.22, together with the results from Key Point 10.7, to find the following integrals.

16 a $\int \frac{1}{x^2+4x+5}\,dx$

b $\int \frac{1}{x^2-6x+10}\,dx$

17 a $\int \frac{1}{x^2+2x+10}\,dx$

b $\int \frac{1}{x^2+4x+20}\,dx$

18 a $\int \frac{1}{x^2+6x+11}\,dx$

b $\int \frac{1}{x^2-10x+30}\,dx$

19 **a** $\int \frac{1}{\sqrt{12-4x-x^2}}\mathrm{d}x$

b $\int \frac{1}{\sqrt{8+2x-x^2}}\mathrm{d}x$

20 **a** $\int \frac{1}{\sqrt{1+4x-x^2}}\mathrm{d}x$

b $\int \frac{1}{\sqrt{2-2x-x^2}}\mathrm{d}x$

For questions 21 to 23, use the technique demonstrated in Worked Example 10.23 to find the following integrals.

21 **a** $\int \frac{2x+4}{x^2+4x+3}\mathrm{d}x$

b $\int \frac{x+1}{x^2+x-6}\mathrm{d}x$

22 **a** $\int \frac{4}{x^2-2x-3}\mathrm{d}x$

b $\int \frac{1}{x^2+5x+6}\mathrm{d}x$

23 **a** $\int \frac{3}{2x^2+x-1}\mathrm{d}x$

b $\int \frac{2}{3x^2+4x+1}\mathrm{d}x$

24 Find the equation of the tangent to the graph of $y = 2\sec^2 x$ at the point where $x = \frac{\pi}{4}$.

25 Find the equation of the normal to the graph of $y = \tan 2x$ at the point $x = \frac{\pi}{6}$.

26 Find the gradient of the graph of $y = \arcsin x - \arccos x$ at the point where it crosses the y-axis.

27 Given that $y = 3\arctan\left(\frac{x}{2}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and simplify your answer.

28 A curve has equation $y = \tan x + \cot x$.

a Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -4\cot 2x$.

b Hence find the coordinates of the stationary points on the graph of $y = \tan x + \cot x$ for $0 < x < \pi$.

29 Find the coordinates of the point on the graph of $y = 3^x$ where the gradient equals ln 81.

30 Evaluate $\int_0^{\frac{\pi}{6}} \sec^2 2x\,\mathrm{d}x$.

31 Find the exact area enclosed by the graph of $y = 2^x$, the x-axis, the y-axis and the line $x = 3$.

32 The area enclosed by the graph of $y = \frac{6}{\sqrt{1-x^2}}$, the x-axis, the y-axis and the line $x = a$ equals π. Find the value of a.

33 A curve has gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\pi\sec^2(\pi x)$ and passes through the point $\left(\frac{1}{4}, 5\right)$. Find the equation of the curve.

34 Show that the function $f(x) = \tan x - \cot x$ is increasing in the interval $0 < x < \frac{\pi}{2}$.

35 **a** Write $\frac{3}{x^2-x-2}$ in partial fractions.

b Hence find $\int \frac{3}{x^2-x-2}\mathrm{d}x$, giving your answer in the form $\ln|f(x)| + c$.

36 **a** Express $\frac{x-6}{x^2-4}$ in partial fractions.

b Hence evaluate $\int_0^1 \frac{x-6}{x^2-4}\mathrm{d}x$, giving your answer as a single logarithm.

37 Use the quotient rule to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$.

38 Prove that if $y = \operatorname{cosec} x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\cot x \operatorname{cosec} x$.

39 Prove that if $f(x) = \cot x$, then $f'(x) = -\operatorname{cosec}^2 x$.

40 Use the change of base rule to prove that if $f(x) = \log_a x$, then $f'(x) = \frac{1}{x\ln a}$.

41 Use implicit differentiation to prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$.

42 Prove that if $y = \arctan x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$.

43 Water drips into a container at the rate given by $\frac{1}{1+4t^2}$ litres per hour, where t is time measured in hours. Initially there was 120 ml of water in the container. Find the volume of the water in the container after 15 minutes.

44 **a** Differentiate $\arcsin(\sqrt{x})$.

b Hence find the exact value of $\int_0^1 \frac{1}{\sqrt{x-x^2}}\,\mathrm{d}x$.

45 a Differentiate $\ln(\sec x)$.

b Hence find the exact area enclosed by the graph of $y = 2\tan 3x$, the x-axis and the line $x = \frac{\pi}{9}$.

46 Evaluate $\int_{3}^{4.5} \frac{1}{\sqrt{(6x - x^2)}}\,dx$.

47 Find the following integrals.

a $\int \frac{1}{x^2 + 6x + 5}dx$ b $\int \frac{1}{x^2 + 6x + 18}dx$ c $\int \frac{2x}{x^2 + 6x + 18}dx$

48 Use the fact that $\int \frac{1}{\sqrt{1 - x^2}}dx = \arcsin x + c$ to prove that

$$\int \frac{1}{\sqrt{a^2 - (x + b)^2}}dx = \arcsin\left(\frac{x + b}{a}\right) + c.$$

49 a Write $4x^2 - 8x + 29$ in the form $(ax - b)^2 + c$. b Hence find $\int \frac{1}{4x^2 - 8x + 19}dx$.

50 a Differentiate $x \arcsin x$. b Hence find $\int \arcsin x\, dx$.

51 Use L'Hôpital's rule to find $\lim_{x \to 0} \frac{\arctan x}{x}$.

52 Use L'Hôpital's rule to find $\lim_{x \to 0} \frac{2^x - 1}{x}$.

10G Integration by substitution

One of the most powerful tools for finding integrals is to use a substitution.

In Key Point 21.7 of the Mathematics: analysis and approaches SL you met integrals of the form $\int g'(x)f'(g(x))dx$ such as $\int 3x^2\sin(x^3)dx$. These could all be done using substitutions, too. For anything which is not within a constant factor of this form, you can expect to be given the substitution required.

WORKED EXAMPLE 10.24

Use the substitution $u = 1 + 2x$ to find $\int x\sqrt{1 + 2x}\,dx$.

You need to express all the xs in terms of u. This includes the dx!

If $u = 1 + 2x$, then $\frac{du}{dx} = 2$

Therefore, $dx = \frac{1}{2}du$

Also, $x = \frac{1}{2}(u - 1)$

Substitute all this into the original integral…

$$\int x\sqrt{1 + x}dx = \int \frac{1}{2}(u - 1)\sqrt{u}\,\frac{1}{2}du$$

… and simplify before integrating

$$= \int \frac{1}{4}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + c$$

The final answer needs to be in terms of x, so replace u by (1 + 2x)

$$= \frac{1}{10}(1 + 2x)^{\frac{5}{2}} - \frac{1}{6}(1 + 2x)^{\frac{3}{2}} + c$$

If you are working with a definite integral, then you have the option of replacing the limits with the corresponding values of the new variable. This often makes the final evaluation easier.

WORKED EXAMPLE 10.25

Use the substitution $x = 2\sin u$ to find $\int_0^1 \frac{1}{(4-x^2)^{1.5}}\,dx$.

Differentiate the substitution to express dx in terms of du. It is not necessary to express u in terms of x

If $x = 2\sin u$, then $\frac{dx}{du} = 2\cos u$

So, $dx = 2\cos u\,du$

Use the limits for x to find the limits for u

Limits:

when $x = 0$, $u = 0$

when $x = 1$, $u = \frac{\pi}{6}$

Substitute in the original integral

$$\int_{x=0}^{x=1} \frac{1}{(4-x^2)^{1.5}}\,dx = \int_{u=0}^{u=\frac{\pi}{6}} \frac{1}{(4-4\sin^2 u)^{1.5}} 2\cos u\,du$$

Simplify as far as possible before attempting to integrate. The denominator can be simplified by using $1 = \sin^2\theta = \cos^2\theta$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{(4\cos^2 u)^{1.5}} 2\cos u\,du$$

$4^{1.5} = 4\sqrt{4} = 8$ and $(a^2)^{1.5} = a^3$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{8\cos^3 u} 2\cos u\,du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 u}\,du$$

Recognize the standard integral $\int \frac{1}{\cos^2 u}\,du = \int \sec^2 u\,du = \tan u$

$$\frac{1}{4}[\tan u]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}} - 0\right) = \frac{1}{4\sqrt{3}}$$

Tip

Conventionally, if nothing else is written to define the limits, it is assumed that they describe the variable that the integral is with respect to. For example, if there is a dx the limits are describing x.

Tip

We could have kept the limits in terms of x but then at the end we would have had to substitute back for $u = \sin^{-1}\left(\frac{x}{2}\right)$.

Exercise 10G

For questions 1 to 5, use the technique demonstrated in Worked Example 10.23 to find the indefinite integrals using the given substitution.

1 a $\int x\sqrt{x+1}\,\mathrm{d}x$, $u = x + 1$

b $\int x\sqrt{x-2}\,\mathrm{d}x$, $u = x - 2$

2 a $\int x(2x+1)^5\,\mathrm{d}x$, $u = 2x + 1$

b $\int x(3x-2)^7\,\mathrm{d}x$, $u = 3x - 2$

3 a $\int \frac{x}{\sqrt{2+x}}\,\mathrm{d}x$, $u = 2 + x$

b $\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$, $u = x - 1$

4 a $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x+1}\,\mathrm{d}x$, $u = \mathrm{e}^x + 1$

b $\int \frac{\mathrm{e}^{3x}}{\mathrm{e}^x-1}\,\mathrm{d}x$, $u = \mathrm{e}^x - 1$

5 a $\int \frac{\cos^3 x}{1+\sin x}\,\mathrm{d}x$, $u = 1 + \sin x$

b $\int \frac{\sin^3 x}{1+\cos x}\,\mathrm{d}x$, $u = 1 + \cos x$

For questions 6 to 9, use the technique demonstrated in Worked Example 10.24 to find the given definite integrals.

6 a $\int_1^3 x(x-1)^4\,\mathrm{d}x$, $x = u + 1$

b $\int_2^3 x(x-2)^5\,\mathrm{d}x$, $x = u + 2$

7 a $\int_1^4 \frac{1}{2\sqrt{x}+x}\,\mathrm{d}x$, $x = u^2$

b $\int_1^9 \frac{1}{\sqrt{x}+3x}\,\mathrm{d}x$, $x = u^2$

8 a $\int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x$, $x = \tan u$

b $\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$, $x = \sec u$

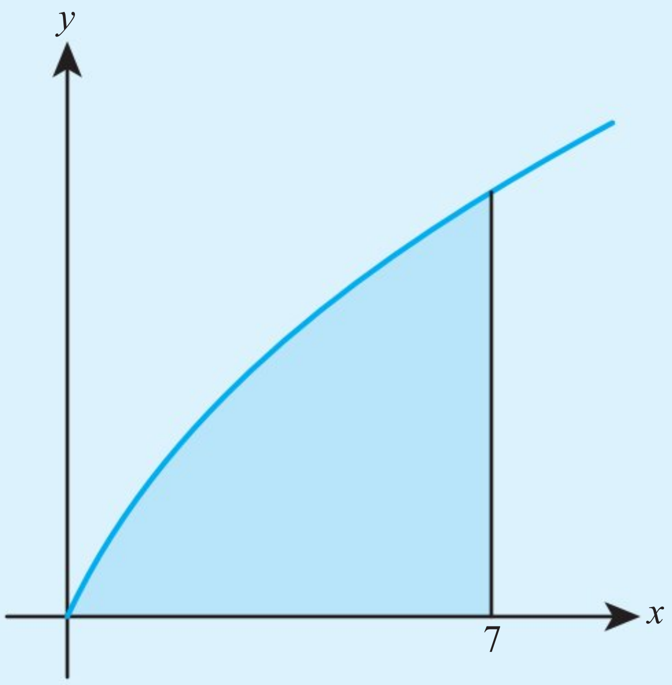
9 a $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$, $x = \sin^2 u$

b $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x$, $x = \sin^2 u$

10 Use the substitution $u = x - 2$ to evaluate $\int_2^3 x(x-2)^3\,\mathrm{d}x$.

11 Use the substitution $u = x + 1$ to find the exact value of $\int_0^1 x^2\sqrt{x+1}\,\mathrm{d}x$.

12 The diagram shows the graph of $y = \frac{x}{\sqrt{x+2}}$.



Use the substitution $u = x + 2$ to find the shaded area.

13 Use the substitution $x = u^2$ to find $\int \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$.

14 Use integration by substitution to find the exact value of $\int_0^5 \frac{x}{\sqrt{x+4}}\,dx$.

15 **a** Write $\frac{1}{u(u+1)}$ in partial fractions.

b Use the substitution $u = e^x$ to find $\int \frac{1}{e^x+1}\,dx$.

16 Use the substitution $u = e^x - 1$ to find the exact value of $\int_{\ln 3}^{\ln 5} \frac{e^{2x}}{e^x - 1}\,dx$.

17 Use the substitution $u = \ln x$ to find $\int \frac{(\ln x)^2}{x}\,dx$.

18 Use the substitution $u = \tan x$ to find $\int \sec^4 x\,dx$.

19 Use the substitution $x = \ln(\sec u)$ to find $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$.

20 Use the substitution $u = \sin x$ to find the exact value of

$$\int_0^{\frac{\pi}{6}} \frac{3\cos x}{10 - \cos^2 x}\,dx.$$

Give your answer in the form $\arctan\left(\frac{p}{q}\right)$, where p and q are integers.

21 Use the substitution $u = 1 + e^x$ to find $\int \frac{1}{1+e^x}\,dx$.

22 Use a substitution of the form $x = a \sin u$ to find $\int \frac{1}{\sqrt{8-25x^2}}\,dx$.

23 Use the substitution $x = \sin u$ to evaluate $\int_0^1 \sqrt{1-x^2}\,dx$.

24 **a** Given that $x = e^u - e^{-u}$, find an expression for e^u in terms of x.

b Use the substitution $x = e^u - e^{-u}$ to find the exact value of $\int_0^1 \frac{1}{\sqrt{4+x^2}}\,dx$.

Tip

The integral in question 17 can also be done by reverse chain rule, because $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Tip

Also try doing the integral in question 18 by writing $\sec^4 x = \sec^2 x \sec^2 x$ and using the identity $\sec^2 x = 1 + \tan^2 x$.

10H Integration by parts

You have already seen that many products of two functions can be integrated using a substitution or the reverse chain rule. However, those techniques are often not useful for integrals where the two functions seem completely unrelated, for example, $\int x \sin x\,dx$ or $\int x^2 e^x\,dx$. In such cases you can sometimes use **integration by parts**, which is the reverse of the product rule.

KEY POINT 10.8

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Proof 10.6

Prove the integration by parts formula.

Start from the product rule for differentiation: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

Integrate both sides: Integrating:

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

Rearrange: So,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

WORKED EXAMPLE 10.26

Find $\int x\sin x\,dx$.

To use integration by parts, you need to choose which part is u and which part is $\frac{dv}{dx}$: Let $u = x$ and $\frac{dv}{dx} = \sin x$

Then $\frac{du}{dx} = 1$ and $v = -\cos x$

Since differentiating x gives a constant, choosing $u = x$ seems more likely to simplify the integral

Use the integration by parts formula: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int x\sin x\,dx = x(-\cos x) - \int -\cos x \bullet 1\,dx$$

Make sure you get the minus signs right:

$$= -x\cos x + \int \cos x\,dx$$

Do not forget $+c$!

$$= -x\cos x + \sin x + c$$

Tip

Notice that $+c$ is not required when finding v from $\frac{dv}{dx}$, but is required in the final answer.

For most integrals of the form $\int x^n\,f(x)dx$, taking $u = x^n$ in integration by parts leads to a simpler integral, because $\frac{du}{dx}$ has a lower power of x. However, this does not work if you do not know how to integrate $f(x)$. A typical example is when $f(x) = \ln x$. The following example also shows you how to deal with the limits in a definite integral efficiently.

WORKED EXAMPLE 10.27

Use integration by parts to evaluate $\int_1^e x^3 \ln x \, dx$.

You may not know how to integrate $\ln x$, but you can integrate x^3 and differentiate $\ln x$

Let $u = \ln x$ and $\frac{dv}{dx} = x^3$.

Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{4}x^4$.

Use the integration by parts formula:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_1^e x^3 \ln x \, dx = \left[\ln x \frac{1}{4}x^4\right]_1^e - \int_1^e \left(\frac{1}{4}x^4\right)\frac{1}{x}\,dx$$

Apply the limits to the first part and simplify the integrand

$$= \left(\frac{1}{4}e^4 \ln e - \frac{1}{4}\ln 1\right) - \int_1^e \frac{1}{4}x^3\,dx$$

$\ln e = 1$ and $\ln 1 = 0$

$$= \frac{1}{4}e^4 - \left[\frac{1}{16}x^4\right]_1^e$$

$$= \frac{1}{4}e^4 - \left(\frac{1}{16}e^4 - \frac{1}{16}\right)$$

$$= \frac{3}{16}e^4 + \frac{1}{16}$$

Repeated integration by parts

It may be necessary to use integration by parts more than once. As long as the integrals are becoming simpler each time, you are on the right track.

WORKED EXAMPLE 10.28

Find $\int x^2 e^x dx$.

Differentiating x^2 decreases the power, so take $u = x^2$ to make the integral simpler

Let $u = x^2$ and $\frac{dv}{dx} = e^x$

Then $\frac{du}{dx} = 2x$ and $v = e^x$

Use the integration by parts formula:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x \; e^x \, dx$$

The remaining integral is a product, so use integration by parts again

Let $u = 2x$ and $\frac{dv}{dx} = e^x$

Then $\frac{du}{dx} = 2$ and $v = e^x$

So,

$$\int 2xe^x \, dx = 2xe^x - \int e^x \cdot 1 dx$$

$$= 2xe^x - e^x + c$$

Substitute this into the main calculation above and simplify. Remember that there was a minus sign in front of the second integral

Hence,

$$\int x^2 e^x dx = x^2 e^x - (2xe^x - e^x + c)$$

$$= x^2 e^x - 2x \; e^x + e^x + c$$

Sometimes it may look like the integration by parts is not making the integral any simpler. However, in some cases applying integration by parts twice still enables you to get to the answer.

Tip

Notice that taking $u = \mathrm{e}^x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$ also works. In either case though, it is important to be consistent when doing the second integration by parts – in this case, you would need $u = \mathrm{e}^x$ on the second application as well.

WORKED EXAMPLE 10.29

Find $\int \mathrm{e}^x \sin x \,\mathrm{d}x$.

This is a product, so integration by parts might be helpful. e^x has the obvious integral e^x, so try $u = \sin x$

Let $u = \sin x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$.

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$ and $v = \mathrm{e}^x$

Use the formula:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \sin x \times \mathrm{e}^x - \int \mathrm{e}^x \cos x \,\mathrm{d}x$$

The second integral is of a similar type as the original one, so repeat the process, this time with $u = \cos x$

Let $u = \cos x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ and $v = \mathrm{e}^x$

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x = \mathrm{e}^x \cos x - \int \mathrm{e}^x(-\sin x)\,\mathrm{d}x$$

$$= \mathrm{e}^x \cos x + \int \mathrm{e}^x \sin x \,\mathrm{d}x$$

We have obtained the original integral $\int \mathrm{e}^x \sin x \,\mathrm{d}x$!

This look like we are going around in circles. However, substituting the last line back enables us to continue

Hence,

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x \,\mathrm{d}x$$

The required integral appears on both sides of the equation, but it has a minus sign on the right. This means that we can rearrange the last equation to make the required integral the subject

Adding $\int \mathrm{e}^x \sin x \,\mathrm{d}x$ to both sides:

$$2\int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x$$

Do not forget to add the constant at the end

And so,

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \frac{1}{2}(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x) + c$$

Exercise 10H

For questions 1 to 3, use the technique demonstrated in Worked Example 10.25 to find the following integrals.

1 a $\int x\cos 2x\,dx$

b $\int x\cos 3x\,dx$

2 a $\int x\sin\left(\frac{x}{2}\right)dx$

b $\int x\sin\left(\frac{x}{3}\right)dx$

3 a $\int xe^{-2x}dx$

b $\int xe^{-3x}dx$

For questions 4 to 6, use the technique demonstrated in Worked Example 10.26 to find the following integrals.

4 a $\int x\ln x\,dx$

b $\int x^2\ln x\,dx$

5 a $\int \frac{1}{x^2}\ln x\,dx$

b $\int \frac{1}{x^3}\ln x\,dx$

6 a $\int \sqrt{x}\ln x\,dx$

b $\int \frac{1}{\sqrt{x}}\ln x\,dx$

For questions 7 to 9, use integration by parts twice, as demonstrated in Worked Example 10.27, to find the following integrals.

7 a $\int x^2e^{3x}dx$

b $\int x^2e^{-2x}dx$

8 a $\int x^2\sin 2x\,dx$

b $\int x^2\sin 3x\,dx$

9 a $\int x^2\cos\left(\frac{x}{3}\right)dx$

b $\int x^2\cos\left(\frac{x}{2}\right)dx$

10 Use integration by parts to find the exact value of $\int_0^1 xe^{2x}\,dx$.

11 Use integration by parts to evaluate $\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

12 Use integration by parts to find $\int 2xe^{-3x}dx$.

13 Use integration by parts to evaluate $\int_1^e x^5\ln x\,dx$.

14 Using integration by parts, show that $\int_1^2 x^2\ln 2x\,dx = \frac{8}{3}\ln 2$

15 Evaluate $\int_1^4 \frac{\ln x}{\sqrt{x}}dx$.

16 Find $\int x^2e^{-x}\,dx$.

17 Evaluate $\int_0^{\pi} x^2\sin x\,dx$.

18 Find $\int (2x+1)\ln x\,dx$.

19 a Write down $\int \sec^2 2x\,dx$.

b Hence find $\int x\sec^2 2x\,dx$.

20 a Show that $\int \frac{x^2}{1+x^2}dx = 1 - \arctan x + c$.

b Hence find $\int x\arctan x\,dx$.

21 a Differentiate $\ln(\sec x)$.

b Hence find $\int_0^{\frac{\pi}{4}} \sin x\ln(\sec x)dx$.

22 Let $I = \int e^x\cos x\,dx$ and $J = \int e^x\sin x\,dx$.

a Use integration by parts to show that $I = e^x\cos x + J$ and find a similar expression for J in terms of I.

b Hence find $\int e^x\cos x\,dx$.

23 By writing $\ln x$ as $1 \bullet \ln x$, use integration by parts to find $\int \ln x\,dx$.

24 Use integration by parts to show that $\int (\ln x)^2\,dx = x((\ln x)^2 - 2\ln x + 2) + c$.

25 Use integration by parts to show that $\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(x^2+1) + c$.

26 Find $\int e^{3x}\sin 2x\,dx$.

27 Find $\int \cos 3x e^{-x}\,dx$.

28 Let $I_n = \int x^n e^x\,dx$.

a Use integration by parts once to show that $I_n = x^n e^x - nI_{n+1}$.

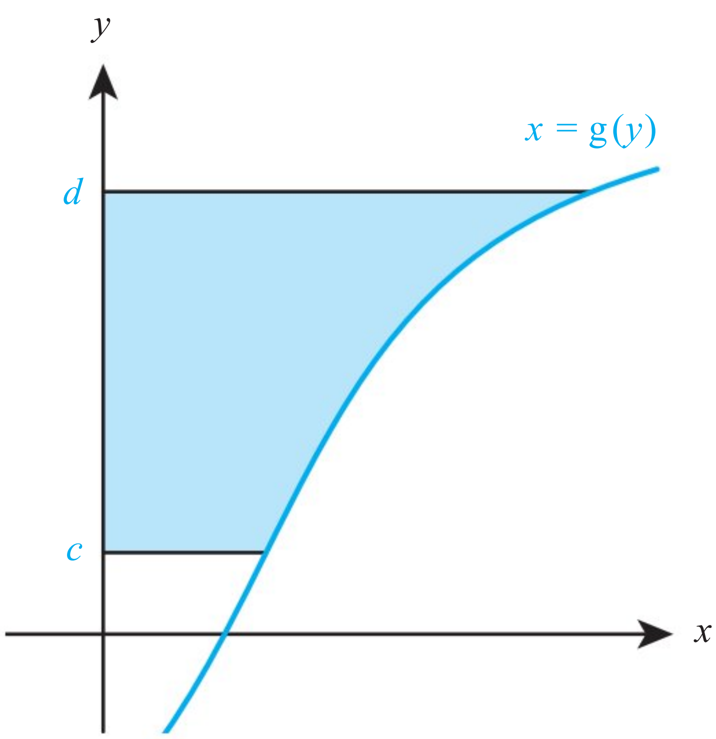
b Hence evaluate $\int_0^1 x^3 e^x\,dx$.

10I Further geometric interpretation of integrals

Area of the region enclosed by a curve and the y-axis

You already know that the area between a curve and the x-axis is given by $\int_a^b f(x)\,dx$.

The diagram below shows the area between a curve with equation $y = f(x)$ and the y-axis.



If you imagine swapping the x and the y axes, you can find the area by using integration as before. However, the equation needs to be for x in terms of y and the limits needs to be the y-values.

KEY POINT 10.9

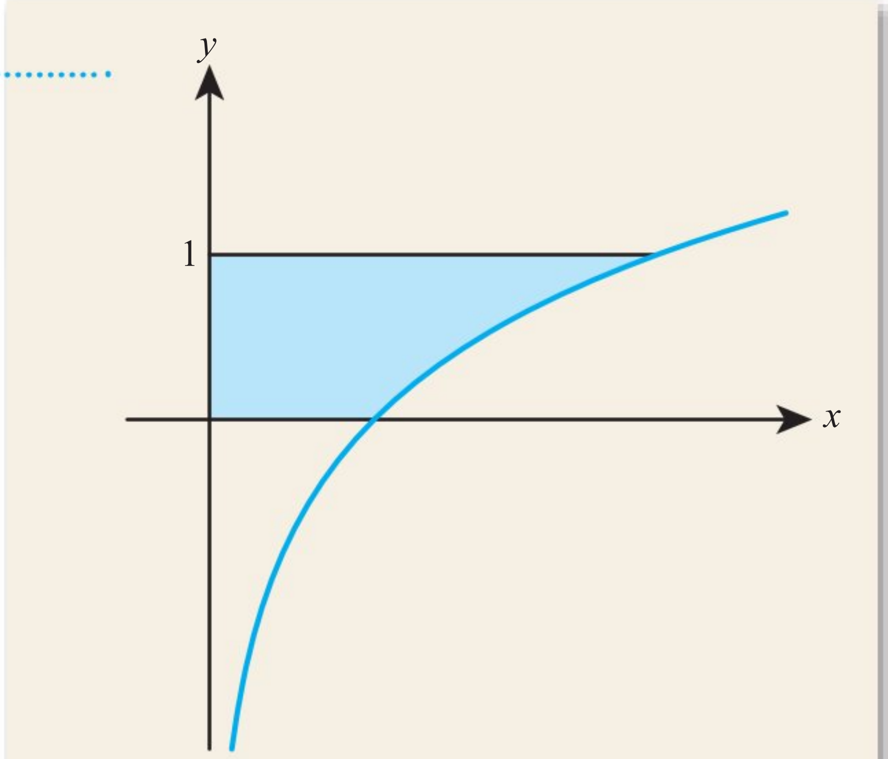
The area bounded by a curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by

$$\int_c^d g(y)\,dy$$

WORKED EXAMPLE 10.30

Find the area enclosed by the curve $y = \ln x$, the y-axis and the lines $y = 0$ and $y = 1$.

A sketch is always helpful to make sure that you are finding the correct area



Rearrange the equation to express x in terms of y: $y = \ln x$ so $x = e^y$

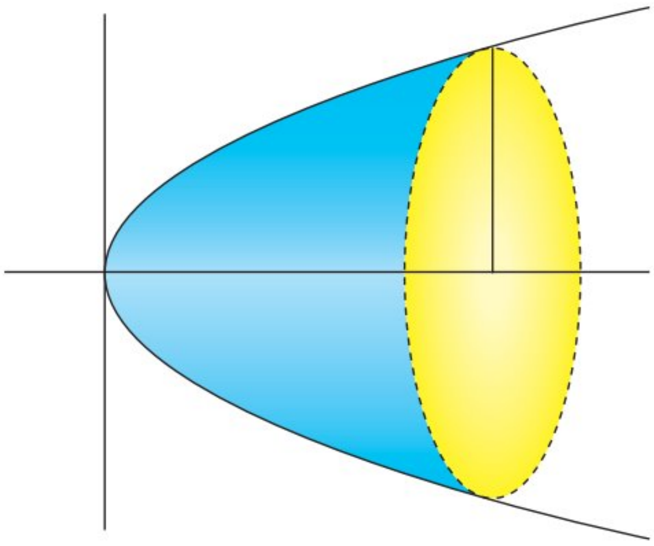
Area $= \int_c^d g(y)\,dy$. The limits for y are given in the question: Area $= \int_0^1 e^y\,dy$

Evaluate the integral using your GDC: $= 1.72$ (3 s.f.)

Volumes of revolution about the x-axis or y-axis

The objects in Figure 10.1 on page 294 are examples of solids of revolution.

When a part of a curve is rotated 360° about the x-axis (or the y-axis) it forms a shape known as a **solid of revolution**. The volume of this solid is a **volume of revolution**.



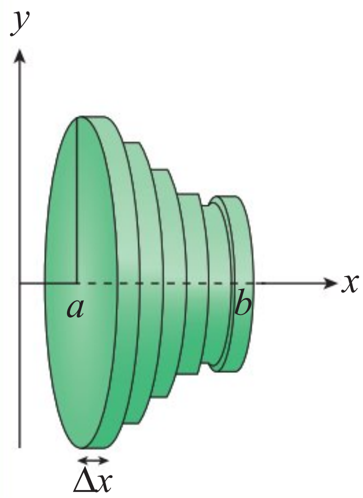
KEY POINT 10.10

The volume of revolution formed when the part of the curve $y = f(x)$, between $x = a$ and $x = b$, is rotated around the x-axis is given by $V = \int_a^b \pi y^2\,dx$.

The proof of this result is based on the same idea as calculating the area under a curve: split up the volume into lots of small parts and add them up.

TOOLKIT: Proof

Prove that the volume of revolution when $y = f(x)$ (for $a < x < b$) is rotated around the x axis is given by $\int_a^b \pi y^2 \, dx$.

The volume can be approximated by a sum of small cylinders. It is useful to sketch a diagram to illustrate this.	The volume can be split up into small cylinders each of length Δx: 
The volume of each cylinder is given by area of the cross-section × length The length of each cylinder is Δx. The radius is equal to the y-coordinate of a point on the curve.	The volume of each cylinder is $\pi y^2 \Delta x$
You can now add up the cylinders to approximate the volume of revolution.	The total volume is approximately: $V \approx \sum_a^b \pi y^2 \Delta x$
The approximation becomes more accurate as Δx gets smaller. In the limit when $\Delta x \to 0$, the sum becomes an integral.	$V = \lim_{\Delta x \to 0} \sum_a^b \pi y^2 \Delta x$ $= \int_a^b \pi y^2 \, dx$

WORKED EXAMPLE 10.31

Find the volume of revolution when the curve $y = x^3$, $0 < x < 2$, is rotated around the x-axis. Give your answer in terms of π.

The volume is given by $\int_0^2 \pi y^2 \, dx$

$$V = \int_0^2 \pi (x^3)^2 \, dx$$

$$= \pi \int_0^2 x^6 \, dx$$

Evaluate the integral on your GDC, then multiply by π

$$= \frac{128}{7}\pi$$

You are the Researcher

There are also formulae to find the surface area of a solid formed by rotating a region around an axis. Some particularly interesting examples arise if we allow one end of the region to tend to infinity. For example, rotating the region formed by the lines $y = \frac{1}{x}$, $x = 1$ and the x-axis results in a solid called the Gabriel's Horn, or Torricelli's trumpet, which has a finite volume but infinite surface area!

When a curve is rotated around the y-axis, you can obtain the formula for the resulting volume of revolution simply by swapping x and y.

KEY POINT 10.11

The volume of revolution formed when the part of the curve $y = \mathrm{f}(x)$, between $y = c$ and $y = d$, is rotated around the y-axis is given by $V = \int_c^d \pi x^2 \, \mathrm{d}y$.

Notice that, to use this formula, you need to write x in terms of y and find the limits on the y-axis.

WORKED EXAMPLE 10.32

Find the volume of revolution when the curve $y = x^2 - 1$, $1 < x < 5$, is rotated around the y-axis. Give your answer in terms of π.

You need to express x^2 in terms of y → $y = x^2 - 1$, so $x^2 = y + 1$

Find the limits for y → Limits:

when $x = 1$, $y = 0$

when $x = 5$, $y = 24$

The volume is given by $\int_0^{24} \pi x^2 \, \mathrm{d}y$ → $V = \int_0^{24} \pi(y+1) \, \mathrm{d}y$

Evaluate the integral on GDC → $= 312\pi$

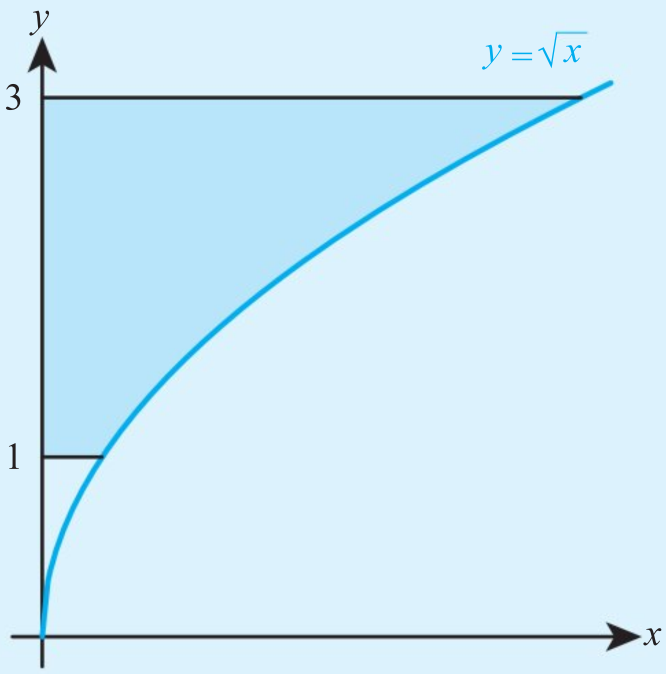
You are the Researcher

An alternative formula for the volume of revolution when $y = \mathrm{f}(x)$ $a < x < b$ is rotated around the y axis is given by $\int_a^b 2\pi xy \, \mathrm{d}x$. Can you justify this and find any applications?

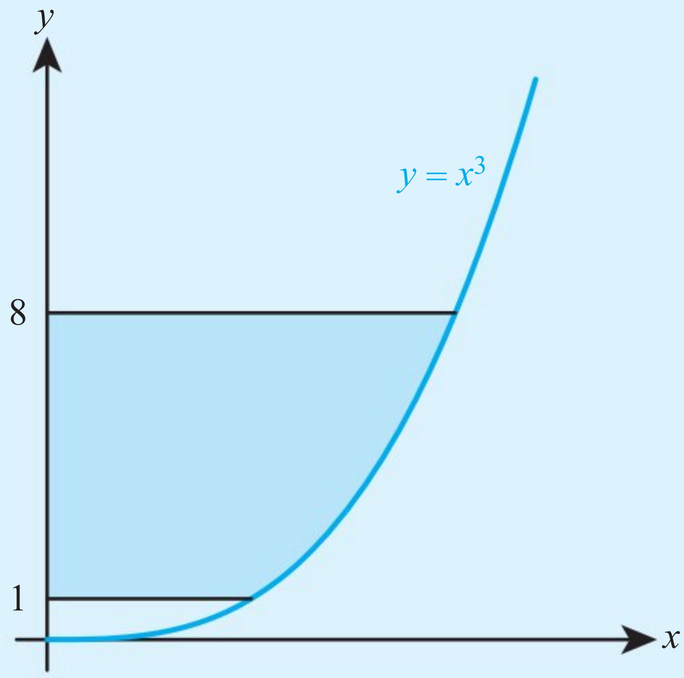
Exercise 10I

For questions 1 to 4, use the technique demonstrated in Worked Example 10.30 to find the area between the given curve, the y-axis and the lines $y = c$ and $y = d$.

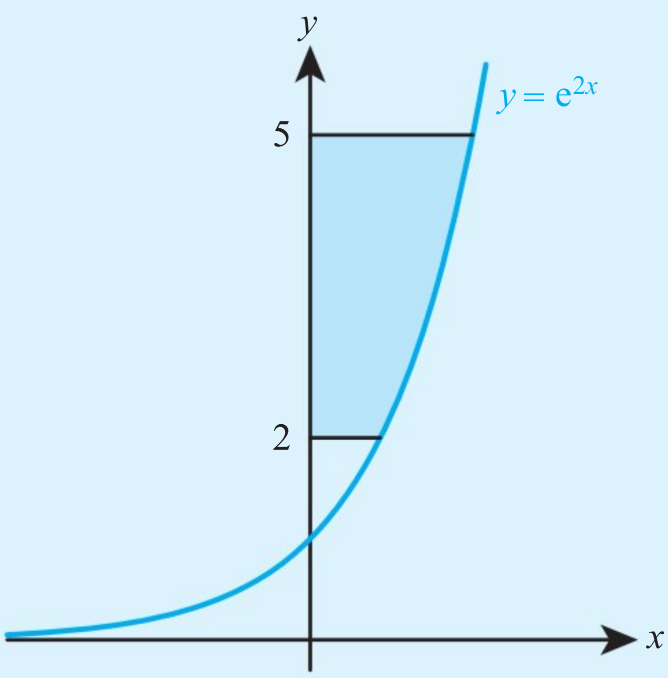
1 a



b



2 a



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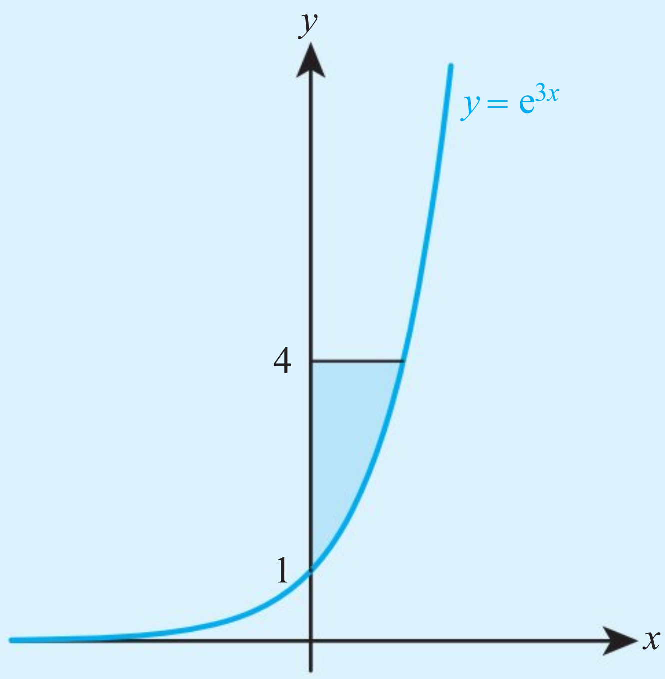
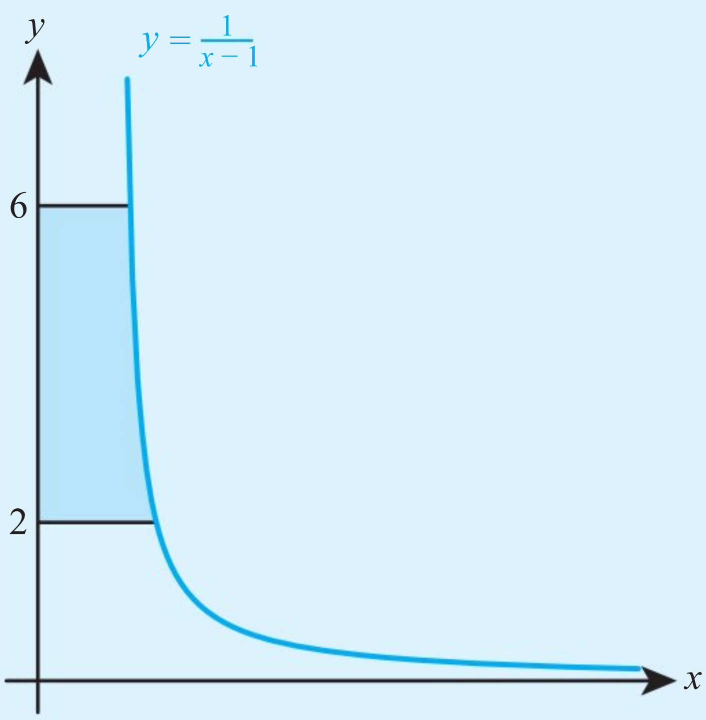
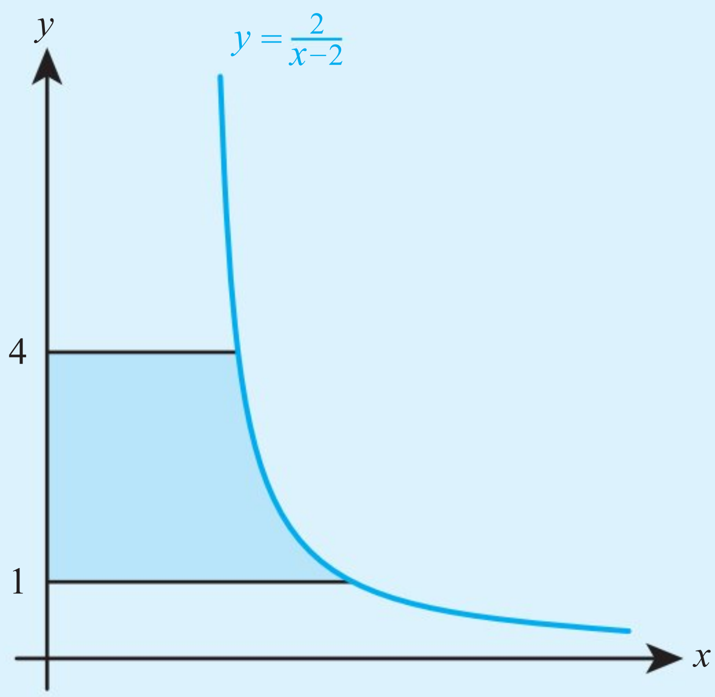


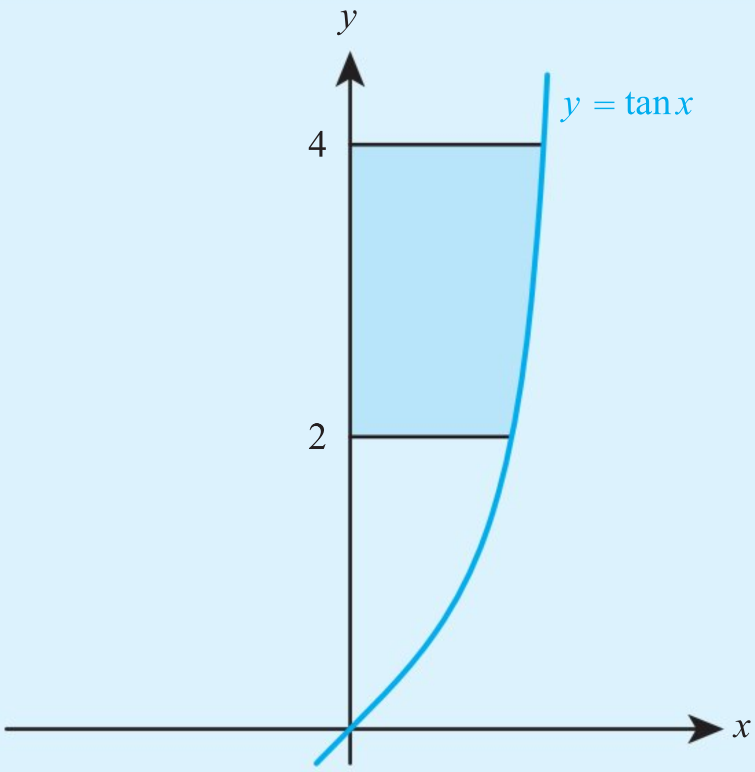
3 a



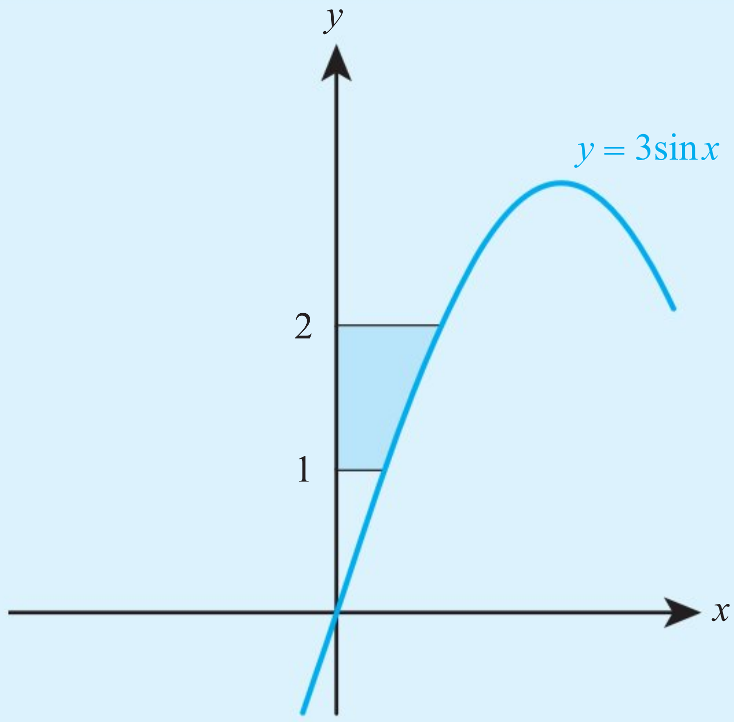
b



4 **a**



b



For questions 5 to 8, use the technique demonstrated in Worked Example 10.31 to find the volume of revolution formed when the given part of the curve is rotated 360° about the x-axis. Give your answer to 3 significant figures.

5 **a** $y = 3x^2$ between $x = 0$ and $x = 3$

b $y = 2x^3$ between $x = 0$ and $x = 2$

6 **a** $y = x^2 + 3$ between $x = 1$ and $x = 2$

b $y = x^2 - 1$ between $x = 2$ and $x = 4$

7 **a** $y = e^{2x}$ between $x = 0$ and $x = \ln 2$

b $y = e^{3x}$ between $x = 0$ and $x = \ln 2$

8 **a** $y = \frac{2}{x+1}$ between $x = 1$ and $x = 3$

b $y = \frac{3}{x+2}$ between $x = 0$ and $x = 2$

For questions 9 to 12, use the technique demonstrated in Worked Example 10.32 to find the volume of revolution formed when the given part of the curve $y = g(x)$, for $x < a < b$, is rotated 360° about the y-axis.

9 **a** $g(x) = 4x^2 + 1$, $a = 0$, $b = 2$

b $g(x) = \frac{x^2 - 1}{3}$, $a = 1$, $b = 4$

10 **a** $g(x) = \ln x + 1$, $a = 1$, $b = 3$

b $g(x) = \ln(2x - 1)$, $a = 1$, $b = 5$

11 **a** $g(x) = \cos x$, $a = 0$, $b = \frac{\pi}{2}$

b $g(x) = \tan x$, $a = 0$, $b = \frac{\pi}{4}$

12 **a** $g(x) = \frac{1}{x-5}$, $a = 6$, $b = 8$

b $g(x) = \frac{1}{x-2}$, $a = 3$, $b = 8$

13 The shaded region in the diagram is bounded by the curve $y = \frac{1}{x}$, the y-axis and the lines $y = 1$ and $y = 5$.

a Find the area of the shaded region.

b Find the volume of the solid generated when the shaded region is rotated about the y-axis.

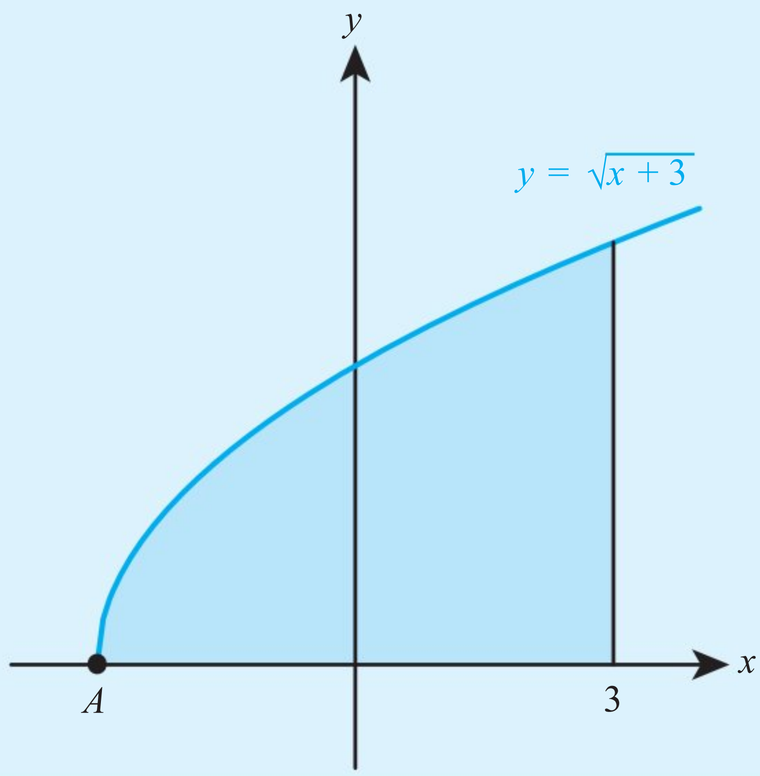
14 The part of the curve with equation $y = \frac{1}{x}$ between $x = 1$ and $x = a$ is rotated 360° about the x-axis. The volume of the resulting solid is $\frac{2\pi}{3}$. Find the value of a.

14 The part of the parabola $y = x^2$ between $x = 0$ and $x = a$ is rotated about the y-axis. The volume of the resulting solid is 8π. Find the value of a.

15 The diagram shows the curve with equation $y = \sqrt{x + 3}$.

a Write down the x-coordinates of point A.

b The region bounded by the curve, the x-axis and the line $x = 3$ is rotated completely about the x-axis. Find the volume of the resulting solid.

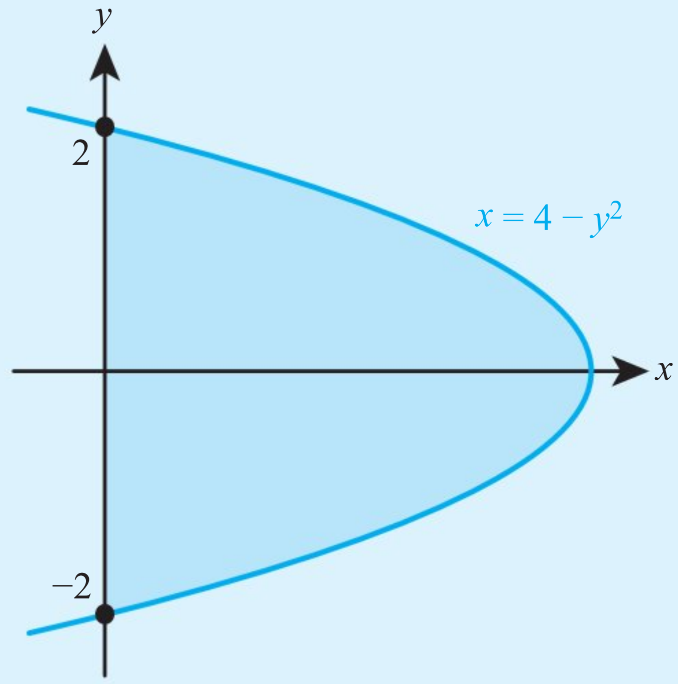


17 The diagram shows the region bounded by the y-axis and the curve with equation $x = 4 - y^2$.

Find

a the area of the region

b the volume of the solid generated when the region is rotated about the y-axis.

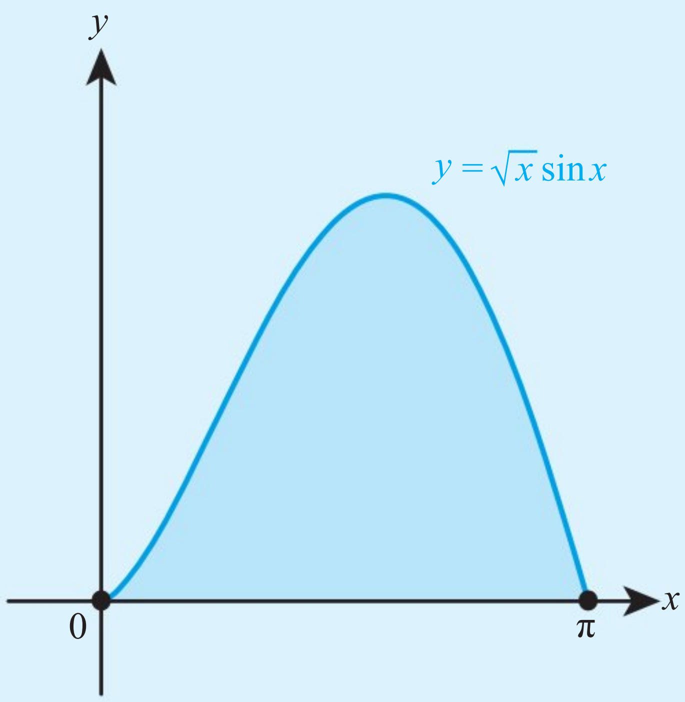


18 The diagram shows the region bounded by the curve $y = \sqrt{x}\sin x$ and the x-axis.

Find

a the area of the region

b the volume of the solid generated when the region is rotated through 2π radians about the x-axis.



19 The curve in the diagram has equation $y = \sqrt{x^3 + 9}$, which intersects the coordinate axes at the points $A\left(-\sqrt[3]{9}, 0\right)$ and $B(0, 3)$. Region R is bounded by the curve, the x-axis and the y-axis.

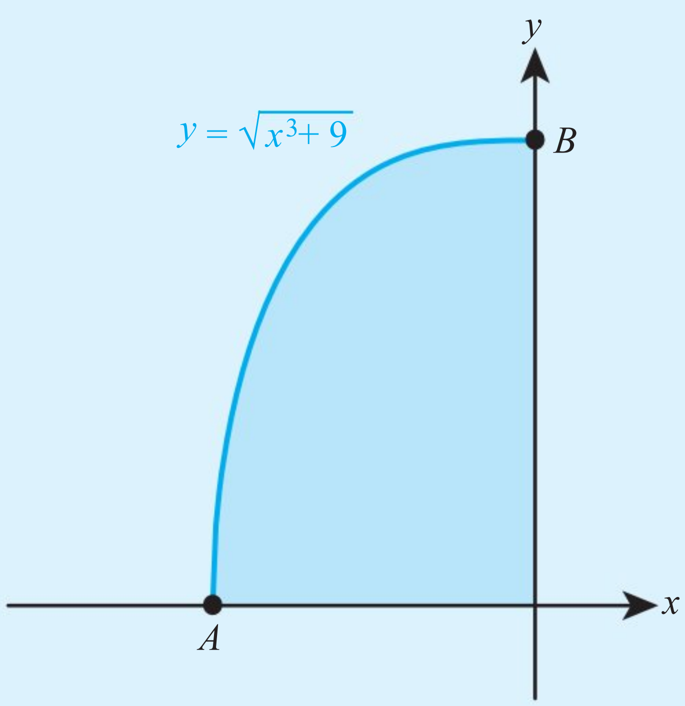
a Show that $x^2 = \sqrt[3]{(y^2 - 9)^2}$.

b Find the area of R.

c Find the volume of revolution generated when R is rotated fully about the

i x-axis

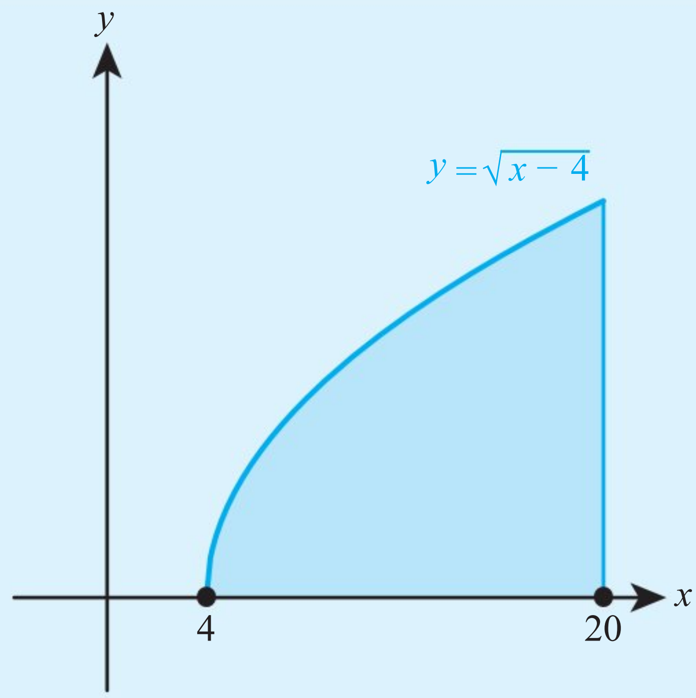
ii y-axis.



20 The part of the curve $y = \frac{1}{x}$, between $x = 1$ and $x = 3$, is rotated about the y-axis. Find the volume of the resulting solid.

21 The diagram shows the part of the curve $y = \sqrt{x - 4}$ between $x = 4$ and $x = 20$.

The region bounded by the curve, the line $x = 20$ and the x-axis is rotated 360° about the y-axis. Find the resulting volume of revolution. Give your answer to the nearest integer.



22 The part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated around the x-axis. Find the exact value of the volume generated.

23 **a** Sketch the curve with equation $y = \sqrt{x}$.

b The part of the curve between $x = 0$ and $x = 9$ is rotated about the x-axis. Find the volume of the resulting solid.

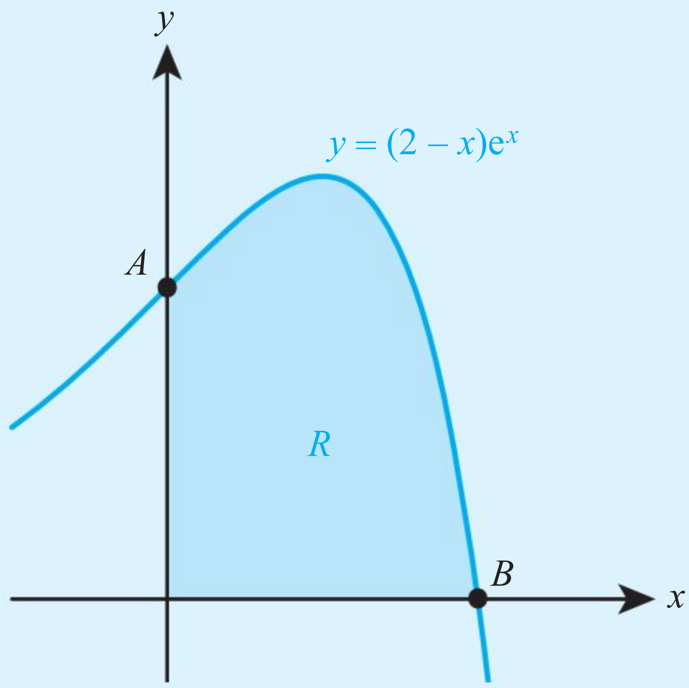
c Find the volume of the solid generated when the same part of the curve is rotated about the y-axis.

24 The diagram shows the graph of $y = (2 - x)e^x$. The region R is bounded by the curve and the coordinate axes.

a Find the coordinates of the points A and B.

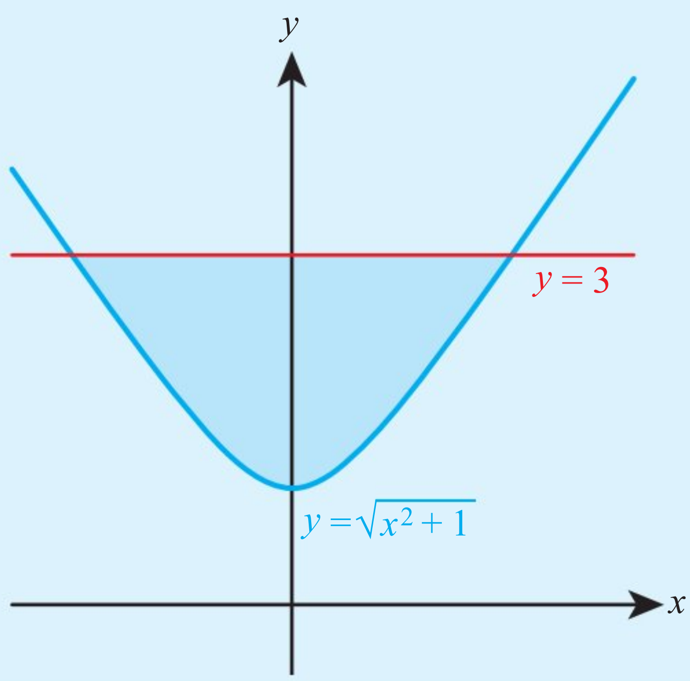
b Find the area of R.

c Find the volume of the solid generated when R is rotated 360° about the x-axis.



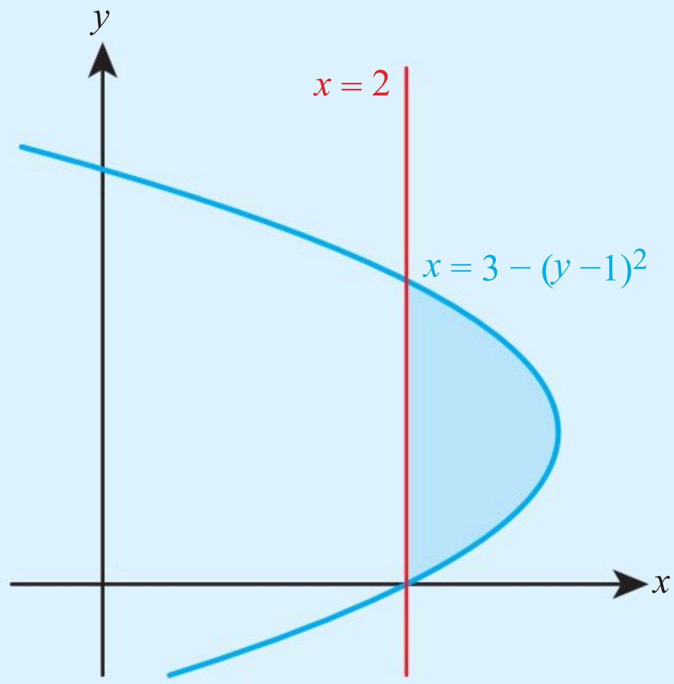
25 The region between the curve $y = \sqrt{x^2 + 1}$ and the line $y = 3$, shown in the diagram, is rotated fully about the y-axis.

Find the volume of the resulting solid, giving your answer as a multiple of π.



26 The diagram shows the region bounded by the curve with equation $x = 3 - (y - 1)^2$ and the line $x = 2$.

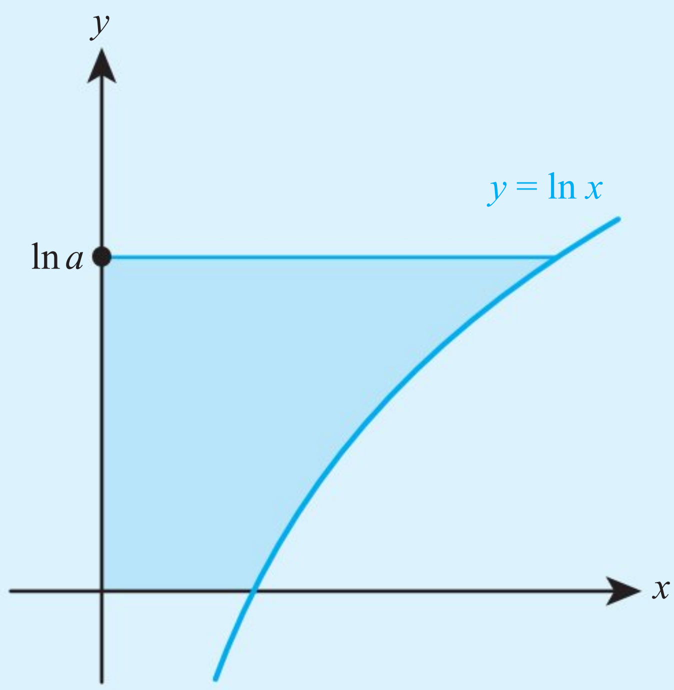
The region is rotated around the y-axis. Find the volume of the resulting solid, giving your answer as a multiple of π.



27 The diagram shows the region bounded by the graph of $y = \ln x$, the coordinates axes and the line $y = \ln a$.

a Find, in terms of a, the area of the shaded region.

b Hence show that $\int_1^a \ln x \, dx = a(\ln a - 1) + 1$.



28 **a** Find the equation of the line passing through the points $(r, 0)$ and $(0, h)$, giving your answer in the form $ax + by = c$.

b By considering the solid formed when a part of this line is rotated about the y-axis, show that the volume of a cone with radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

29 **a** Write down the equation of a circle with centre at the origin and radius r.

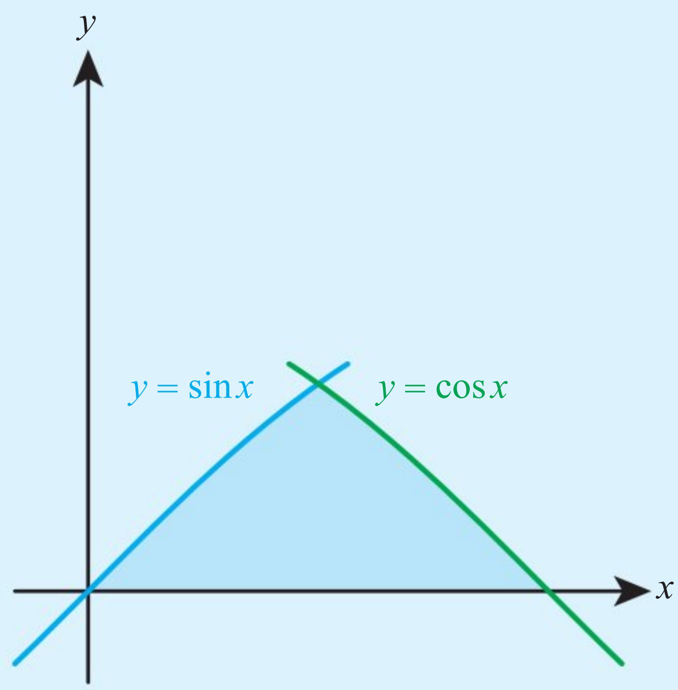
b By considering a suitable solid of revolution, prove that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

30 **a** Find the coordinates of the intersection points of the curves $y = x^2$ and $y = \sqrt{x}$.

b Find the volume of revolution generated when the region between the two curves is rotated about the x-axis.

31 The region shown in the diagram is bounded by the x-axis and the curves $y = \sin x$ and $y = \cos x$.

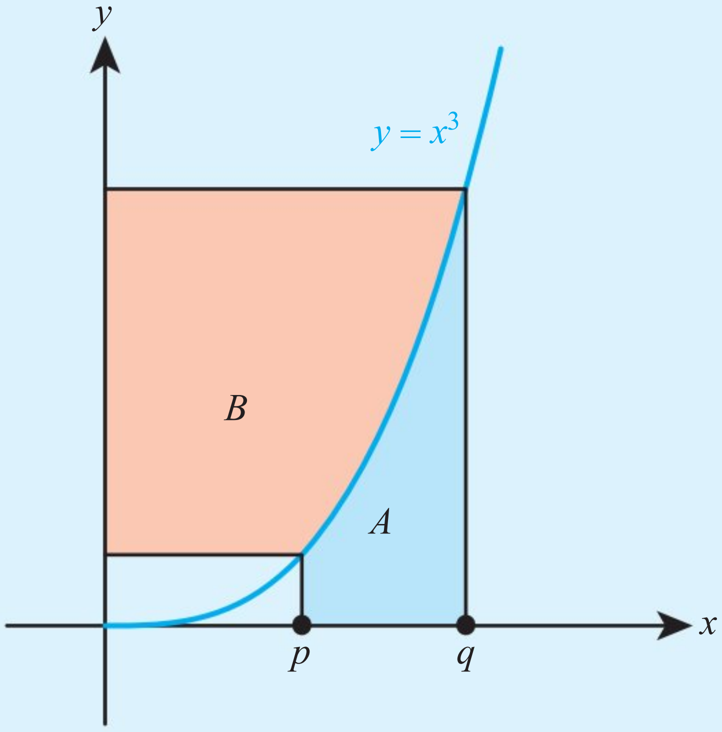
Find the exact volume of the solid generated when the region is rotated around the x-axis.



32 **a** Find the points of intersection of the curve $y = x^2$ and the line $y = 2x$.

b The region bounded by the line and the curve is rotated fully about the y-axis. Find the volume of the resulting solid.

33 The diagram shows the graph of $y = x^3$ and two regions, A and B.



Show that the ratio (area of A) : (area of B) is independent of p and q, and find the value of this ratio.

34 **a** The graph of $y = \ln x$ is translated 2 units to the right. Write down the equation of the resulting curve.

b Hence find the exact volume generated when the region bounded by $x = 1$, $y = 1$ and the curve $y = \ln x$, for $1 \leqslant x \leqslant e$, is rotated around the line $x = -2$.

35 **a** Sketch the graph of $y = \cos x$ for $-\pi \leqslant x \leqslant \pi$.

The curve from part **a** is rotated through 2π radians about the line $y = -1$.

b Show that the volume of the resulting solid is given by

$$\frac{1}{2}\pi\int_{-\pi}^{\pi}(\cos 2x + 4\cos x + 3)\,dx$$

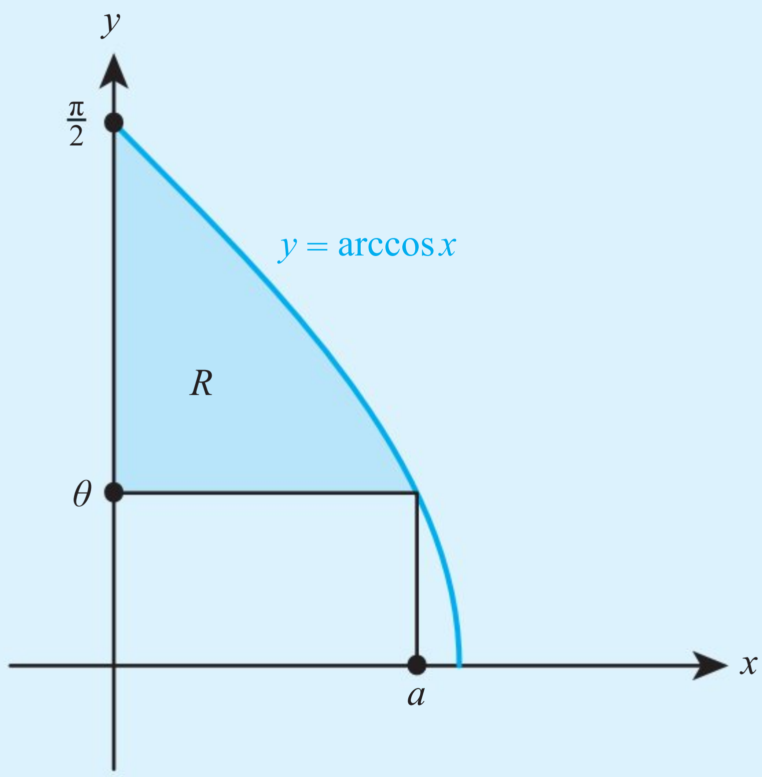
c Find the exact value of the volume.

36 The diagram shows the graph of $y = \arccos x$ for $x > 0$. The region R is bounded by the curve, the y-axis and the line $y = \theta$.

a Find, in terms of θ, the area of R.

b Write down an expression for $\sin\theta$ in terms of a.

c Hence show that $\int_0^a \arccos x\,dx = 1 + a\arccos a - \sqrt{1 - a^2}$.



Checklist

- You should be able to use differentiation from first principles:
  - $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- You should be able to use L'Hôpital's rule:
  - If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
  - If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
- You should be able to use implicit differentiation:
  - $\frac{d}{dx} g(y) = g'(y)\frac{dy}{dx}$
- You should be able to use the results given in your formula booklet to differentiate and integrate various functions.
  - Standard integrals can be combined with a linear substitution:
    - — If $\int f(x)\,dx = F(x)$, then $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$.
  - In particular,
    - — $\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,dx = \arcsin\left(\frac{x+b}{a}\right) + c$
    - — $\int \frac{1}{a^2+(x+b)^2}\,dx = \frac{1}{a}\arctan\left(\frac{x+b}{a}\right) + c$
- You should be able to use integration by substitution:
  - Differentiate the substitution to express dx in terms of du.
  - Replace all occurrences of x by the relevant expression in terms of u.
  - Change the limits from x to u.
  - Simplify as much as possible before integrating.
- You should be able to use integration by parts:
  - $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
- You should know that the area bounded by a curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d g(y)\,dy$.
- You should know how to find volumes of revolution:
  - The volume of revolution formed when the part of the curve $y = f(x)$, between $x = a$ and $x = b$, is rotated around the x-axis is given by $V = \int_a^b \pi y^2\,dx$.
  - The volume of revolution formed when the part of the curve $y = f(x)$, between $y = c$ and $y = d$, is rotated around the y-axis is given by $V = \int_c^d \pi x^2\,dy$.

Mixed Practice

1 Find the gradient of the curve $y = \arcsin(3x)$ at the point where $x = \frac{1}{6}$.

2 Evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sec^2(2x)\,dx$.

3 Given that $y = \sec x$,

a find $\frac{d^2y}{dy^2}$

b show that the graph of $y = \sec x$ has no points of inflection.

4 Given that $y = 10^x$, find $\frac{d^3y}{dx^3}$.

5 The part of the graph of $y = \ln x$ between $x = 1$ and $x = 2e$ is rotated 360° around the x-axis. Find the volume generated.

6 A curve has equation $2x^3 - 5y^3 = 11$. Find the gradient of the curve at the point (2, 1).

7 Use the substitution $3x = u$ to find $\int \frac{6}{1 + 9x^2}\,dx$.

8 The side of a cube is increasing at a rate of $0.6\,\text{cm s}^{-1}$. Find the rate of increase of the volume of the cube when the side length is 12 cm.

9 Given that $y = 3\sin(2\pi x)$, find the rate of change of y when $x = \frac{7}{12}$.

10 Find the maximum value of x^2e^{-x} for $-3 \leqslant x \leqslant 3$.

11 A cuboid has a square base of side a cm and height h cm. The volume of the cuboid is $1000\,\text{cm}^3$.

a Show that the surface area of the cuboid is given by $S = 2a^2 + \frac{4000}{a}$.

b Find the value of a for which $\frac{dS}{da} = 0$.

c Show that this value of a gives the minimum value of the surface area, and find this minimum value.

12 **a** Express $\frac{5 - x}{x^2 - x - 2}$ in partial fractions.

b Hence show that $\int_3^5 \frac{5 - x}{x^2 - x - 2}\,dx = \ln\left(\frac{4}{3}\right)$.

13 Find the equation of the normal to the curve $x^3y^3 - xy = 0$ at the point (1, 1).

Mathematics HL November 2010 Paper 2 Q13

14 Let $y(x) = xe^{3x}$, $x \in \mathbb{R}$.

a Find $\frac{dy}{dx}$.

b Prove by induction that $\frac{d^ny}{dx^n} = n3^{n-1}e^{3x} + x3^ne^{3x}$ for $n \in \mathbb{Z}^+$.

c Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. Justify whether any such point is a point of inflection.

d Find the coordinates of any points of inflection on the graph of $y(x)$. Justify whether any such point is a point of inflection.

e Hence sketch the graph of $y(x)$, indicating clearly the points found in parts **c** and **d** and any intercepts with the axes.

Mathematics HL May 2015 Paper 1 TZ1 Q11

15 Use differentiation from first principles to prove that the derivative of $y = x^3 - 3x$ is $3x^2 - 3$.

16 Given that $y = e^{2x+1}$, prove by induction that $\frac{d^n y}{dx^n} = 2^n e^{2x+1}$.

17 If $y = \frac{1}{1-2x}$, prove that $\frac{d^n y}{dx^n} = \frac{2^n n!}{(1-2x)^{n+1}}$.

18 Prove that if $y = x^2 e^x$, then $\frac{d^n y}{dx^n} = (x^2 + 2nx + n(n-1))e^x$.

19 The function f(x) is defined by

$$f(x) = \begin{cases} 2^x & \text{for } x \leqslant 2 \\ k \times 3^x & \text{for } x > 2 \end{cases}$$

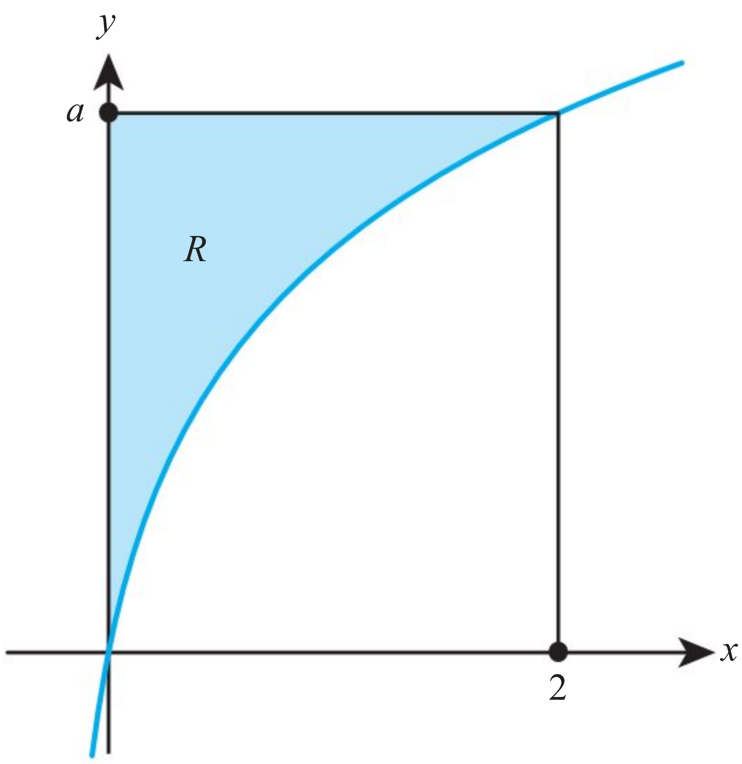
a Find the value of k such that f(x) is continuous.
b For this value of k, determine whether f(x) is differentiable when $x = 2$.

20 Use L'Hôpital's rule to find $\lim_{x \to 0} \frac{3^x - 1}{x}$.

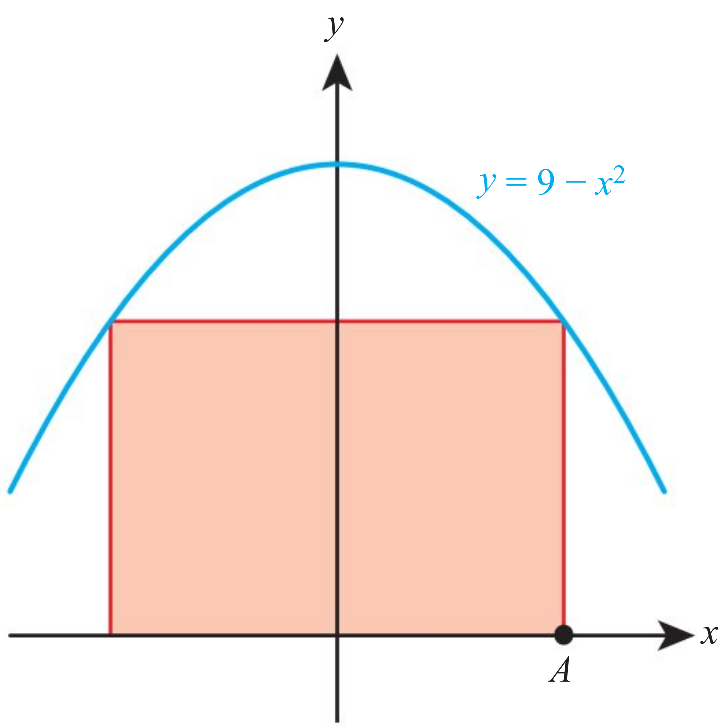
21 Find $\lim_{x \to 0} \frac{\sin x - x}{\tan x - x}$.

22 Show that $\int_0^2 \frac{2x+5}{x^2+4}\,dx = \ln 2 + \frac{5\pi}{8}$.

23 The diagram shows the region R bounded by the curve $y = \ln(5x + 1)$, the y-axis and the line $y = a$.
a Write down the exact value of a.
b Find the area of R.
c The region R is rotated fully about the y-axis. Find the volume of the resulting solid.



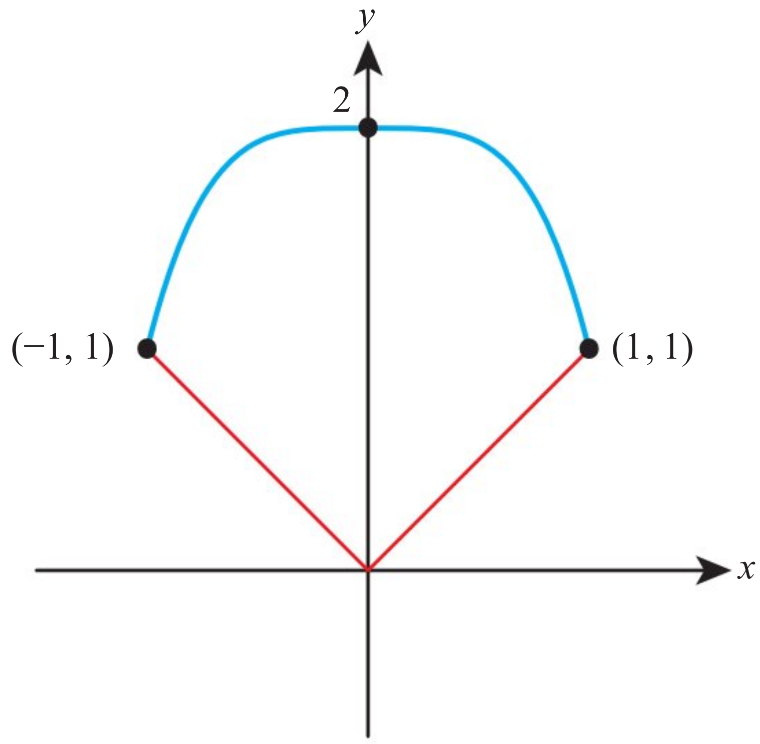
24 A rectangle is drawn inside the finite region bounded by the curve $y = 9 - x^2$ and the x-axis, so that two of the vertices lie on the axis and the other two on the curve.
a Find the coordinates of vertex A so that the area of the rectangle is maximum possible.
b Find the minimum possible area of the rectangle.



25 The part of the curve $y = \ln(x^2)$ between $x = 1$ and $x = e^2$ is rotated 360° around the y-axis. Find the exact value of the resulting volume of revolution.

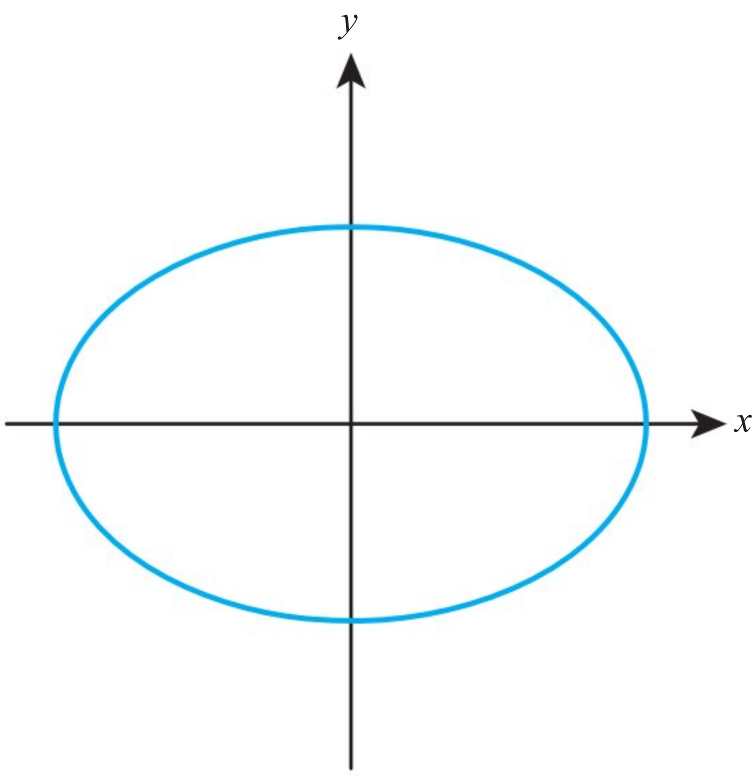
26 The part of the graph $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the resulting volume of revolution.

27 The diagram shows the region bounded by the graphs of $y = 2 - x^4$ and $y = |x|$.



The region is rotated about the y-axis to form a solid of revolution. Find the volume of the solid.

28 The diagram shows an ellipse with equation $4x^2 + 9y^2 = 36$.



Show that the volume generated when the ellipse is rotated around the x-axis is not the same as the volume generated when the ellipse is rotated around the y-axis.

29 Evaluate $\int_1^e x^3 \ln x \, dx$.

30 Use the substitution $u = \sqrt{x+1}$ to find the exact value of $\int_{-1}^{3} \frac{1}{2} e^{\sqrt{x+1}} \, dx$.

31 **a** Use integration by parts to find $\int te^{-t} dt$.

b Hence use the substitution $u = x^2$ to find the exact value of $\int_0^1 2x^3 e^{-x^2} dx$.

32 Use the substitution $u = x^2 + 1$ to find $\int x^3\sqrt{x^2+1}\,dx$.

33 **a** Use a suitable substitution to show that $\int \sqrt{x}e^{\sqrt{x}}\,dx = \int 2u^2e^u\,du$.

b Hence evaluate $\int_0^4 \sqrt{x}e^{\sqrt{x}}\,dx$.

34 Use the substitution $x = \tan\theta$ to show that $\int \frac{1}{1+x^2}\,dx = \arctan x + c$.

35 The function f is defined by $f(x) = \begin{cases} e^{-x^2}(-x^3+2x^2+x), & x \leqslant 1 \\ ax+b, & x>1 \end{cases}$, where a and b are constants.

Find the exact values of a and b if f is continuous and differentiable at $x = 1$.

Mathematics HL May 2014 Paper 3 Q4a

36 The region enclosed between the curves $y = \sqrt{x}e^x$ and $y = e\sqrt{x}$ is rotated through 2π about the x-axis. Find the volume of the solid obtained.

Mathematics HL May 2010 Paper 1 TZ1 Q8

37 **a** Write down the derivative of $\arcsin x$.

b Use integration by parts to find $\int \arcsin x\,dx$.

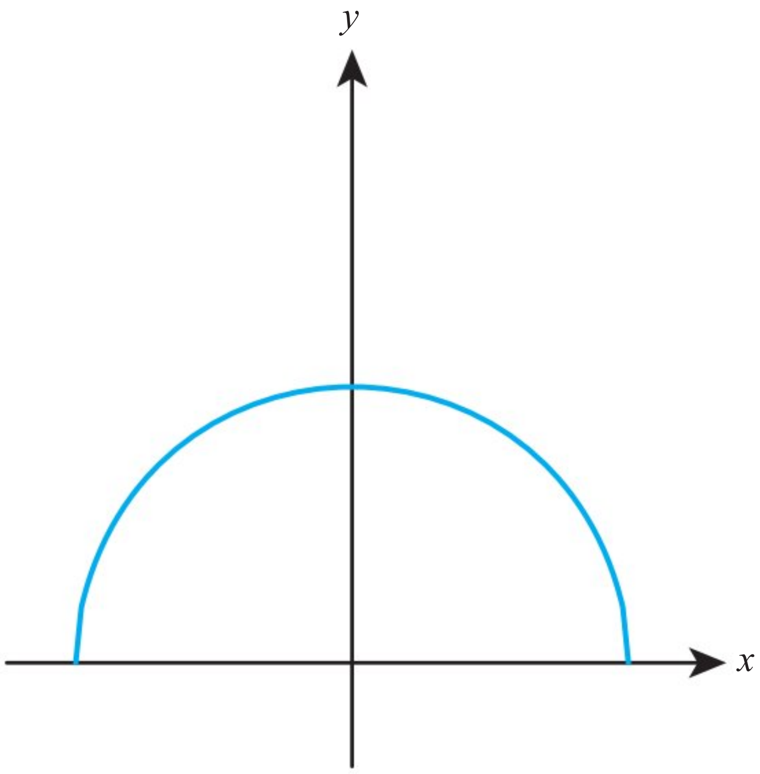
38 **a** Complete the square for $x^2 - 8x + 25$.

b Hence find $\int \frac{2x+7}{x^2-8x+25}\,dx$.

39 A rectangle is inscribed in a semi-circle with radius r and centre at the origin, so that one side of the rectangle lies along the diameter of the semi-circle and the other two vertices lie on its circumference. Show that the maximum possible area of the rectangle is r^2.

40 **a** Use the substitution $x = a\sin\theta$ to find $\int \sqrt{a^2-x^2}\,dx$.

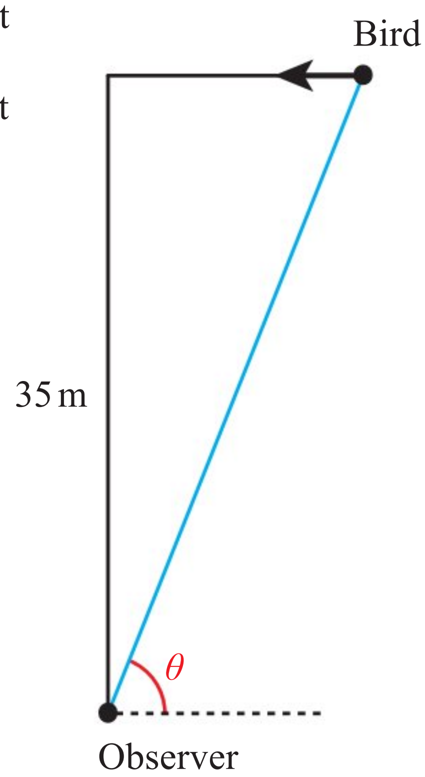
b The diagram shows the semicircle with equation $x^2 + y^2 = 36$.



Use integration to show that the area of the semicircle is 18π.

41 Use the substitution $x = \dfrac{e^u + e^{-u}}{2}$, with $u \geqslant 0$, to evaluate $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2 - 1}}\,dx$.

42 A bird is flying at a constant speed at a constant height of 35 m in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is 1.2 radians and is increasing at $\dfrac{1}{60}$ radians per second. Find the speed, in metres per second, at which the distance between the bird and the observer is decreasing.



43 Find the shortest distance between the point (0, 1) and the curve $y = \sin x$ with x in radians.

44 Find $\displaystyle\lim_{x \to \frac{\pi}{2}} (\tan x - \sec x)$.

45 Find $\displaystyle\lim_{x \to \infty} \left(x \arctan x - \frac{\pi x}{2}\right)$.

46 **a** Show that $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$.

b Prove that $\dfrac{d^n}{dx^n}(\cos x) = \cos\left(x + \dfrac{n\pi}{2}\right)$.

47 Find the stationary point on the curve $e^x + ye^{-x} = 2e^3$.

48 A curve is given by the implicit equation

$x^2 + xy + y^2 = 12$

a Find the coordinates of the stationary points on the curve.

b Show that at the stationary points, $(x + 2y)\dfrac{d^2y}{dx^2} = -2$.

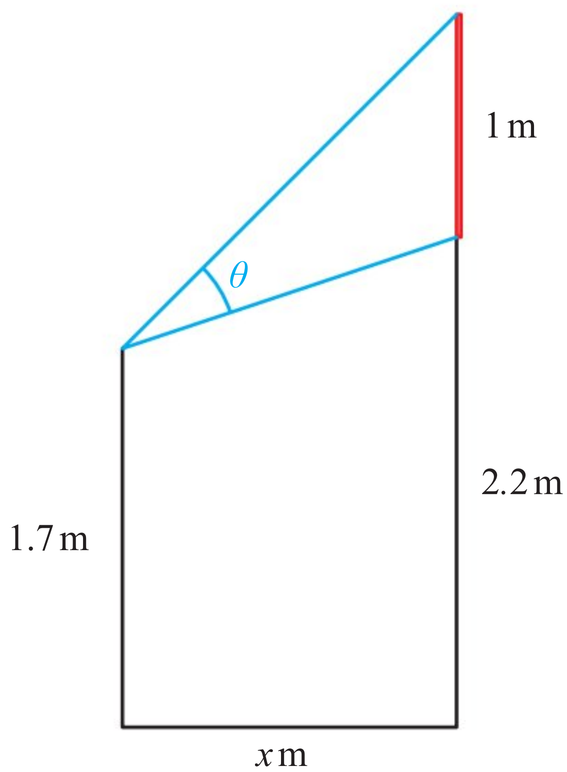
c Hence determine the nature of the stationary points.

49 A painting of height 1 m is hanging on the wall of an art gallery so that the bottom of the painting is 2.2 m above the floor. Alessia is standing so that her eyes are at a height of 1.7 m. She is x m from the wall.

a Show that the angle at which the visitor sees the painting is

$\theta = \arctan\dfrac{1.5}{x} - \arctan\dfrac{0.5}{x}$.

b How far from the wall should Alessia stand so that the painting appears as large as possible? Find the corresponding value of θ.



50 **a** Find the equation of the line passing through points $(a, 0)$ and (b, h), where $a, b, h > 0$.

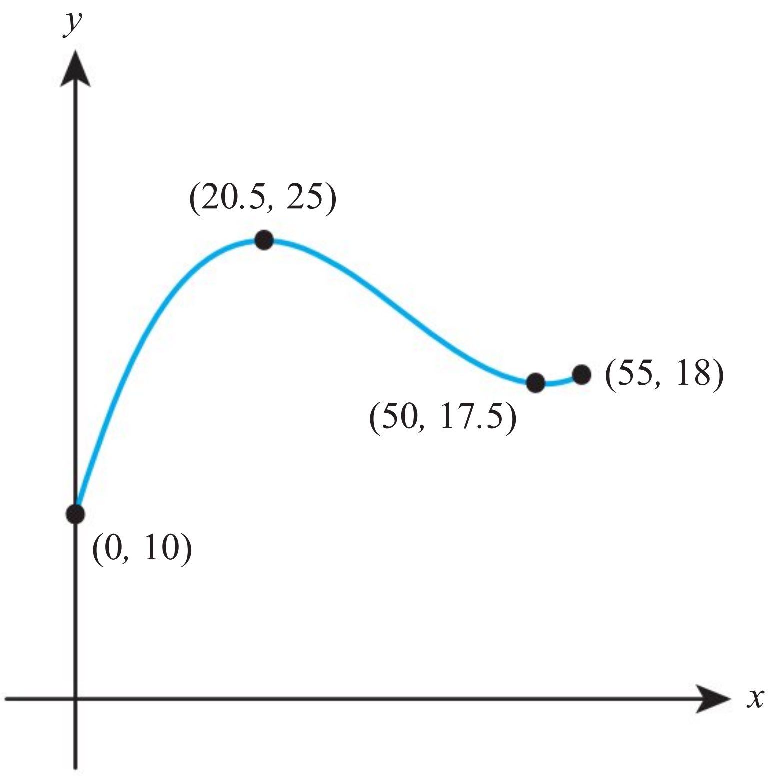
b The part of this line, between $y = 0$ and $y = h$, is rotated about the y-axis. Prove that the volume of the resulting solid is $\dfrac{\pi h}{3}(a^2 + ab + b^2)$.

51 A parabola has equation $y = r^2 - x^2$ and a circle has radius $r > 0$ and centre at the origin.

a Sketch the parts of both curves with $y \geqslant 0$.

b When those parts of the curves are rotated about the y-axis, the resulting volumes are equal. Find the value of r.

52 A large vase can be modelled by a solid revolution formed when the cubic curve, shown in the diagram, is rotated about the x-axis. The units of length are centimetres.



a Find the equation of the curve in the form $y = ax^3 + bx^2 + cx + d$.

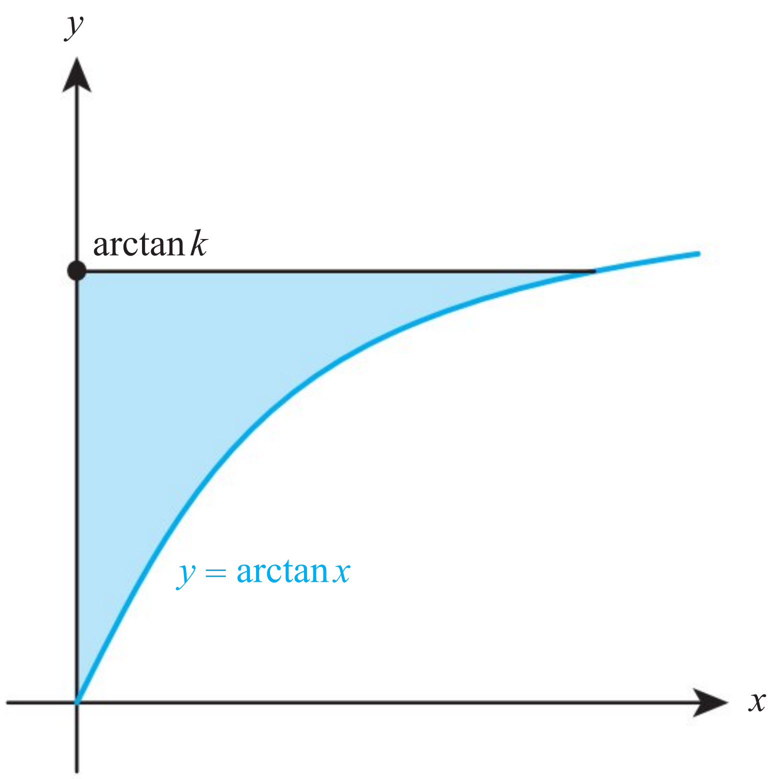
b Hence find the volume of the vase in litres.

53 **a** Given that $\tan\theta = k$, and that $0 < \theta \leqslant \frac{\pi}{2}$, express $\sec\theta$ in terms of k.

The diagram shows the curve with equation $y = \arctan x$ for $x \geqslant 0$. The shaded region is bounded by the curve, the y-axis and the line $y = \arctan k$.

b Show that the area of the shaded region is $\frac{1}{2}\ln(1 + k^2)$.

c Hence find $\int_0^k \arctan x \, dx$.



54 **a** Find all values of x for $0.1 \leqslant x \leqslant 1$ such that $\sin(\pi x^{-1}) = 0$.

b Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2}\sin(\pi x^{-1})\,dx$, showing that it takes different integer values when n is even and when n is odd.

c Evaluate $\int_{0.1}^{1} \left|\pi x^{-2}\sin(\pi x^{-1})\right| dx$.

Mathematics HL May 2013 Paper 1 TZ1 Q10

55 The curve C has equation $2x^2 + y^2 = 18$. Determine the coordinates of the four points on C at which the normal passes through the point $(1, 0)$.

Mathematics HL May 2012 Paper 1 TZ1 Q9

56 The probability density function for the standard normal distribution is given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ for $x \in \mathbb{R}$.

a Explain why this means that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$.

b Use integration to show that, if $Z \sim N(0, 1)$, then $E(Z) = 0$.

c Use integration by parts to show that $\text{Var}(Z) = 1$. You will need to use the result from part **a**.

d Hence explain why, for $X \sim N(\mu, \sigma^2)$, $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$.

11 Series and differential equations

ESSENTIAL UNDERSTANDINGS

- Calculus describes rates of change between two variables and the accumulation of limiting areas.
- Understanding these rates of change and accumulations helps us to model, interpret and analyse real-world problems.
- Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

In this chapter you will learn…

- how to classify differential equations
- how to find approximate solutions to some differential equations
- when and how to solve differential equations by separating variables
- when and how to solve differential equations by using a substitution
- when and how to solve differential equations by using an integrating factor
- how to approximate functions as polynomials, called Maclaurin series
- how to manipulate known Maclaurin series to find new Maclaurin series
- how to use Maclaurin series to find approximate solutions to differential equations.

CONCEPTS

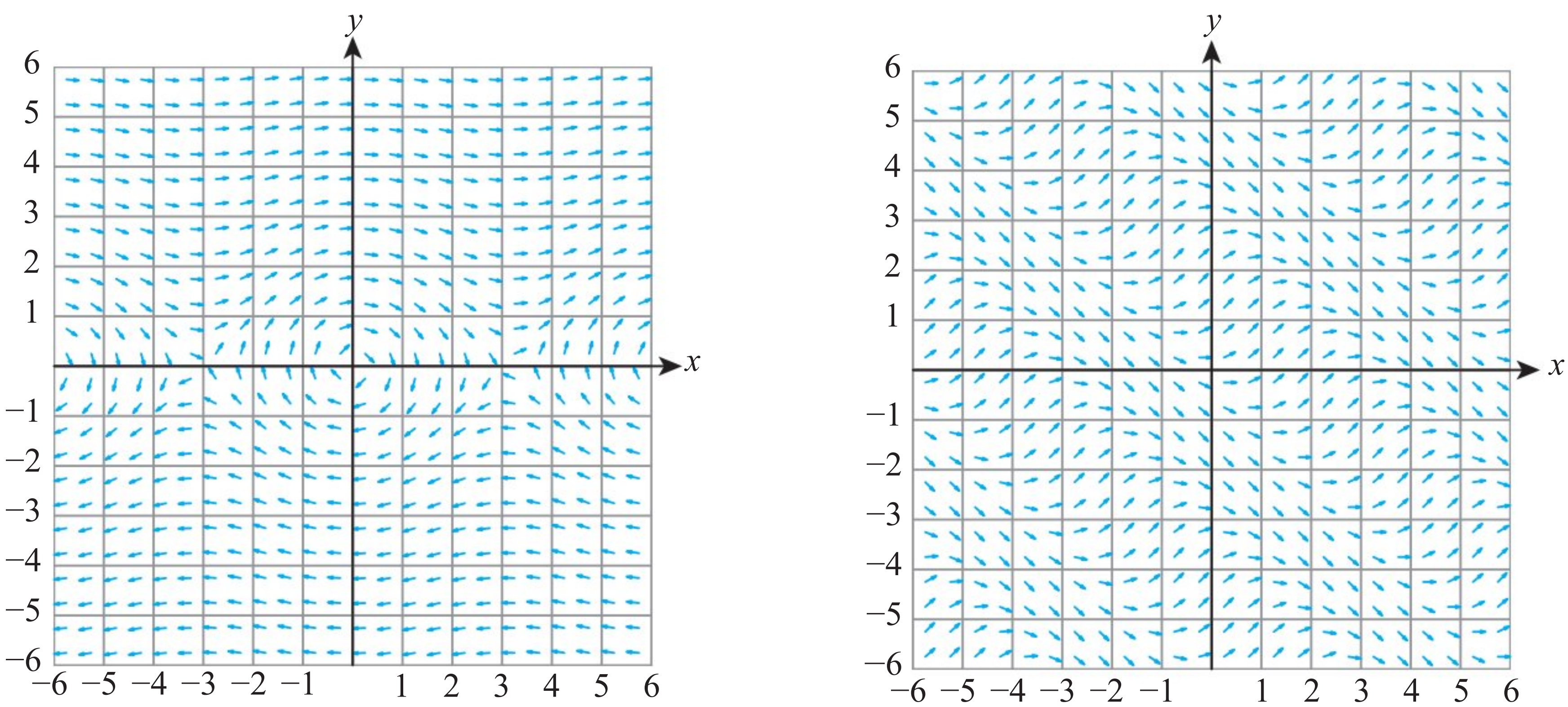
The following concepts will be addressed in this chapter:

- The derivative may be represented physically as a rate of **change** and geometrically as the gradient or slope function.
- A finite number of terms of an infinite series can be a general **approximation** of a function over a limited domain.

LEARNER PROFILE – Caring

When giving to charity, are you influenced more by mathematical measures of the impact your donation will make or by an emotional connection to the cause? Can all outcomes be measured?

Figure 11.1 Representing gradients at different points using direction fields.



PRIOR KNOWLEDGE

Before you start this chapter, you should already be able to complete the following:

1 Find $\int x\cos x^2\,\mathrm{d}x$.

2 If $y^2 + xy = 3\mathrm{e}^x$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

3 If $\mathrm{f}(x) = \sin 2x$, evaluate $\mathrm{f}^{(3)}(0)$.

4 Simplify $\mathrm{e}^{2\ln x}$ if $x > 0$.

5 For the sequence $u_{n+1} = 2u_n - n$ with $u_0 = 2$, use technology to find u_{10}.

6 Simplify $\frac{10!}{8!}$.

In many real-life situations, we know information about how a quantity changes. We can use this to create a model called a differential equation. In this chapter, you will see how some of these differential equations can be solved exactly. However, not every differential equation can be solved in terms of well-known functions. Such is their importance in many real-world situations that even if they cannot be solved exactly, there are methods which are used to find approximate solutions. In this chapter, you will meet two methods of doing this. The first is to consider short, straight-line sections. The second is to approximate functions as polynomials, called Maclaurin series, which turn out to have many other useful applications too.

Starter activity

Look at the diagrams in Figure 11.1. They represent differential equations, and each small line shows what the gradient is at each point. Imagine they represent streamlines in a flowing water. If you were to drop a floating object in at one point, see if you can trace out its path.

Now look at this problem:

Try to sketch curves which have the following properties:

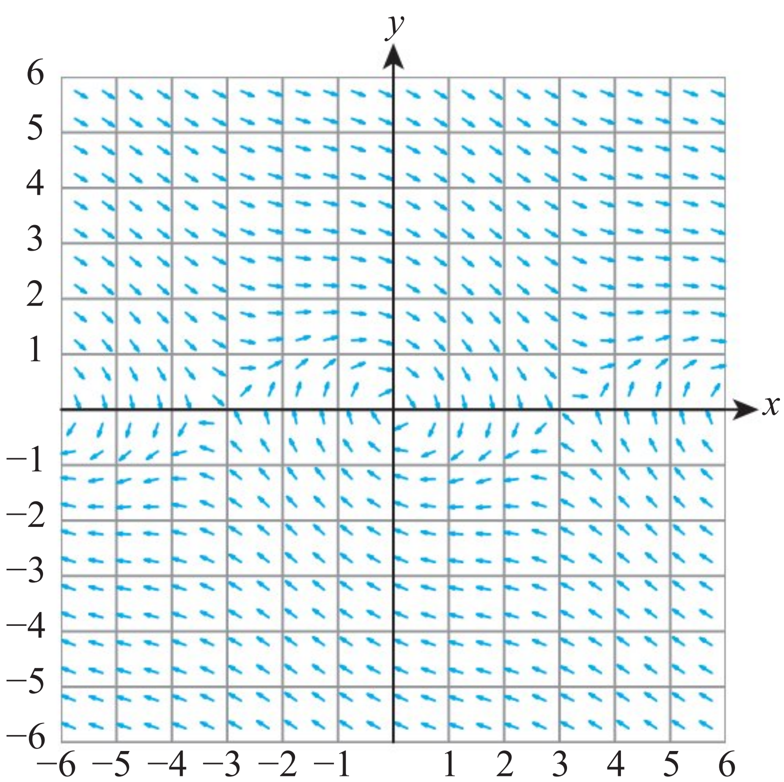
a the gradient at every point is constant

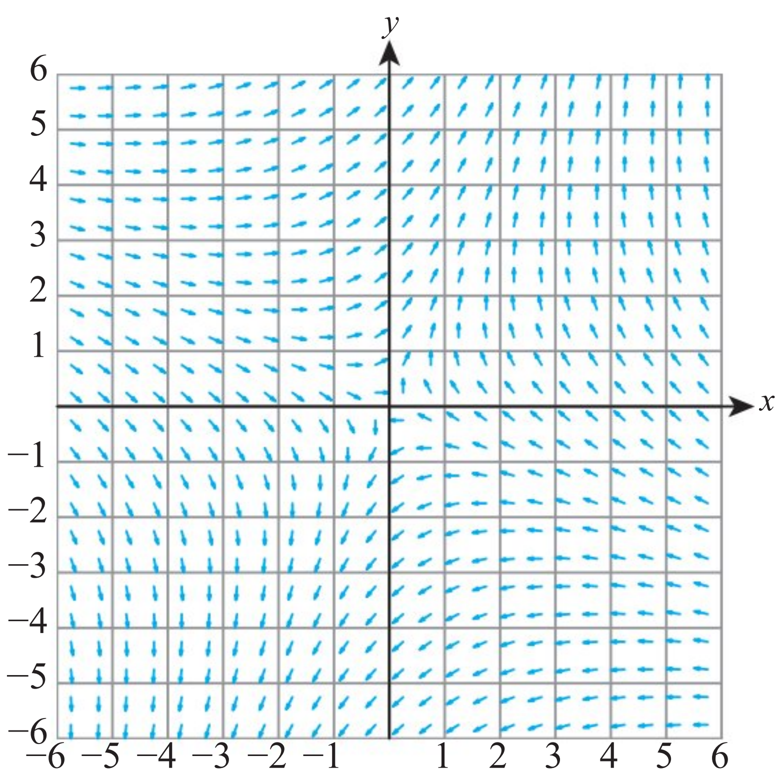
b the gradient at every point equals the y-coordinate

c the gradient at every point equals the x-coordinate

d the gradient at every point is perpendicular to the line connecting the point to the origin

e the gradient at every point equals the distance from the origin.





11A First order differential equations and Euler's method

Terminology of differential equations

A differential equation is an equation including derivatives, such as

A: $\frac{dy}{dx} = x + y$

B: $\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = y^2 - 4$

C: $\frac{d^2y}{dx^2} = 5x + 2y + e^x\frac{dy}{dx}$

These have a huge range of applications in any area where things change – such as economics, science and computing. Many books have been written on how to solve them, and the first step is identifying some ways to classify them.

Typically, the differential equation has an independent variable, x, and a dependent variable, y. Information about the rate of change of y as x changes, $\frac{dy}{dx}$, is known. The act of solving a differential equation means to find y as a function of x (if that is possible – otherwise, you can sometimes leave it as an implicit relationship).

A **first-order differential equation** is one which only goes up to the first derivative, for example A and B above.

A **linear differential equation** is one where neither y nor any of its derivatives are multiplied together or have any non-linear function (such as, $y^2, \sqrt{y}, e^y$) applied to them. For example, A and C above.

You are the Researcher

All the differential equations discussed here are called ordinary differential equations or ODEs. There is a whole other type of differential equation called partial differential equations, which apply to situations that can change in more than one way (for example, over time and over space). Some famous important examples of this include the wave equation and the heat equation.

Not all changes happen continuously. If there are changes that happen discretely (such as once each day), then the appropriate tool is a difference equation.

As with integration, solving differential equations introduces unknown constants. The solution which still includes an unknown constant is called the **general solution**. A first order differential equation will include just one arbitrary constant, but higher order differential equations will include more. To find the values of the constants we use **initial conditions** (values of y or $\frac{dy}{dx}$ at $x = 0$) or **boundary conditions** (values of y or $\frac{dy}{dx}$ at other values of x). Once these conditions have been used the solution with the constants evaluated is called the **particular solution**.

WORKED EXAMPLE 11.1

The rate at which water flows out of a bucket is proportional to the square root of the volume remaining. Write this as a differential equation.

Define the variables ··· Let V = volume of water remaining, and t = time.

Remember that if a quantity is decreasing, the derivative is negative ··· $\frac{dV}{dt} = -k\sqrt{V}$

CONCEPTS – CHANGE

Is it easier to measure a value, or the **change** in a value? Consider the following examples:

- In a car, is it easier to measure the car's speed or its position?
- Is it easier for a geographer to find the total volume of a river or the amount that flows under a bridge each minute?
- Is it easier to find the total number of bees in a country, or estimate the factors affecting how the population grows?
- Is it easier to measure the total number of electrons in a wire, or the current that passes through it?
- Is it easier to measure the total amount of money in a country, or to estimate the amount earnt each year?

If you think about these examples, it might help you to understand why so many models are written in terms of differential equations.

You have already met simple differential equations in your previous study of calculus. If the derivative is given in terms of x only, then we can find the function for y just using integration.

WORKED EXAMPLE 11.2

Solve $\frac{dy}{dx} = e^{2x}$ if $y(0) = 3$.

Because we know the derivative just in terms of x, we can undo the differentiation using integration to write down the general solution of the differential equation ··· $y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$

We can use the initial condition to find the value of the constant ··· When $x = 0$, $y = 3$ so,

$$3 = \frac{1}{2}e^0 + c$$

$$c = \frac{5}{2}$$

Write down the particular solution ··· So, $y = \frac{1}{2}e^{2x} + \frac{5}{2}$

Euler's method

With many differential equations, you cannot find an exact expression for y in terms of x. However, that does not mean that the equation has no solution – it can still be explored graphically. We can approximate the solution by imagining lots of small lines connected together. To do this, we first have to write the differential equation in the form

$$\frac{dy}{dx} = f(x, y)$$

From a given starting point, (x_0, y_0) we can use the differential equation to calculate the gradient at that point, $f'(x_0, y_0)$. We then use a fixed step length, h, to jump to the next value of x but use the gradient at the point we are leaving to determine how much y changes.

Since change in y is given by the change in x multiplied by the gradient, we find that

$$y_1 - y_0 \approx h \times f'(x_0, y_0)$$

We can then repeat this process to get a general iteration formula, called **Euler's method**.

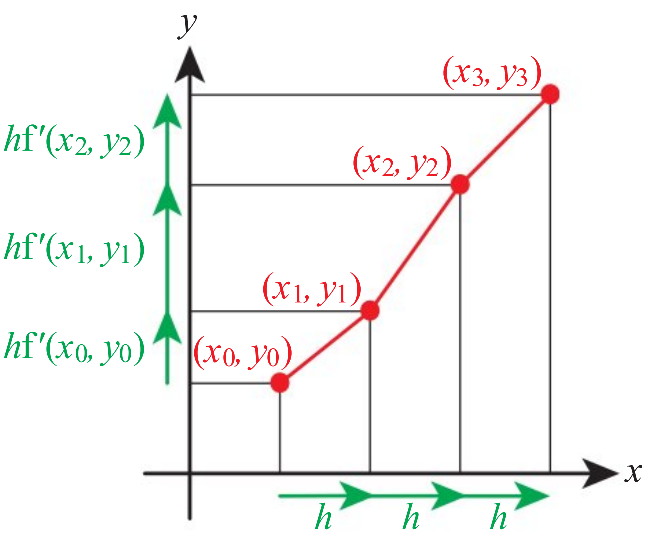
KEY POINT 11.1

- $x_{n+1} = x_n + h$
- $y_{n+1} = y_n + h \times f'(x_n, y_n)$

Tip

Make sure you are familiar with the sequence or iteration function on your calculator for this method. If you need to do many iterations to get to the required value, you can just write down the first few and last few stages of the iteration in your working.

You can visualize this process graphically:



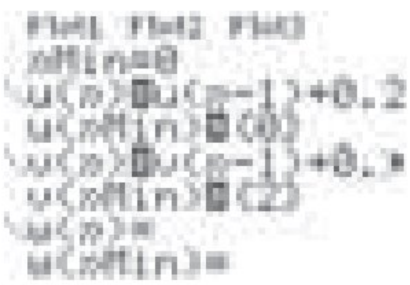
WORKED EXAMPLE 11.3

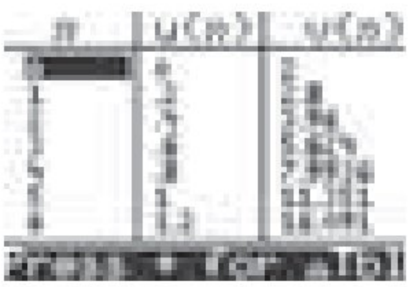
Use Euler's method with a step length of 0.2 to estimate the value of $y(1)$ given that $\frac{dy}{dx} = x + 2y$ and $y(0) = 2$.

Write down the iterative formula for Euler's method

$$y_{n+1} = y_n + 0.2(x_n + 2y_n)$$

Use your GDC to evaluate several iterations, recording the output at each stage





The process produces:

n	x_n	y_n
0	0	2
1	0.2	2.8
2	0.4	3.96
3	0.6	5.624
4	0.8	7.9936
5	1	11.35104

You can read off the value of y when $x = 1$

So, $y(1) \approx 11.4$

Leonard Euler (1707–1783) was born in Switzerland and was one of the most prolific and versatile mathematicians in history. His contributions ranged from celestial mechanics to ship building and music. He introduced the current mathematical meanings of the symbols $f(x)$, e, π and $\sum$ and made some fundamental leaps in the understanding of complex numbers.

He continued writing even after going blind. He did much of the working in his head and instructed his sons to write down his results.

You are the Researcher

There are various improvements that can be made to Euler's method. For example, if the derivative is just a function of x, then you can use the gradient at $\frac{(x_n + x_{n+1})}{2}$.

There are further extensions to something called Runge–Kutta methods which are used in most modern computers programs to solve real world differential equations. They will probably have been used by engineers to study the effect of wind on the next bridge you cross, by animators to make realistic looking hair in CGI graphics and by gaming programmers to make characters run, swim and jump correctly.

TOOLKIT: Problem Solving

Under what conditions will Euler's method underestimate the true value? When will it overestimate the true value?

What can you say about any predictions made by Euler's method in solving the differential equation $\frac{dy}{dx} = x^3$?

TOK Links

Which is better – a perfect solution to equations which vaguely model the real-world situation, or approximate solutions to equations which accurately model the real-world situation? How should we decide when a model, method or solution is good enough?

Exercise 11A

For questions 1 to 6, classify each differential equation as either linear or non-linear and state its order.

1 a $\frac{dy}{dx} + 2y = 4x$

b $\frac{dy}{dx} = y - x$

2 a $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$

b $\frac{d^2y}{dx^2} - 5y = 0$

3 a $f''(x) + xf'(x) = \sin x$

b $f'''(x) - e^x = f(x)$

4 a $\frac{dy}{dx} + y^2 = x$

b $\frac{dy}{dx} = \sin(x + y)$

5 a $\left(\frac{dy}{dx}\right)^2 + y^2 = x^2$

b $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{y} + \frac{1}{x}$

6 a $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 4y = 0$

b $y\frac{dy}{dx} = x$

For questions 7 to 9, use the method demonstrated in Worked Example 11.1 to write the given information as a differential equation.

7 a The growth rate of bacteria in a petri dish is proportional to the number of bacteria (B).

b The rate of increase of height in a human is inversely proportional to the height (h).

8 a The acceleration of a falling meteorite is inversely proportional to the square of its distance from the planet (S).

b The acceleration of a car is proportional to the square root of the time since the car started.

9 a The rate of increase of the number of people with a disease (I) in a population of size N is proportional to the number of people with the disease and proportional to the number of people without the disease.

b The rate of spread of a rumour is proportional to the number of people who know a rumour (R) in a group of size N and the number of people who don't know the rumour, and is inversely proportional to the time (t) since the rumour started.

For questions 10 to 12 find the particular solution of the given differential equation using the initial conditions.

10 a $\frac{dy}{dx} = x^2,\ y(0) = 1$

b $\frac{dy}{dx} = \sin x,\ y(0) = 2$

11 a $\frac{dy}{dx} + e^{2x} = 0,\ y(0) = 0.5$

b $\frac{dy}{dx} + \frac{1}{1+x^2} = 0,\ y(0) = 5$

12 a $\frac{dy}{dx} + \frac{1}{x} = 1,\ y(1) = 2$

b $\frac{dy}{dx} + \frac{1}{x^2} = 3,\ y(1) = 4$

For questions 13 to 16, use Euler's method with a step length of 0.2 to estimate $y(1)$ given that $y(0) = 1$ for each of the following differential equations.

13 a $\frac{dy}{dx} = y$

b $\frac{dy}{dx} = x$

14 a $\frac{dy}{dx} = x + y$

b $\frac{dy}{dx} = x - y$

15 a $\frac{dy}{dx} = \frac{x}{y}$

b $\frac{dy}{dx} = xy$

16 a $\frac{dy}{dx} = x^2 + y^2$

b $\frac{dy}{dx} = ye^x$

17 Consider the differential equation $\frac{d^2y}{dx^2} + 8e^{2x} = 0$.

a Find the general solution to this differential equation.

b Find the particular solution with $y(0) = 1$ and $y'(0) = -2$.

18 Consider the differential equation $\frac{dy}{dx} = 2x$ subject to $y = 0$ when $x = 0$.

a Use Euler's method with step length 0.1 to estimate

i $y(1)$ ii $y(2)$.

b Use Euler's method with step length 0.2 to estimate

i $y(1)$ ii $y(2)$.

c Solve the differential equation exactly. Hence determine which one of your estimates in parts **a** and **b** are furthest away from the true value.

19 For the differential equation $\frac{dy}{dx} = \frac{xy}{x+y}$ subject to $y = 1$ when $x = 0$.

a Use Euler's method with step length 0.5 to estimate the value of y when $x = 10$.

b How could this estimate be improved?

20 The following table shows the speed of a car as it is accelerating, measured at three second intervals.

Time (s)	0	3	6	9	12	15
Speed ($m\,s^{-1}$)	0	6	12	19	24	27

Use Euler's method to estimate the distance travelled by the car in the first 15 seconds.

21 The following table shows the speed of a car as it is decelerating, measured at three second intervals.

Time (s)	0	3	6	9	12	15
Speed ($m\,s^{-1}$)	20	15	10	8	6	5

Use Euler's method to estimate the distance travelled by the car in the first 15 seconds.

22 Use Euler's method with step length of 0.1 to approximately sketch the solution for $1 \leqslant x \leqslant 5$ to $\frac{dy}{dx} = y^2 - x^2$ and $y(1) = 1$.

23 **a** Use Euler's method with step length of 0.1 to approximately sketch the solution for $0 \leqslant x \leqslant 4$ to $\frac{dy}{dx} = \sin(x + y)$ with x and y in radians and $y(0) = 0$.

b Hence estimate, to one decimal place, the largest value of y in this range.

24 The height of a piece of ash (h metres) falling vertically into a fire is modelled by $\frac{dh}{dt} = -0.1h^2 - 0.5t$, where t is in seconds.

Initially the ash is 2 metres above the fire. Use Euler's method with a step length of 0.1 seconds to

a estimate the height of the piece of ash after 1 second

b estimate the time (to the nearest second) it takes to reach the fire.

25 A raindrop is modelled as a perfect sphere. Its volume decreases at a rate proportional to its surface area. When the volume is 0.5cm^3 volume is decreasing at a rate of 0.1cm^3 per minute.

a Find an expression for $\frac{dr}{dt}$, where r is in cm and t is in minutes.

b Hence determine how long it takes to completely evaporate.

26 Consider the differential equation $\frac{d^2y}{dx^2} + xe^{-x^2} = 0$ subject to the initial conditions that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$.

Use Euler's method with step length 0.1 to estimate y when $x = 1$.

27 Consider the differential equation $\frac{d^2y}{dx^2} = 2x + y$ subject to initial conditions that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

Consider an Euler's method approach (applied to second order differential equation) with step length 0.1.

a Show that when $x = 0.1$ this method predicts that $y = 1.2$ and $\frac{dy}{dx} = 2.1$.

b Use this variation on Euler's method to predict $y(1)$.

28 Consider the system of coupled differential equations

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = x - y$$

When $t = 0$, $x = 1$ and $y = 2$.

Use Euler's method with a step length in t of 0.1 to estimate the values of x and y when $t = 1$.

11B Separating variables and homogeneous differential equations

Separation of variables

So far, the only types of differential equation you can solve are of the form $\frac{dy}{dx} = f(x)$, which you did by integrating both sides with respect to x. However, you saw in Section 11A that there are many situations where the right-hand side can be a function of both x and y. In this situation, you cannot integrate the right-hand side with respect to x since y is not a constant. However, if the right-hand side can be separated into a function of x multiplied by a function of y, then a method called separation of variables can be used.

Tip

This looks a lot like you are treating $\frac{dy}{dx}$ as a fraction, and at the moment it is not a problem if you think about it like that, but technically this is not the case. In more advanced work, you will see that you are actually going through the line $\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$.

KEY POINT 11.2

If $\frac{dy}{dx} = f(x)g(y)$, then $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$.

WORKED EXAMPLE 11.4

Find the general solution to $\frac{dy}{dx} = xy$ given that $y > 0$.

Get all the instances of x on one side, all the instances of y on the other and then integrate	$\int \frac{1}{y}\,dy = \int x\,dx$
Perform the integrals on both sides. We only need to put a constant of integration on one side	$\ln\lvert y\rvert = \frac{x^2}{2} + c$
	Since $y > 0$
We can use the fact that $y > 0$ to remove the modulus, then do e to the power of both sides to remove the natural logarithm	$y = e^{\frac{x^2}{2}+c}$
Technically we are already finished, but it can be very useful if you are going to go on to use this solution to rearrange it to make the constant of integration into a multiplicative factor	$= e^{\frac{x^2}{2}}e^c$
	$= Ae^{\frac{x^2}{2}}$

Tip

A common error when solving these differential equations is to just put a '$+ c$' at the end of the solution, but as you can see in Worked Example 11.4, the answer is not $y = e^{\frac{x^2}{2}} + c$.

Be the Examiner 11.1

Find the general solution to $\frac{dy}{dx} = y$ if $y > 0$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\int \frac{1}{y}\,dy = \int 1\,dx$	$\int \frac{1}{y}\,dy = \int 1\,dx$	$\int \frac{1}{y}\,dy = \int 1\,dx$
$\ln y = x + c$	$-\frac{1}{y^2} = x + c$	$\ln y = x + c$
$y = Ae^x$	$y = \sqrt{\frac{1}{c-x}}$	$y = e^x + c$

Tip

When solving these type of differential equations you can often have issues with taking logs of negative numbers unless you are very careful with modulus signs. Normally, if this is closely analysed, it turns out that it does not matter so you will often see that solutions ignore this issue. If in a question you see a condition such as 'for $y > 0$, this is usually there so that you do not have to worry about dealing with modulus signs.

Homogenous differential equations

If the differential equation can be written as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, then it is called a **homogenous differential equation**. In this situation, a clever substitution will turn it into a separable differential equation. It is not obvious (and will be proved later) that you use the substitution $y = vx$ where v is a function of x.

KEY POINT 11.3

If $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, then use the substitution $y = vx$ to form a differential equation in v and x.

This will be a separable differential equation.

Note that if $y = vx$, then we can use the product rule to get that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Tip

It is not always immediately obvious that the expression can be written as $f\left(\frac{y}{x}\right)$. One thing to look out for are fractions where the 'power' of each term in the top and bottom is equal. For example $\frac{x^2+y^2}{2x^2+3y^2}$ or $\frac{xy+y^2}{xy+x^2}$. Dividing top and bottom of each of these expressions by x^2 makes it a bit clearer that it is a function of $\frac{y}{x}$.

WORKED EXAMPLE 11.5

Solve $\frac{dy}{dx} = \frac{y+x}{x}$ for $x > 0$.

We could write $\frac{dy}{dx} = \frac{\frac{y}{x}+1}{1}$ so the right-hand side is a function of $\frac{y}{x}$.

Use $y = vx$.

Therefore, $\frac{dy}{dx} = x\frac{dv}{dx} + v$

Substituting in:

Therefore, a substitution $y = vx$ is appropriate: $x\frac{dv}{dx} + v = \frac{vx+x}{x} = v+1$

Some algebraic neatening is required to isolate $\frac{dv}{dx}$: $x\frac{dv}{dx} = v+1-v$

$= 1$

So,

Now separate variables: $\int 1\,dv = \int \frac{1}{x}dx$

Conduct the integration. Since $x > 0$ we can drop the modulus sign in the log: $v = \ln x + c$

Substitute back for y using $v = \frac{y}{x}$: $\frac{y}{x} = \ln x + c$

$y = x\ln x + cx$

Proof 11.1

Prove that if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, then the substitution $y = vx$ creates a separable differential equation.

Eliminate $\frac{dy}{dx}$ using $\frac{dy}{dx} = v + x\frac{dv}{dx}$: $v + x\frac{dv}{dx} = f(v)$

Eliminate y using $\frac{y}{x} = v$

Get all of the references to v on the left-hand side: $x\frac{dv}{dx} = f(v) - v$

Write the left-hand side explicitly in a separable form: $\frac{dv}{dx} = (f(v) - v) \times \frac{1}{x}$

Which is a product of a function of v and a function of x.

Links to: Physics

Unfortunately, the term 'homogeneous differential equation' can also mean something entirely different. It describes a special type of linear differential equation – often applied to second order differential equations with constant coefficients. These are hugely important in physics. See if you can find out how they can be used to model damped simple harmonic motion. How does the model change if inhomogeneous differential equations are used instead?

Exercise 11B

For questions 1 to 4, use separation of variables, as demonstrated in Worked Example 11.4, to find the general solution of the following differential equations.

1 a $\frac{dy}{dx} = 2y, \quad y > 0$

b $\frac{dy}{dx} = -y, \quad y > 0$

2 a $\frac{dy}{dx} = y + 1, \quad y > -1$

b $\frac{dy}{dx} = 1 - y, \quad y < 1$

3 a $\frac{dy}{dx} = x^2 y^2$

b $\frac{dy}{dx} = x^3 y^3, \quad y > 0$

4 a $\frac{dy}{dx} = \frac{y}{x}, \quad x, \ y > 0$

b $\frac{dy}{dx} = \frac{x}{y}, \quad x, \ y > 0$

For questions 5 to 7, use the substitution $y = vx$, as demonstrated in Worked Example 11.5, to find the general solution of the following differential equations.

5 a $\frac{dy}{dx} = 1 + \frac{2y}{x}$

b $\frac{dy}{dx} = 1 - \frac{y}{x}$

6 a $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \quad x, \ y > 0$

b $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}, \quad x, \ y > 0$

7 a $\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}}$

b $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}, \quad x, \ y > 0$

8 Find the general solution of the differential equation $\frac{dy}{dx} = xy^2$.

9 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{\cos x}{y^2}$.

10 Solve the differential equation $\frac{dy}{dx} = 2x\mathrm{e}^{-y}$.

11 Given that $\frac{dy}{dx} = y\sec^2 x$, and that $y = 4$ when $x = 0$, find an expression for y in terms of x.

12 Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{9x^2}{4y}$ with the initial condition $y = 3$ when $x = 0$. You may leave your answer in the form $f(y) = g(x)$.

13 Find the general solution of the differential equation $x\frac{dy}{dx} = y$. Give your answer in the form $y = f(x)$.

14 Find the general solution of the differential equation $\frac{1}{y^2}\frac{dy}{dx} = 2x$.

15 For the differential equation $y^2\frac{dy}{dx} = 3x$ find the particular solution of the equation with $y = 3$ when $x = 2$.

16 a Find the general solution of the differential equation $\frac{dy}{dx} = 2(x + 2)(y - 1)$.

b Find the particular solution of the differential equation with $y = 2$ when $x = 0$.

17 Given that $\sec x\frac{dy}{dx} = \cos^2 y$, use separation of variables to show that $\sin x - \tan y = c$ for some constant c.

18 The mass (m grams) of a radioactive substance decays at a rate proportional to the current mass. Initially, the mass of the substance is 25 g and the rate of decay is $5\,\mathrm{g\,s^{-1}}$.

a Find the constant k such that $\frac{dm}{dt} = -km$.

b Find an expression for the mass of the substance after t seconds.

c How long does it take for the mass to decay to half of its initial value?

19 The population of bacteria, N thousand, grows at a rate proportional to its size. The initial size of the population is 2000 and the initial rate of increase is 500 bacteria per minute.

a Find the constant k such that $\frac{dN}{dt} = kN$.

b Find the size of the population after 10 minutes, giving your answer to the nearest thousand.

20 A balloon is being inflated at a rate inversely proportional to its current volume. Initially the volume of the balloon is $300\,\mathrm{cm}^3$ and it is increasing at the rate of $10\,\mathrm{cm^3\,s^{-1}}$.

a Show that $\frac{dV}{dt} = \frac{3000}{V}$.

b Find the volume of the balloon after t seconds.

21 An object of mass 1 kg falls vertically through the air. Taking into account air resistance, the acceleration of the object is given by $\frac{dv}{dt} = 10 - 0.1v$, where v is the velocity in $m\,s^{-1}$.

a Given that the object starts from rest, find an expression for the velocity at time t seconds.

b Find the distance travelled by the object in the first three seconds.

22 Variables x and y satisfy the differential equation $y\frac{dy}{dx} = 4e^{-2x}$. When $x = 0$, $y = -2$. Find an expression for y in terms of x.

23 Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$.

24 Given that $\frac{dy}{dx} = 2e^{x-2y}$ and that $y = 0$ when $x = 0$, express y in terms of x.

25 Given that $\frac{dy}{dx} = xy + 2x + y + 2$, and that $y = 0$ when $x = -3$, show that $y = A\sqrt{e^{x^2+2x-3}} + B$, where A and B are constants to be found.

26 The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{\sin x}{y}$. When $x = 0$, $y = 10$. Find an expression for y in terms of x.

27 **a** Use separation of variables to show that the general solution of the differential equation $\frac{dy}{dx} = \frac{\cos x}{\sin y}$ can be written as $\sin x + \cos y = c$

b A particular solution of the differential equation satisfies $0 \leqslant x \leqslant \pi$ and $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and has $y = \frac{\pi}{3}$ when $x = \frac{\pi}{6}$. Find the two possible values of y when $x = \frac{\pi}{2}$.

28 Use the substitution $y = vx$ to find the general solution of the differential equation $x\frac{dy}{dx} = 2x + 3y$. Give your answer in the form $y = f(x)$.

29 **a** Show that $xy\frac{dy}{dx} = x^2 + y^2$ is a homogeneous differential equation.

b Find the general solution of the equation, giving your answer in the form $y^2 = g(x)$.

30 **a** Explain why $\frac{dy}{dx} = \frac{(2x + y)^2}{4x^2}$ is a homogeneous differential equation.

b Use a substitution $y = vx$ to find the solution of the equation which satisfies $y = 0$ when $x = 1$.

31 Find the general solution of the homogeneous differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$.

32 **a** Find the general solution of the differential equation $x\frac{dy}{dx} = 2x - y$.

b Find the particular solution for which $y = 5$ and $x = 1$.

33 Given that $\frac{dy}{dx} = \frac{3y}{x} + \frac{4y}{x^2}$, and that $y = 8$ when $x = 2$, solve the differential equation to find an expression for y in terms of x.

34 Find the general solution of the differential equation $x\frac{dy}{dx} = y^2 + 9$.

35 Given that $(1 + x^2)\frac{dy}{dx} = 2x\sqrt{1 - y^2}$, and that $x = 0$ when $y = 0$, find an expression for y in terms of x.

36 Find the particular solution of the differential equation $(1 - x^2)\frac{dy}{dx} = xy + y$ which satisfies the initial condition $y = 2$ when $x = 0$.

37 Find the general solution of the differential equation $\cos^2 x\frac{dy}{dx} = \sec y$, giving your answer in the form $y = f(x)$.

38 The size of the population (N) satisfies the differential equation $\frac{dN}{dt} = 0.6N - 0.002N^2$, where time is measured in months. The initial size of the population is 200.

a Solve the differential equation to find the size of the population at time t.

b Sketch the graph of N against t and describe the long term behavior of the population.

39 A particle moves with acceleration $a = 10 - 0.1v^2$, where v is the velocity measured in $m\,s^{-1}$. The initial velocity of the particle is zero.

a By solving a differential equation, show that $v = \frac{10(e^{2t} - 1)}{e^{2t} + 1}$.

b Use the substitution $u = e^{2t} + 1$ to show that the displacement of the particle from the starting position at time t is $x = 5\ln\left(\frac{(e^{2t} + 1)^2}{4e^{2t}}\right)$.

40 **a** Show that the substitution $z = \frac{1}{y}$ transforms the equation $\frac{dy}{dx} = xy(y-1)$ into the equation $\frac{dz}{dx} = x(z-1)$.

b Hence find the particular solution of the differential equation which satisfies the initial condition $y = \frac{1}{3}$ when $x = 0$.

41 **a** Show that the substitution $u = y + \frac{1}{x}$ transforms the equation $x^2\frac{dy}{dx} = xy - x + 2$ into the equation $x\frac{du}{dx} = u - 1$.

b Hence find the general solution of the equation $x^2\frac{dy}{dx} = xy - x + 2$.

42 Use the substitution $z = 2x - 3y$ to find the solution of the differential equation $(2x - 3y + 3)\frac{dy}{dx} = 2x - 2y + 1$, given that $y = 1$ when $x = 1$. Leave your answer in the form $(2x - 3y + 3)^2 = f(x)$.

43 **a** Show that the substitution $x = u - 1$, $y = v + 3$ transforms the differential equation $\frac{dy}{dx} = \frac{4x - y + 7}{2x + y - 1}$ into a homogeneous differential equation.

b Hence find the general solution of the differential equation, giving your answer in the form $f(x, y) = c$

44 Use the substitution $z = x + y$ to find the general solution of the differential equation $\frac{dy}{dx} = \cos^2(x + y) - 1$.

45 Variables x and y satisfy the differential equation $\sqrt{1 - x^2}\frac{dy}{dx} = \sqrt{1 - y^2}$. A particular solution has $y = \frac{1}{2}$ when $x = \frac{\sqrt{3}}{2}$. Show that this particular solution can be written in the form $y = x\sqrt{A} + B\sqrt{1 - x^2}$, stating the values of the constants A and B.

46 The von Bertalanffy model for the growth of an organism states surface area promotes growth, but volume restricts it. This can be written as a differential equation in terms of the mass, M of the organism:

$\frac{dM}{dt} = \alpha M^{\frac{2}{3}} - \beta M$, where α and β are constants.

a Show that if the solution has a point of inflection, it occurs when $M = \frac{8\alpha^3}{27\beta^3}$.

b Use the substitution $v = M^{\frac{1}{3}}$ to solve the differential equation.

c Find in terms of α and β, the long term value of M.

d Sketch the solution if M is initially very small.

11C Integrating factors

Differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$ can be solved using an **integrating factor**. This is a function, $\mu(x)$ which you multiply both sides by to turn the left-hand side into a perfect product rule derivative – this is an expression that can be written in the form $\frac{d}{dx}(f(x, y))$. Once you have the left-hand side in this form, you can integrate both sides with respect to x to solve the differential equation.

TOOLKIT: Problem Solving

When you do the indefinite integral to find the integrating factor, you do not need to include the arbitrary constant. Can you explain why this is the case?

KEY POINT 11.4

If $\frac{dy}{dx} + P(x)y = Q(x)$, then multiplying by the integrating factor $\mu(x) = e^{\int P\,dx}$ turns the equation into

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

Key Point 11.4 is proved on page 363 in Proof 11.2.

WORKED EXAMPLE 11.6

Solve $\frac{dy}{dx} + \frac{2y}{x} = x^3$ for $x > 0$.

This is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{2}{x}$ and $Q(x) = x^3$. We can then find the integrating factor. It is worth using a few rules of logarithms and exponents to neaten it up

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

By multiplying both sides by the integrating factor, you form a perfect product rule derivative. After some practice, you might just be able to skip this line and go straight to the next one

Multiplying both sides by the integrating factor:

$$x^2\frac{dy}{dx} + 2xy = x^2 \times x^3$$

So,

$$\frac{d}{dx}(x^2 y) = x^5$$

Then integrate both sides with respect to x

$$x^2 y = \frac{x^6}{6} + c$$

Rearrange to get y

$$y = \frac{x^4}{6} + \frac{c}{x^2}$$

Proof 11.2

Prove Key Point 11.4.

After multiplying $\frac{dy}{dx} + Py = Q$ by the integrating factor, μ, the left hand side will be $\mu\frac{dy}{dx} + \mu Py$ but we want it to be the perfect derivative, $\frac{d}{dx}(\mu y)$

We want to find an expression for μ such that

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \mu Py$$

Expanding the left hand side using the product rule, this becomes

$$\mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + \mu Py$$

This will be true if

You can subtract $\mu\frac{dy}{dx}$ from both sides

$$y\frac{d\mu}{dx} = \mu Py$$

Therefore,

We want a solution which is true for all y, so we can divide through by y

$$\frac{d\mu}{dx} = \mu P$$

We can solve this by separating variables

$$\int \frac{1}{\mu} d\mu = \int P\, dx$$

We are looking for any function that works, so we can choose a function which is positive, therefore dropping the modulus when integrating

$$\ln \mu = \int P dx$$

So, $\mu = e^{\int P dx}$ as required.

Exercise 11C

For questions 1 to 4, use the techniques demonstrated in Worked Example 11.6 to find the general solution to each differential equations.

1 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x$

b $\frac{\mathrm{d}y}{\mathrm{d}x} - y = \mathrm{e}^x$

2 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x$

b $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2$

3 **a** $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x$

b $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2y}{x} = x$

4 **a** $y' + y\tan x = 2\sin x$

b $y' + y\tan x = \cos x$

5 **a** Find the integrating factor for the first order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \mathrm{e}^x$.

b Hence find the general solution of the equation.

6 Use the integrating factor method to find the general solution of the equation $\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \mathrm{e}^x$.

7 **a** Find the integrating factor for the first order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \mathrm{e}^{-x^2} + 8x$.

b Hence find the general solution of the equation.

8 Use integrating factor to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = 5\mathrm{e}^{-2x^2} + 12x$.

9 The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}y = 2x^2$ can be solved using the integrating factor method.

a Show that the integrating factor is $\frac{1}{x^3}$.

b Hence find the general solution of the equation.

10 **a** Find the integrating factor for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 12x$.

b Hence find the particular solution of the equation given that $y = 5$ when $x = 1$.

11 Use the integrating factor method to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 2\mathrm{e}^{\cos x}$.

12 Variables x and y satisfy the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 10xy = 3\mathrm{e}^{-5x^2}$. When $x = 0,\ y = 4$. Use the integrating factor method to find the solution of the differential equation.

13 Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{1}{x^3}$.

14 For the first order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sec x$,

a show that the integrating factor is $\sec x$.

b Hence show that the general solution can be written as $y = \sin x + A\cos x$.

15 **a** Show that the integrating factor for the equation $\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\tan x = \sec^2 x$ is $\cos^2 x$.

b Hence find the particular solution of the differential equation with $y = 8$ when $x = \frac{\pi}{4}$.

16 **a** Find the integrating factor for the equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \cos x$.

b Hence find the general solution of the equation.

17 Given that $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{1}{x^2}$, and that $y = 2$ when $x = 1$, find an expression for y in terms of x.

18 Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x+2} = x - 1$.

Question 16 can also be solved by separating variables. Which method do you find simpler?

19 Find the particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy}{x^2+1} = 2x$ with the initial condition $y = 0$ when $x = 0$.

20 Find the general solution of the differential equation $x\frac{dy}{dx}+2y=4$.

21 Find the general solution of the differential equation $(x-1)\frac{dy}{dx}+y=6x$.

22 Show that the general solution of the differential equation $\frac{dy}{dx}-y\tan x=\cos x$ can be written in the form $y=A\sin x+(Bx+c)\sec x$.

23 Variables y and y satisfy the differential equation $x^2\frac{dy}{dx}+2xy=e^x$. When $x=1$, $y=0$. Find the exact value of y when $x=2$.

24 Find the general solution of the differential equation $\cos x\frac{dy}{dx}-2y\sin x=3$.

25 Find the particular solution of the differential equation $(\tan x)\frac{dy}{dx}+y=\tan x$ such that $y=1$ when $x=\frac{\pi}{4}$.

26 Find the general solution of the equation $x^2\frac{dy}{dx}-2xy=\frac{x^4}{x-3}$.

27 Given that $\cos x\frac{dy}{dx}+y\sin x=\cos^2 x$ and that $y=2$ when $x=0$, find y in terms of x.

28 Find the general solution of the differential equation $\left(x-\frac{1}{x}\right)\frac{dy}{dx}+2y=1$

29 a The velocity ($v\,\text{m s}^{-1}$) of an object falling through a liquid is modelled by $\frac{dv}{dt}=10-2v$, subject to an initial speed $v_0\,\text{ms}^{-1}$.

i Find an expression for the speed at time t seconds.

ii What is the long term value of v (the terminal velocity)?

b If, instead, the liquid is warmed as the object falls through it, the motion is modelled by $\frac{dv}{dt}=10-\frac{v}{t+1}$.

i Find an expression for the speed at time t seconds for the new model.

ii Show that, for this new model, the long term effective acceleration is $5\,\text{m s}^{-2}$.

30 a Find $\frac{d}{dx}\left(x^2y^2\right)$.

b Hence solve, for $y>0$, $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{xy}$.

31 Consider the differential equation $3xy^2\frac{dy}{dx}+y^3=e^x$.

a Show that the substitution $z=y^3$ transforms this equation into the equation $x\frac{dz}{dx}+z=e^x$.

b Hence find an expression for y in terms of x.

32 a Show that the substitution $z=y^2$ transforms the equation $2y\frac{dy}{dx}-\frac{y^2}{x}=x^2$ into a linear differential equation.

b Hence find an expression for y in terms of x, given that $y=-2$ when $x=2$.

33 Use the substitution $y=\sqrt{u}$ to find the general solution of the differential equation $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$, where $y>0$.

34 a Use the substitution $z=\sin y$ to transform the equation $\cos y\frac{dy}{dx}+\tan x\sin y=2\cos^2 x$ into a linear differential equation.

b Given that $y=\frac{\pi}{6}$ when $x=\frac{\pi}{4}$, show that $\sin 2x-\sin y=\frac{\sqrt{2}}{2}\cos x$.

35 Consider the differential equation $\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}=x$.

a Show that the substitution $z=x\frac{dy}{dx}$ transforms the equation into $\frac{dz}{dx}-\frac{2}{x}z=x^2$.

b Find an expression for z in terms of x.

c Hence find the general solution of the differential equation for x and y.

36 The number of parasites (P) in a sheep's coat at a time t after the infestation starts is modelled by the differential equation

$$\frac{dP}{dt} = \alpha e^{-\gamma t} - \beta P.$$

a Assuming that $\beta \neq \gamma$

i solve the differential equation, assuming that the original value of P is negligible

ii find the t-coordinate of the maximum point (you may assume that any stationary point is a maximum point)

iii sketch the graph of P against t.

b Solve the differential equation in the situation where $\beta = \gamma$.

37 The decay sequence of bismuth-210 (Bi) is show below:

$$^{210}\text{Bi} \rightarrow {}^{210}\text{Po} \rightarrow {}^{206}\text{Pb}$$

The rate at which bismuth is turned into polonium (Po) is given by

$$\frac{d[\text{Bi}]}{dt} = -k_1[\text{Bi}]$$

where k_1 is called the 'rate constant' of this conversion.

Initially, the amount of bismuth is $[\text{Bi}]_0$ and there is no polonium (Po) or lead (Pb).

a Solve this differential equation to find the amount of bismuth after time t.

b The rate of conversion of polonium into lead has a rate constant k_2. Explain why the rate of production of polonium is given by

$$\frac{d[\text{Po}]}{dt} = k_1[\text{Bi}] - k_2[\text{Po}].$$

c Solve this differential equation.

d Hence find the amount of lead as a function of time.

e What is the long term amount of lead?

11D Maclaurin series

You have already met infinite binomial expansions, for example

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

In turns out that many other functions can be represented by infinite series. These are called **Maclaurin series**.

Suppose that a function $f(x)$ can be represented by the infinite series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

We want the sum of the series (when it converges) to equal the value of the function. Substituting $x = 0$, you can see that $a_0 = f(0)$.

The derivatives of the function and the series should be equal as well. Differentiation gives:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots$$

Substituting $x = 0$ again gives $a_1 = f'(0)$.

Comparing the second derivatives,

$f''(x) = 2a_2 + 6a_3x + \dots$

and so $a_2 = \frac{f''(0)}{2}$.

You can see that, after differentiating n times, the first term of $f^{(n)}(x)$ is $n!a_n$ and all other terms contain x. Setting $x = 0$ gives $a_n = \frac{f^{(n)}(0)}{n!}$.

KEY POINT 11.5

The Maclaurin series for $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

This can be written using sigma notation as

$$f(x) = \sum_{n=0} \frac{f^{(n)}(0)}{n!}x^n$$

WORKED EXAMPLE 11.7

Find the first three non-zero terms of the Maclaurin series for e^x.

First we need to calculate the derivatives

$f'(x) = e^x$
$f''(x) = e^x$

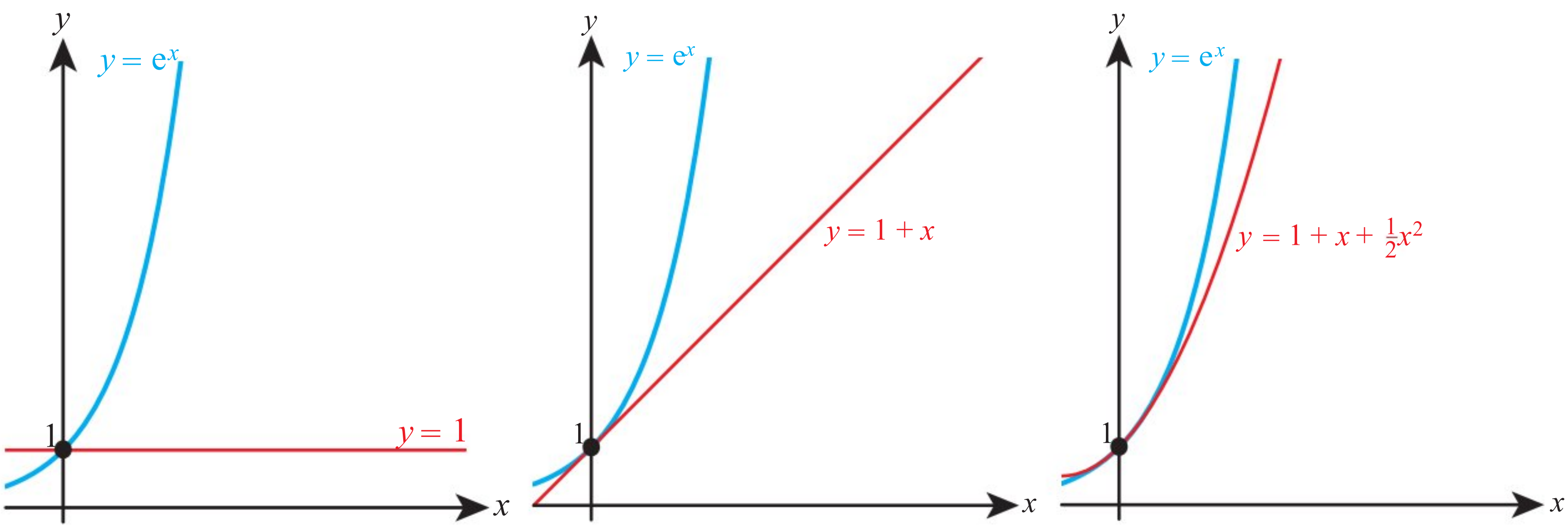
Then evaluate the function and its derivatives at $x = 0$

$f(0) = e^0 = 1$
$f'(0) = e^0 = 1$
$f''(0) = e^0 = 1$

Substitute into the Maclaurin series formula (Key Point 11.5)

$$e^x \approx 1 + 1 \times x + 1 \times \frac{x^2}{2}$$
$$= 1 + x + \frac{x^2}{2}$$

You can also think graphically about Maclaurin series. The constant term matches the value of the function at $x = 0$. The linear term matches the derivative of the function at $x = 0$. The quadratic term matches the curvature of the function at $x = 0$:



Standard Maclaurin series

You can similarly find Maclaurin series for other standard functions. These are given in the formula booklet.

KEY POINT 11.6

- $e^x = 1 + x + \frac{x^2}{2!} + \ldots$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

You also know another example of a Maclaurin expansion, the binomial series $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \ldots$ for $p \in \mathbb{Q}$.

TOOLKIT: Problem Solving

What does it mean to say that a function is represented by an infinite series?

1 Use graphing software to draw the graph of $y = e^x$.

  a On the same screen draw the graphs of $y = 1 + x$, $y = 1 + x + \frac{x^2}{2}$ and $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. (Each of these is the start of the Maclaurin series for e^x with an increasing number of terms.) What do you notice?

  b For each of the three polynomials above, find the percentage errors when it is used to approximate e^1 and e^2.

2 Repeat question 1 for $y = \sin x$ and the first two, three and four terms of its Maclaurin series. Why can a polynomial function never be a good approximation to $\sin x$ for large values of x?

3 Now draw the graph of $y = \ln(1+x)$ and the Maclaurin series with increasing number of terms. Can you find a polynomial that gives a good approximation when $x = 3$? Why do you think that is?

You are the Researcher

As you might have found in the above Problem Solving activity, the Maclaurin series of a function does not always agree with the original function. Finding the interval over which it does work is called finding the radius of convergence. You might like to look into how the radius of convergence is found.

TOOLKIT: Problem Solving

It turns out that Maclaurin series also work with complex numbers. For example, what is the Maclaurin series for $\cos x + i\sin x$?

Maclaurin series for more complicated functions

You can replace x in a Maclaurin series by another expression to get the Maclaurin series for a composite function.

WORKED EXAMPLE 11.8

Find the Maclaurin series for $\sin(2x^2)$ up to the term in x^6.

Write down the Maclaurin series for $\sin(x)$, using the dummy variable y in place of x for now...

$$\sin y = y - \frac{y^3}{6} + \frac{y^5}{120} + \ldots$$

Then substitute in $y = 2x^2$. You can stop once you get a term which will simplify to include x^6

$$\sin(2x^2) \approx (2x^2) - \frac{(2x^2)^3}{6}$$

$$= 2x^2 - \frac{8x^6}{6}$$

$$= 2x^3 - \frac{4x^6}{3}$$

TOOLKIT: Problem Solving

Is it possible to replace x by a more complicated expression, or even another Maclaurin series?

a Try replacing x by $1 + x$ in the Maclaurin series for $\sin x$ to get a series for $\sin(1 + x)$. Is it possible to write down the first three terms of the resulting series?

b The Maclaurin series for $\cos x$ begins $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$. By letting $y = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$, find the Maclaurin series for $\ln(\cos x)$ up to the term in x^6.

c Find the first three non-zero terms in the Maclaurin series for $\frac{e^x}{\cos x}$.

Hint: write $\frac{1}{1+y}$ as $(1+y)^{-1}$.

You are the Researcher

Notice that there is no Maclaurin series for $\ln x$. This is because the function is not defined for $x = 0$. There is a related type of series, called Taylor series, which can be used to represent functions close to points other than $x = 0$.

You can multiply two Maclaurin series to get a series for the product of two functions. For example, the series for $e^x \ln(1 + x)$, up to the term in x^2, is

$$\left(1 + x + \frac{x^2}{2} + \ldots\right)\left(x - \frac{x^2}{2} + \ldots\right) = x - \frac{x^2}{2} + x^2 + \ldots = x + \frac{x^2}{2} + \ldots$$

Notice that you only need to write the terms up to x^2.

You can also divide two Maclaurin series, by turning division into multiplication.

WORKED EXAMPLE 11.9

Find the Maclaurin series for $\frac{e^x}{1+x}$ up to the term in x^2. Hence, estimate $\frac{e^{0.1}}{11}$ to four decimal places.

Products are much easier to work with so we will think about this as $e^x \times \frac{1}{1+x}$. You can quote standard Maclaurian Series. You need terms up to x^2 to find the result up to x^2

$$e^x = 1 + x + \frac{x^2}{2} \ldots$$

$$(1+x)^{-1} = 1 - x + x^2 \ldots$$

$$e^x \times \frac{1}{1+x} \approx \left(1 + x + \frac{x^2}{2}\right) \times (1 - x + x^2)$$

You could multiply out all the terms, but it is quicker to focus on all the ways to form a constant, a linear term and a quadratic term. Any higher order terms are not required

$$= 1 \times 1 + (1 \times -x + x \times 1) + \left(1 \times x^2 + x \times -x + \frac{x^2}{2} \times 1\right)$$

$$= 1 + 0.5x^2$$

We can now substitute $x = 0.1$ into our series to find an approximation

$$\text{So, } \frac{e^{0.1}}{1.1} \approx 1.005$$

$$\text{Therefore, } \frac{e^{0.1}}{11} \approx 0.1005$$

CONCEPTS – APPROXIMATION

How accurate does an **approximation** need to be? How do you decide how many terms are needed in your Maclaurin series expansion? Many computers and calculators use Maclaurin series to calculate values of functions. How many terms of the Maclaurin series are needed to agree with your calculator's value of sin(0.1) to 8 decimal places? If a ship is 2 km away from a lighthouse at an angle of 0.5 radians clockwise from north, how many terms of the Maclaurin series are needed to find out how far east it is of the lighthouse to the nearest metre?

Differentiating and integrating Maclaurin series

You can differentiate and integrate Maclaurin series. For example, you can check that differentiating the series for $\sin x$ gives the series for $\cos x$.

In Section 11E you will use differentiation and integration of Maclaurin series to find approximate solutions of differential equations.

WORKED EXAMPLE 11.10

Find the first three terms of the Maclaurin series for $\frac{1}{1+x^2}$. Hence, find the first three non-zero terms in the Maclaurin series for $\arctan x$.

Use the standard Maclaurin series for $(1+x)^n$

$$(1+x)^{-1} = 1 - x + x^2 + \ldots$$

$$\text{So, } (1+x^2)^{-1} = 1 - x^2 + x^4 \ldots$$

$\arctan x$ is the integral of $\frac{1}{1+x^2}$

$$\arctan x = \int \frac{1}{1+x^2}\,dx$$

You can integrate the Maclaurin series term by term

$$\approx x - \frac{x^3}{3} + \frac{x^5}{5} + c$$

You need a suitable value to substitute in to find c

$\arctan 0 = 0$ therefore $c = 0$, so

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}$$

L'Hôpital's rule was covered in Section 10B.

Evaluation of limits using Maclaurin series

You already know how to evaluate limits of the form $\frac{0}{0}$ using L'Hôpital's rule. Some such limits can also be evaluated by considering Maclaurin series.

WORKED EXAMPLE 11.11

By using the Maclaurin series for $\cos x$, evaluate $\lim_{x\to 0}\frac{\cos x - 1}{x^2}$.

Write down the first three terms of the Maclaurin series for $\cos x$

$$\frac{\cos x - 1}{x^2} = \frac{\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\right)-1}{x^2}$$

then simplify the top

$$= \frac{-\frac{x^2}{2}+\frac{x^4}{24}-\ldots}{x^2}$$

and divide by x^2

$$= -\frac{1}{2}+\frac{x^2}{24}-\ldots$$

As $x \to 0$, all the terms apart from the first one tend to 0, so

Consider what happens when $x \to 0$

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

See if you can obtain the same answer by using L'Hôpital's rule twice.

TOOLKIT: Proof

Consider the following proof of Euler's formula.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\ldots$$

$$\text{So, } e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}\ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\right)$$

$$= \cos x + i\sin x$$

At first this seems very plausible, but mathematicians like to ask hard questions about plausible assumptions made in proofs. Here are examples of the types of questions which need to be asked about these proofs:

1 What does it mean to raise a number to a complex power? Is it allowed?

The 'usual' answer to this can feel like a bit of a disappointment, but gives you a lot of insight into how mathematical theories are developed. We do not really have a conceptual basis for raising a number to a complex power in the same way that usual powers are described using repeated multiplication. We define e^z using the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\ldots$$

This has the advantage that it guarantees consistency with real number powers, but it does not really provide an explanation or deeper conceptual basis. There is also a TOK issue here – have we proved anything about e raised to a complex number if we are just defining it to be the case?

2 The Maclaurin series for e^x was constructed for real x – can we replace x by a complex number and it will still work?

Consider the binomial expansion of $(1+x)^{-1} = 1+x+x^2 \ldots$

We know that this only works for $|x|<1$. We can still put $x=2$ into the right-hand side, but it does not converge and it does not equal the left hand side at this value.

In the same way, just because the Maclaurin series for e^x converges for all real x, does not necessarily mean it still converges for all complex values. It turns out that it does, but to establish this needs some complex analysis called the ratio test.

3 Can we rearrange the terms to form the sine and cosine Maclaurin Series?

This is a very subtle issue. We are used to being able to rearrange and regroup sums in any way we want, and with finite sums this works. However, with infinite sums this 'obvious' rule breaks down. For example, consider the infinite sum

$S = 1 - 1 + 1 - 1 + 1 \ldots$

If we group the terms in pairs,

$S = (1 - 1) + (1 - 1) + (1 - 1) \ldots$

$0 + 0 + 0 \ldots$

$= 0$

If we group the terms in a different way,

$S = 1 + (-1 + 1) + (-1 + 1) \ldots$

$= 1 + 0 + 0 \ldots$

$= 1$

It turns out that we can only rearrange and regroup the terms if the sum satisfies a condition called absolute convergence. This is actually the case for the e^z Maclaurin Series.

Exercise 11D

For questions 1 to 4, use the technique demonstrated in Worked Example 11.7 to find the first three non-zero terms of the Maclaurin expansion of the given function

1 a $\sin x$

b $\cos x$

2 a $(1-x)^{-1}$

b $\ln(1+x)$

3 a $(1+x)^{\frac{1}{2}}$

b $(1+x)^{-\frac{1}{2}}$

4 a $\arcsin x$

b $\arccos x$

Tip

Notice that the series in questions 2a and 3 are the binomial expansion of $(1 + x)^n$.

For questions 5 to 8, use the technique demonstrated in Worked Example 11.8 to find the Maclaurin expansion up to including the term stated.

5 a $e^{\frac{-x}{2}}$ up to x^3

b $e^{\frac{-x}{3}}$ up to x^3

6 a e^{3x^2} up to x^6

b e^{2x^3} up to x^9

7 a $\cos(3x^2)$ up to x^8

b $\cos(2x^3)$ up to x^{12}

8 a $\ln(1+4x)$ up to x^3

b $\ln(1-3x)$ up to x^3

For questions 9 to 11, use the technique demonstrated in Worked Example 11.9, together with the standard Maclaurin series from Key Point 11.6, to find the first three non-zero terms in the Maclaurin series of the following functions.

9 a $e^x \sin x$

b $e^x \cos x$

10 a $\sqrt{1+x}\sin x$

b $\sqrt[3]{1+x}\cos x$

11 a $\dfrac{\ln(1+x)}{1+x}$

b $\dfrac{\ln(1+x)}{\sqrt{1+x}}$

12 a Find the first three derivatives of $f(x) = \tan x$.

b Hence write down the first two non-zero terms in the Maclaurin series of $\tan x$.

c Find the percentage error when your series from part b is used to approximate $\tan(0.5)$.

13 a Find the first three derivatives of $f(x) = \sec x$.

b Hence find the first two non-zero terms in the Maclaurin expansion of $\sec x$.

c Use your series to find an approximate value of $\sec(0.2)$. What is the percentage error in this approximation?

14 Use standard Maclaurin series to find the first three non-zero terms in the Maclaurin expansion of $x\cos 3x$.

15 a Write down the first three terms of the Maclaurin series for $\ln(1-x)$.

b Hence find an approximate value of $\ln(0.9)$, giving your answer in the form $\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$.

16 a By first finding suitable derivatives, find the first four terms in the Maclaurin expansion of $\ln(e+x)$.

b Use your expansion to find an approximate value of $\ln(e+1)$. Give your answer in terms of e.

c Find the percentage error in your approximation.

17 Find the Maclaurin series for $\ln(1+x+x^2)$ up to and including the term in x^3.

18 Find the Maclaurin series for $(1+x)\sin 2x$ up to and including the terms in x^4.

19 Find the Maclaurin series, up to and including the term in x^3, for

a $(1+x+x^2)e^x$

b e^{x+x^2}.

20 By finding suitable derivatives, find the Maclaurin series for $\sqrt{\cos x}$ up to an including the terms in x^2.

21 a Write down the first three non-zero terms of the Maclaurin series for $\ln(1-3x)$.

b Hence evaluate $\lim\limits_{x\to 0}\dfrac{\ln(1-3x)}{2x}$.

22 Use Maclaurin series to find $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$.

23 a Find the first three non-zero terms in the Maclaurin expansion of $\tan x$.

b Hence find the Maclaurin series for $e^x \tan x$ up to including the term in x^4.

24 a By using the standard series for $\sin x$ and $\cos x$, find the Maclaurin expansion of $\sin x \cos x$ up to the term in x^3.

b Find the same Maclaurin expansion by using a double angle formula first.

25 a By first using laws of logarithms, find the first two non-zero terms in the Maclaurin series of $\ln\sqrt{\dfrac{1+x}{1-x}}$.

b Use your expansion to find an approximate value of $\ln 3$, giving your answer as a fraction.

26 a Write down the first two non-zero terms of the Maclaurin expansion of $\sin 2x$.

b Hence find an approximation to the positive root of the equation $\sin 2x = 7x^3$.

27 a Find the first three derivatives of $\arcsin x$.

b Hence find the first two non-zero terms in the Maclaurin expansion of $\arcsin x$.

c Write down the first two non-zero terms in the Maclaurin expansion of $\sin 2x$.

d Hence find an approximate value of the positive root of the equation $\arcsin x = \sin 2x$.

28 a Find the Maclaurin expansion of $\ln(e+x)$ up to and including the term in x^3.

b By setting $x = e$ in your expansion, show that $\ln 2 \approx \dfrac{5}{6}$.

29 a Find the first three non-zero terms in the Maclaurin expansion of e^{-x^2}.

b Hence find an approximate value of $\int_0^1 e^{-x^2}\,dx$.

c Find the percentage error in your approximation.

d Compare your answer to part c with the accuracy of the approximation obtained for $\int_0^{0.1} e^{-x^2}\,dx$. Explain your answer.

30 a Find the first three non-zero terms in the Maclaurin expansion of $e^{-x}\tan x$.

b Hence find an approximate value of $\int_0^1 e^{-x}\tan x\,dx$.

c Find $\lim_{x\to 0}\frac{e^{-x}\tan x}{2x}$.

31 a Write down the first two non-zero terms of the binomial expansion of $\frac{1}{\sqrt{1-x^2}}$.

b Hence find the Maclaurin expansion of $\arcsin x$, up to and including the term in x^3.

c By setting $x = \frac{1}{2}$ in your expansion, show that $\pi \approx \frac{25}{8}$.

32 a Show that $(1+x)e^{-2x} \approx 1 - x + \frac{2}{3}x^3$.

b Hence find an approximate value of $\int_0^1 (1+x)e^{-2x}\,dx$.

33 a Write down the first three non-zero terms of the Maclaurin series for $x\cos x$.

b Hence find $\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}$.

34 Use Maclaurin series to find $\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$.

35 a Find the first four non-zero terms in the Maclaurin expansion of $\ln(1+x^2)$.

b Hence find $\lim_{x\to 0}\ln\frac{(1+x^2)}{1-\cos x}$.

36 Find the first three non-zero terms of the Maclaurin series for $\frac{e^{-x}}{\sqrt{1+x}}$.

37 a Use the Maclaurin series for $\ln(1+x)$ to find the first four terms of the Maclaurin series for $\ln(2+x)$.

b Hence find the first four terms of the Maclaurin series for $\ln\left[(2+x)^2(1-x)^3\right]$.

38 a Use the Maclaurin series for $\sin x$ to find the first three non-zero terms in the Maclaurin series for $x\sin x$.

b Hence find $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

39 a Write down the first four non-zero terms in the Maclaurin expansion of $x\ln(1+x)$.

b Hence find $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)} - \frac{1}{x}\right)$.

40 a Write 2^x in the form e^{kx}.

b Hence find the first three terms in the Maclaurin series for 2^x.

c Find $\lim_{x\to 0}\frac{2^x-1}{3^x-1}$.

41 Let $f(x) = \ln(e^x \sec x)$.

a Show that $f'(x) = 1 + \tan x$ and find the next three derivatives of $f(x)$.

b Hence find the Maclaurin series for $f(x)$ up to and including the term in x^4.

c Find $\lim_{x\to 0}\frac{1}{x^3}\ln((1-x)e^x\sec x)$.

42 a Write down the first three non-zero terms in the Maclaurin series for $\sin x$.

b Hence show that the first four non-zero terms in the Maclaurin series for $e^{\sin x}$ are $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4$.

43 a Write down the Maclaurin series, up to an including the term in x^5, of

i $\ln(1+x)$ ii $\arctan x$.

b Hence find the Maclaurin series for $\arctan(\ln(1+x))$ up to including the term in x^5.

44 **a** Write down the coefficient of x^{2n+1} in the Maclaurin series for $\sin x$.

b Hence find the coefficient of x^{2n} in the Maclaurin series for $\frac{\sin x}{x}$.

c Show that $\frac{\sin x}{x} - \cos x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{(2n+1)!}x^{2n}$.

45 **a** Show that the coefficient of x^{2n} in the Maclaurin expansion of e^{2x^2} is $\frac{2^n}{n!}$.

b Find the coefficient of x^{2n} in the Maclaurin expansion of $e^{2x^2} - 2\cos x$.

46 **a** Write down the coefficient of x^n in the Maclaurin expansion of $\ln(1+x)$.

b Hence show that $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}$.

47 The function f(x) has the Maclaurin series starting $3 - \frac{x}{4} + \frac{3x^2}{2} + \ldots$

Find the equation of the tangent of the graph of $y = f(x)$ at $x = 0$.

48 The first three non-zero terms of the Maclaurin series for f(x) are $2 - 4x^2 + \frac{3}{2}x^3$. Show that f(x) has a stationary point at $x = 0$ and determine the coordinates and the nature of this stationary point.

49 The function f(x) has the Maclaurin series

$$f(x) = \sum_{0}^{\infty}\frac{3n+2}{4(2n+1)!}x^n.$$

Find the value of

a f(0) **b** f″(0).

50 Given that $f(x) = xe^x$,

a prove by induction that $f^n(x) = (n+x)e^x$.

b Hence find the first five non-zero terms in the Maclaurin series for f(x).

11E Using Maclaurin series to solve differential equations

Not all differential equations can be solved exactly, but one useful tool for investigating their solution is to see if there is a Maclaurin series solution to the differential equation. These can then be used to find approximate solutions.

When doing this, it is useful to remember the following expressions for derivatives of differential equations.

KEY POINT 11.7

If

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots = \sum_{k=0} a_kx^k$$

then

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \ldots = \sum_{k=0}(k+1)a_{k+1}x^k$$

and

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 \ldots = \sum_{k=0}(k+1)(k+2)a_{k+2}x^k$$

The basic strategy is to substitute these into the equation and compare coefficients to find a relationship between the coefficients of the Maclaurin series. This is sometimes called a **recurrence relation**.

WORKED EXAMPLE 11.12

Find a recurrence relation to describe the coefficients in the Maclaurin series to solve the differential equation $\frac{d^2y}{dx^2} = -y$.

Substituting in the expressions for y and its second derivative from Key Point 11.7

$$\sum_{k=0}(k+1)(k+2)a_{k+2}x^k = -\sum_{k=0}a_kx^k$$

Comparing coefficients of x^k

$$(k+1)(k+2)a_{k+2} = -a_k$$

We can compare coefficients to find a recursion relation between a_{k+2} and a_k

So,

$$a_{k+2} = -\frac{1}{(k+1)(k+2)}a_k$$

With the recurrence relation you could now write down a general solution. For the second order differential equation in Worked Example 11.12 there needs to be two arbitrary constants. We can use a_0 and a_1 as these constants and write everything in terms of them. If you have some initial conditions, then you can use them to find the particular solution.

WORKED EXAMPLE 11.13

Given that $y(0) = 0$ and $y'(0) = 1$ find the particular solution to the differential equation in Worked Example 11.12.

Substituting in the initial conditions

$$y(0) = a_0 = 0$$
$$y'(0) = a_1 = 1$$

Use the recursion relation to find the other coefficients. It is best to just substitute in a few to get a sense of the pattern

$$a_2 = -\frac{1}{2}a_0 = 0$$
$$a_3 = -\frac{1}{6}a_1 = -\frac{1}{6}$$
$$a_4 = \frac{1}{12}a_2 = 0$$
$$a_5 = -\frac{1}{20}a_3 = \frac{1}{120}$$

You can summarize the results. You might recognize this series as being the one for sin x, but usually it will not be a recognizable series

So, for even n, $a_n = 0$

For odd n, $a_n = \frac{(-1)^n}{n!}$

Therefore, $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots$

Tip

To prove your assertion in Worked Example 11.13 you could use induction.

TOOLKIT: Problem Solving

Can you turn the Maclaurin series solution in Worked Example 11.13 into a closed, finite form?

WORKED EXAMPLE 11.14

Find, in terms of a Maclaurin series, the general solution to $\frac{dy}{dx} = x + y$.

Substituting in the expressions for y and its derivative from Key Point 11.7

$$\sum_{k=0}(k+1)a_{k+1}x^k = x + \sum_{k=0}a_k x^k$$

Adding on the x means that two cases need to be considered separately. x^1 terms act slightly differently to all the others

For $k \neq 1$

$$(k+1)a_{k+1} = a_k \quad (*)$$

If $k = 1$, we get:

$$2a_2 = 1 + a_1 \quad (**)$$

For a first order differential equation we need to write everything in terms of one constant. We could make various decisions, but the standard one is to write everything in terms of a_0 To help make this clear we can rename a_0

Let $a_0 = A$

Then from (*):

$$a_1 = a_0 = A$$

From (**)

$$2a_2 = 1 + A$$

$$a_2 = \frac{1+A}{2}$$

Using (*) repeatedly:

$$3a_3 = a_2$$

$$a_3 = \frac{1+A}{6}$$

$$4a_4 = a_3$$

$$a_4 = \frac{1+A}{24}$$

In general:

$$y = A + Ax + \sum_{k=2}\frac{1+A}{k!}x^k$$

Tip

The solution to Worked Example 11.14 looks a little simpler if you set a_1 to be A. You have the freedom to do this, and although the answer looks different it is equivalent.

If you are not interested in the general expression for the solution, but only want the first few terms, there is an easier way. We can take the differential equation differentiate it again to find expressions for the further derivatives which can then be substituted into the general expression for the Maclaurin series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)\ldots$$

This can also be used with non-linear equations:

WORKED EXAMPLE 11.15

a Find the first three terms of the Maclaurin series solution to $y\frac{dy}{dx} = e^{2y}$ with initial condition $y(0) = 1$.

b Hence, estimate the value of $y(0.1)$.

a When $x = 0$:

Substitute in the initial values to find the initial value of $\frac{dy}{dx}$

$$1 \times \frac{dy}{dx} = e^{2\times 1}$$

$$\text{So } \frac{dy}{dx} = e^2$$

Using the product rule and implicit differentiation:

We want an expression for $\frac{d^2y}{dx^2}$. We can differentiate the differential equation with respect to x using the product rule and implicit differentiation:

We choose $u = y$ and $v = \frac{dy}{dx}$

Then $u' = \frac{dy}{dx}$ and $v' = \frac{d^2y}{dx^2}$.

On the right-hand side, we use

$\frac{d}{dx}(e^y) = \frac{dy}{dx}\frac{d}{dy}(e^{2y}) = 2\frac{dy}{dx}e^{2y}$

$$\frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 2\frac{dy}{dx}e^{2y}$$

When $x = 0$:

Substitute in the values of y and $\frac{dy}{dx}$ when $x = 0$ and rearrange to find $\frac{d^2y}{dx^2}$

$$e^2 \times e^2 + 1\frac{d^2y}{dx^2} = 2e^2 \times e^2$$

$$\text{So, } \frac{d^2y}{dx^2} = e^4$$

So, the Maclaurin series solution is

Substitute the values of the derivatives into the Maclaurin series:

$f(0) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)\ldots$

$$y = 1 + e^2x + \frac{e^4}{2}x^2\ldots$$

b When $x = 0.1$:

Substitute $x = 0.1$ into your answer

$$y \approx 1 + 0.1e^2 + \frac{0.01e^4}{2} \approx 2.01$$

LEARNER PROFILE – Reflective

You now have two different ways to solve differential equations approximately – Maclaurin series and Euler's method. Why do we need two methods? Can you think about situations when one method is better than another?

Exercise 11E

For questions 1 to 3, use the method demonstrated in Worked Example 11.12 to find a recurrence relation to describe the coefficients in the Maclaurin series to solve the given differential equations.

1 **a** $\frac{dy}{dx} = y$

b $\frac{dy}{dx} = -2y$

2 **a** $\frac{d^2y}{dx^2} + 4y = 0$

b $\frac{d^2y}{dx^2} - 3y = 0$

3 **a** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

b $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$

For questions 4 to 6, use the method demonstrated in Worked Example 11.15 to find the first three terms of the Maclaurin series solution of the following differential equations.

4 **a** $\frac{dy}{dx} = y + 2,\ y(0) = 1$

b $\frac{dy}{dx} = y^3 + 1,\ y(0) = 1$

5 **a** $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 2$

b $\frac{dy}{dx} = x^2 - y^2,\ y(0) = 3$

6 **a** $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0,\ y(0) = 1,\ y'(0) = 2$

b $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y^2 = 0,\ y(0) = -1,\ y'(0) = 2$

7 Given that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ and $y(0) = 1,\ y'(0) = 1$, find the first three terms of the Maclaurin series for y.

8 **a** Given that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ and $y(0) = 1,\ y'(0) = 2$ find the first three terms of the Maclaurin series for y.

b Hence estimate the value of $y(1)$.

9 **a** Find, in terms of a Maclaurin series, the first three non-zero terms of the Maclaurin series for $\sqrt{1+x^3}$.

b Hence find the first four terms of the Maclaurin series for the general solution to $\frac{dy}{dx} = \sqrt{1+x^3}$.

c If $y(0) = 1$, estimate to five decimal places the value of $y(0.1)$.

10 Find, in terms of a Maclaurin series, the first four terms of the solution of $\frac{d^3y}{dx^3} + y = 0$ with $y(0) = 1,\ y'(0) = 2,\ y''(0) = 4$.

11 **a** Find the first two non-zero terms in the Maclaurin series solution if $y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = 0$.

b Hence estimate the value of $y(0.5)$.

12 Find the first four terms of the Maclaurin series solution to $y'' + y^2 = 0$ if $y(0) = 1$ and $y'(0) = -1$.

13 Find the first four terms of the Maclaurin series solution to

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + y^2 = t$$

if, when $t = 0$, $y = -2$ and $\frac{dy}{dt} = 3$.

14 **a** Write down, using sigma notation, the Maclaurin series of xe^x.

b Find the Maclaurin series for the general solution to the differential equation $\frac{dy}{dx} = xe^x$.

c If $y(0) = 1$, use the first three non-zero terms of the Maclaurin estimate the value of $y(0.5)$.

15 Find the first three terms of the Maclaurin series solution to $\frac{dy}{dx} + e^y = \cos x$ given that $y(0) = 1$.

16 The differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ has initial conditions $y(0) = 0,\ y'(0) = 2$.

a Show that the Maclaurin series coefficients satisfy the recurrence relation

$$a_{k+2} = \frac{4(k+1)a_{k+1} - 4a_k}{(k+2)(k+1)}.$$

b Prove using induction that the Maclaurin series solution is

$$y = \sum_{k=0} \frac{2^k x^{k+1}}{k!}.$$

17 a Find, in the form of sigma notation, the Maclaurin series solution to

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$$

given that $y(0) = 1$ and $y'(0) = 0$.

b Hence find in a closed form (that is, not as a sum) the particular solution to the differential equation.

18 Find the Maclaurin series solution to $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = 0$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$.

19 Find the general Maclaurin series solution to $y'' + xy' + y = 0$.

20 The Legendre differential equation is given by

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

where l is a constant.

Consider a solution of the form $y = \sum_{r=0} a_r x^r$.

a Show that $a_{r+2} = \frac{r(r+1) - l(l+1)}{(r+1)(r+2)}a_r$.

b Find the finite polynomial solution with $y(1) = 1$ if

i $l = 1$ ii $l = 2$.

You are the Researcher

The solutions in **20 b** are called Legendre polynomials. They are used in many situations in physics and they are also an example of orthogonal functions, a very important type of functions which mathematically act a bit like basis vectors.

21 The Hermite polynomial differential equation is given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ky = 0$$

where k is a constant.

Consider a solution of the form $y = \sum_{r=0} a_r x^r$.

a Find a relationship between a_r and a_{r+2}.

b When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$. Find a condition on k for the solution to be a finite polynomial and find the polynomial solutions associated with the two smallest such values of k.

You are the Researcher

The solutions of the type found in **21 b** are called Hermite polynomials. They have many applications from quantum mechanics to probability.

22 The Chebyshev differential equation is given by

$(1 - x^2)f''(x) - xf'(x) + k^2f(x) = 0$.

Consider a solution of the form $y = \sum_{r=0} a_r x^r$.

a Show that $a_{r+2} = \frac{r^2 - k^2}{(r+1)(r+2)}a_r$.

b A Chebyschev polynomial is a polynomial solution to the above differential equation with the property that on the interval $-1 \leqslant x \leqslant 1$ the function varies between -1 and 1. Find the quadratic and cubic Chebyschev polynomials with the greatest possible leading coefficient, under the constraints given.

You are the Researcher

For $|x| \leqslant 1$ the polynomials found in **22 b** can be written as $\cos(k \arccos x)$. Can you find how this differential equation can be linked to trigonometry?

23 a Show that the differential equation

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - y = 0$$

does not have a (non-zero) Maclaurin series solution.

b By considering solutions of the form

$y = x^m(a_0 + a_1x + a_2x^2 \ldots)$,

find a general solution to the equation in part a.

Checklist

- You should know that solutions to differential equations can be approximated using Euler's method with step length h:
  - $x_{n+1} = x_n + h$
  - $y_{n+1} = y_n + h \times f'(x_n, y_n)$
- You should know that differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ can be solved by separation of variables:
  - $\int \frac{1}{g(y)} dy = \int f(x) dx$
- You should know that a homogeneous differential equation can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ and can be solved using the substitution $y = vx$.
- You should know that first order linear differential equations are of the form $\frac{dy}{dx} + P(x)y = Q(x)$. They can be solved by using an integrating factor:
  - $\mu(x) = e^{\int P(x)\,dx}$
  - $\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$
- A Maclaurin series can be used to approximate a function for values of x close to zero. It is given by:
  - $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$
- You should be familiar with the following Maclaurin series, which are given in the formula booklet:
  - $e^x = 1 + x + \frac{x^2}{2!} + \ldots$
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
  - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
  - $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \ldots$ for $p \in \mathbb{Q}$
- You should know other Maclaurin series can be found by substitution, differentiation, integration or multiplying two series.
- You should know Maclaurin series can be used to find approximate solutions to differential equations. This can be done by repeatedly differentiating the differential equation and substituting in values at 0. Alternatively, you can use:
  - $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots = \sum_{k=0} a_k x^k$

  then
  - $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \ldots = \sum_{k=0} (k+1)a_{k+1}x^k$

  and
  - $\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 \ldots = \sum_{k=0} (k+1)(k+2)a_{k+2}x^k$

Mixed Practice

1 **a** Use separation of variables to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 3y\cos 2x$ for $y > 0$.

b Find the particular solution given that $y = 5$ when $x = 0$.

2 Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 3x^2 y$ by separating variables.

3 **a** Find the integrating factor for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 y = 6x^2$.

b Hence find the particular solution of the equation which satisfies $y = 1$ when $x = 0$.

4 The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2xy}{x^2+1} = 1$ can be solved using an integrating factor.

a Show that the integrating factor is $\frac{1}{(x^2+1)}$.

b Hence find the general solution of the equation.

5 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \mathrm{e}^x$ with initial condition $y(0) = 2$.
Use Euler's method with step length 0.1 to estimate the value of $y(2)$.

6 Find the first three non-zero terms in the Maclaurin expansion of $(1-x)^2 \sin x$.

7 **a** Write down the first three terms of the Maclaurin series for e^x.

b Hence show that $\sqrt[3]{\mathrm{e}} \approx \frac{25}{18}$.

8 **a** Write down the first three non-zero terms of the Maclaurin series for $\arctan 2x$.

b Hence find $\lim_{x\to 0} \frac{2x - \arctan 2x}{x^3}$.

9 **a** Find the first two terms of the Maclaurin series of $\mathrm{e}^{(x^2)}$.

b Hence find the first four non-zero terms of the general Maclaurin series solution to $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{e}^{(x^2)}$.

10 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \cos^2 x$, given that $y = 2$ when $x = 0$.

a Use Euler's method with a step length of 0.1 to find an approximation to the value of y when $x = 0\ 3$.

b **i** Show that the integrating factor for solving the differential equation is $\sec x$.
ii Hence solve the differential equation, giving your answer in the form $y = \mathrm{f}(x)$.

Mathematics HL May 2013 Paper 3 Q2

11 The rate of change of mass (R) of a radioactive substance is proportional to the amount of substance remaining. This can be written as the differential equation
$\frac{\mathrm{d}R}{\mathrm{d}t} = -kR$.

a Solve this differential equation, given that the initial mass of the substance is R_0.

b Find the time taken for the mass of the substance to halve from its initial mass.

c What does the fact that your answer to part **b** is independent of R_0 tell you?

12 **a** Use Euler's method with step length 0.25 to sketch the solution in $0 < x < 10$ to $\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{y} = \sqrt{x}$ with initial condition $y(0) = 0$.

b Hence estimate, to one decimal place, the minimum value of x on this curve.

13 Consider the differential equation $\frac{dy}{dx} = e^{y-x}$ with the initial condition $y(0) = -1$.

a Use Euler's method with step length 0.1 to estimate the value of $y(2)$.

b Solve the equation exactly to find the error in your estimate in part **a**.

c How could you decrease the error in your estimate?

14 Consider the differential equation $\frac{dy}{dx} - 2xy = e^{x^2}$ with the initial condition $y(0) = -1$.

a Use Euler's method with step length 0.05 to estimate the value of $y(1)$.

b Solve the equation exactly to find the error in your estimate in part **a**.

15 Find the particular solution of the differential equation $\cos^2 x \frac{dy}{dx} + y = 1$ for which $y = 3$ when $x = 0$.

16 Given that $(x^2+1)\frac{dy}{dx} = 2(y^2+1)$ and that $y = 0$ when $x = 0$, show that $y = \frac{2x}{1-x^2}$.

17 Find the particular solution of the differential equation $\frac{dy}{dx} - 4xy = e^{2x^2}$ given that $y = 4$ when $x = 0$.

18 Two variables satisfy the differential equation $\frac{dy}{dx} = \frac{3y}{x^2}$. When $x = 1$, $y = 2$.

a Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.3$. Give your answer to two decimal places.

b Solve the differential equation.

c Hence find the percentage error in your approximation from part **a**.

d How can the accuracy of your approximation be improved?

19 **a** Find the particular solution of the differential equation $(x-1)\frac{dy}{dx} = \cos^2 y$ given that $y = 0$ when $x = 0$.

b Find the percentage error when the value of y at $x = 0.5$ is approximated using Euler's method with step length 0.1.

20 The growth rate of a population of insects depends on its current size but also varies according to the time of year. This can be modelled by the differential equation $\frac{dN}{dt} = 0.2N\left(1 + 2\sin\left(\frac{\pi t}{6}\right)\right)$, where N thousand is the population size and t is the time in months since the measurements began. The initial population is 2000. Solve the differential equation to find the size of the population at time t.

21 **a** Write the differential equation $\frac{dy}{dx} = 2x(1+x^2-y)$ in the form $\frac{dy}{dx} + \mathrm{P}(x)y = \mathrm{Q}(x)$.

b Hence find the general solution of the equation.

22 **a** Show that $x^2\frac{dy}{dx} = 3xy + 2y^2$ is a homogeneous differential equation.

b Find the particular solution of the equation which satisfies $y = 4$ when $x = 2$.

23 Find the term in x^2 in the Maclaurin expansion of $e^{2x}(1-2x)^{\frac{1}{3}}$.

24 Use the first three terms of the binomial expansion of $\frac{1}{1-x}$ to find the first three non-zero terms of the Maclaurin series for $\ln(1-x)$.

25 Find the term in x^4 in the Maclaurin expansion of $\frac{\cos x}{\sqrt{1-x^2}}$.

26 **a** Given that $y = \arcsin x$, write down $\frac{dy}{dx}$ and find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dy^3}$.

b Hence find the Maclaurin series for $\arcsin x$ up to an including the term in x^3.

c Find $\lim_{x\to 0} \frac{\arcsin x - \sin x}{x^3}$.

27 Find the first three non-zero terms of the Maclaurin series solution to $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y^2 = x$, given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 4$.

28 Find the first three terms of the Maclaurin series solution to $\frac{dy}{dx} + xe^y = x + 2$, given that $y(0) = 1$.

29 A differential equation is given by $\frac{dy}{dx} = \frac{y}{x}$, where $x > 0$ and $y > 0$.

a Solve this differential equation by separating the variable, giving your answer in the form $y = f(x)$.

b Solve the same differential equation by using the standard homogeneous substitution $y = vx$.

c Solve the same differential equation by the use of an integrating factor.

d If $y = 20$ when $x = 2$, find y when $x = 5$.

Mathematics HL November 2012 Paper 3 Q1

30 Consider the differential equation $y\frac{dy}{dx} = \cos 2x$.

a **i** Show that the function $y = \cos x + \sin x$ satisfies the differential equation.

ii Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

iii For which value of the constant of integration does your solution coincide with the function given in part **i**?

b A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve C.

i Determine the equation of C in the form $y = g(x)$, and state the range of the function g.

A region R in the xy-plane is bounded by C, the x-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

ii Find the area of R.

iii Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians.

Mathematics HL May 2013 Paper 2 TZ2 Q12

31 Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ that has $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$, simplified as far as possible.

32 **a** Show that $x\frac{dy}{dx} = y + \frac{x}{\ln y - \ln x}$ is a homogeneous differential equation.

b Given that $y = 1$ when $x = 1$, solve the differential equation, giving your answer in the form $y\left(\ln\left(\frac{y}{x}\right) - 1\right) = f(x)$.

33 **a** Write $\frac{1+v}{9-v^2}$ in partial fractions.

b Show that $\frac{dy}{dx} = \frac{y}{x} + \frac{9x+y}{x+y}$ is a homogeneous differential equation.

c Use a suitable substitution to show that $(y-3x)(y^2-9x^2)$ is constant.

34 Use the substitution $z = 2x - 3y$ to find the particular solution of the differential equation $(2x-3y+3)\frac{dy}{dx} = 2x - 3y + 1$ for which $y = 1$ when $x = 1$. Give your answer in the form $Ae^{y-x} = f(x, y)$.

35 Use the substitution $y = u^2$ to find the solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \sqrt{\frac{y}{x}}$, given that $x \geqslant 1$ and that $y = 0$ when $x = 1$.

36 Use Maclaurin series to show that, for small values of x, $\ln\left(\frac{6+5x}{6-5x}\right) \approx \frac{5}{3}x$.

37 **a** Find the first three derivatives of $\ln(\sin x + \cos x)$.

b Use Maclaurin series to show that $\lim_{x\to 0} \frac{\ln(\sin x + \cos x)}{\arctan 2x} = \frac{1}{2}$.

38 Let $f(x) = e^{\sin x}$.

a Write down $f'(x)$ and show that $f''(x) = (\cos^2 x - \sin x)e^{\sin x}$.

b Given that $f'''(0) = 0$ and $f^{(4)}(0) = -3$, Find the Maclaurin expansion of $f(x)$ up to and including the term in x^4.

c Use your expansion from part **b** to find $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x^3}$.

39 Let $g(x) = \sin x^2$, where $x \in \mathbb{R}$.

a Using the result $\lim_{t\to 0} \frac{\sin t}{t} = l$, or otherwise, calculate $\lim_{x\to 0} \frac{g(2x - g(3x))}{4x^2}$.

b Use the Maclaurin series of $\sin x$ to show that $g(x) = \sum_{n=0}^{\infty} (-1)^x \frac{x^{4n+2}}{(2n+1)!}$.

c Hence determine the minimum number of terms of the expansion of $g(x)$ required to approximate the value of $\int_0^1 g(x)dx$ to four decimal places.

Mathematics HL November 2013 Paper 3 Q4

40 Find the Maclaurin series solution to $y'' - 2xy' + y = 0$ given that $y(0) = 0$ and $y'(0) = 1$.

Analysis and approaches HL:

Practice Paper 1

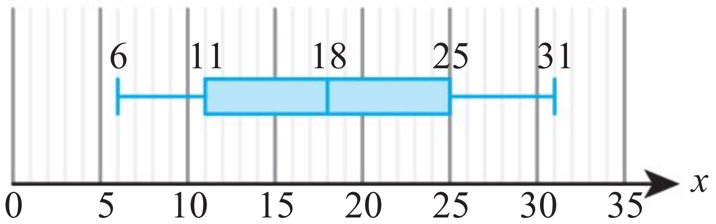
Non-calculator.
2 hours, maximum mark for the paper [110 marks].

Section A

1 Find the equation of the normal to the graph of $y = \arctan(3x)$ at the point where $x = \frac{1}{3}$.

[5]

2 The box plot shows the distribution of lengths of leaves, in cm, of a certain plant.



a Given that a leaf is longer than 11 cm, find the probability that it is longer than 25 cm.

b Five leaves are selected at random. Find the probability that exactly two of them have length between 11 cm and 25 cm.

[4]

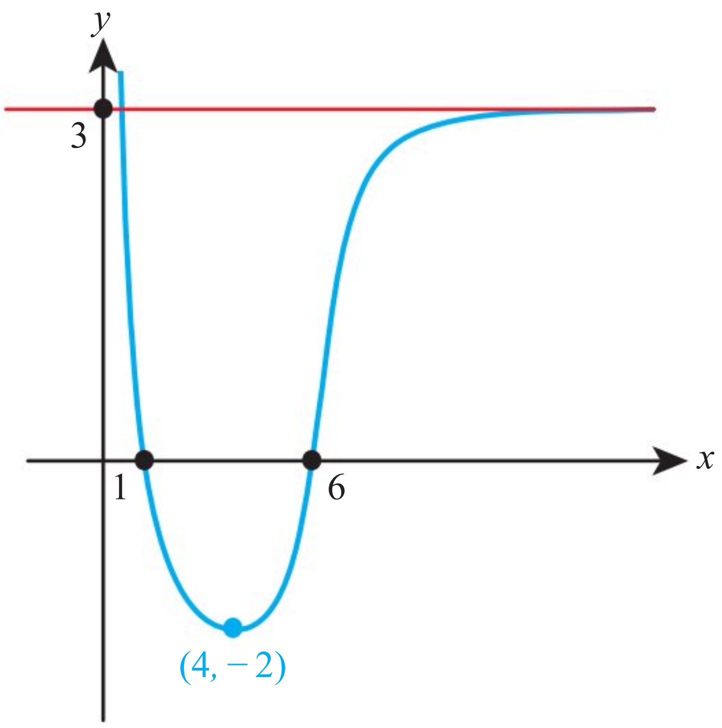
3 Given that $z = 1 + 2\mathrm{i}$ and $w = 2 - \mathrm{i}$,

a calculate $\frac{z}{w^*}$

b find the real values of p and q such that $pz + qw = \mathrm{i}$.

[6]

4 The graph of $y = \mathrm{f}(x)$ is shown in the diagram.



On separate diagrams, sketch the graphs of

a $y = |\mathrm{f}(x)|$ **b** $y = \frac{1}{\mathrm{f}(x)}$.

[6]

5 If $f(x) = \dfrac{6}{5-\sqrt{x}}$ and $g(x) = \ln(x+7)$, find the exact solution of $(g \circ f)^{-1}(x) = 4$.

[5]

6 **a** Express $\dfrac{x+3}{x^2-1}$ in partial fractions.

b Hence find the exact value of $\displaystyle\int_5^7 \frac{x+3}{x^2-1}\,dx$, giving your answer in the form $\ln\left(\frac{a}{b}\right)$, where $a, b \in \mathbb{N}$.

[7]

7 **a** Let $t = \tan x$. Write down an expression for $\tan(2x)$ in terms of t.

b Find the two solution of the equation $\tan(2x) = 1$ with $x \in (0, \pi)$.

c Hence find the exact value of $\tan\left(\frac{\pi}{8}\right) + \tan\left(\frac{5\pi}{8}\right)$.

[7]

8 Prove by induction that $13^n - 7^n - 6^n$ is divisible by 7 for all $n \in \mathbb{N}$.

[7]

9 Find the coordinates of the stationary points on the curve with equation $y^3 + x^2 + 2xy = 0$.

[7]

Section B

10 *[18 marks]*

Three planes are given by their Cartesian equations

Π_1: $x + 2y - z = 15$

Π_2: $x + y + kz = a$

Π_3: $3x + 4y + 5z = a^2$

a Given that Π_1 and Π_2 are perpendicular, find the value of k.

[2]

b **i** Show that, for all values of a, the three planes do not intersect at a single point.

ii There are two values of a for which the three planes intersect along a line. Show that one of those values is 5 and find the other one.

iii In the case $a = 5$ find the equation of the line of intersection of the three planes.

[9]

c Line L has equation $\mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-6\end{pmatrix}$. It intersects plane Π_1 at A and Π_2 at B.

Given that $a = 5$,

i find the distance AB

ii find the sine of the acute angle between L and Π_3.

[7]

11 *[18 marks]*

A discrete random variable X_t has the probability distribution given by

$$P(X_t = k) = \frac{At^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

a Use the Maclaurin series for e^x to show that $A = e^{-t}$.

[2]

b Write down the value of $P(X_t = 0)$.

[1]

A continuous random variable T takes values $t > 0$ and satisfies $P(T > t) = P(X_t = 0)$ for $t > 0$.

c **i** Write down the function $F(t)$ such that $F(t) = P(T < t)$.

ii Hence show that the probability density function of T is

$$f(t) = \begin{cases} 0 & \text{for } t \leqslant 0 \\ e^{-t} & \text{for } t > 0 \end{cases}$$ [4]

d The random variable T is used to model the time, in minutes, between customers arriving at a shop. Find

i the probability that more than 3 minutes pass between two successive customers arriving

ii the median time between two successive customers.

[5]

e The model for the time between successive customers is modified to take into account the fact that the gap is never longer than 5 minutes. The new random variable modelling the time between two customers, M minutes, has the probability density function

$$g(t) = \begin{cases} Cf(t) & \text{for } 0 < t \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

i Find the value of C.

ii Find the mean time between two successive customers according to the new model.

[6]

12 *[19 marks]*

Define $f(x) = \dfrac{\sin x}{\sin x + \cos x}$ for $0 \leqslant x \leqslant 2\pi$.

a **i** Write $\sin x + \cos x$ in the form $R\sin(x + \theta)$ where $R > 0$ and $\theta \in \left(0, \dfrac{\pi}{2}\right)$.

ii Hence find the equations of the vertical asymptotes of the graph of $y = f(x)$.

iii Find $f'(x)$ and hence show that $f(x)$ has no stationary points.

iv Sketch the graph of $y = f(x)$.

[8]

b Define $S = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x}\,dx$ and $C = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin x + \cos x}\,dx$.

i Evaluate $C + S$ and show that $C - S = \dfrac{1}{2}\ln 2$.

ii Hence find the area bounded by the graph of $y = f(x)$, the x-axis and the line $x = \dfrac{\pi}{4}$.

[11]

Analysis and approaches HL:

Practice Paper 2

Calculator.
2 hours, maximum mark for the paper [110 marks].

Section A

1 Starting on his 18th birthday, Morgan puts \$100 into a savings account earning 4% interest annually. On each birthday he adds \$100 to his account.
u_n is the amount in dollars in the account on his nth birthday, so $u_{18} = 100$ and $u_{19} = 204$.

a Write an expression for u_{n+1} in terms of u_n.

b On what birthday will Morgan first have more than \$2000 in his account.

c What percentage of the amount in the account at the birthday in part **b** is due to interest?

[6]

2 If $x = \log_{10} a$ and $y = \log_{10} b$, determine an expression in terms of x and y for

a $\log_{10} (100ab^2)$

b $\log_b a$.

[5]

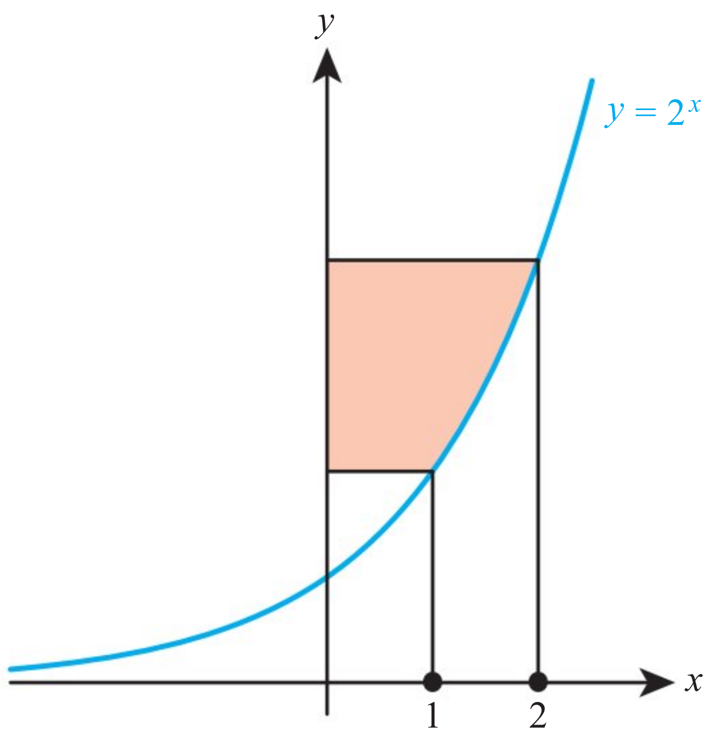
3 An arithmetic sequence has fifth term 5 and tenth term −15.

a Find the first term and the common difference.

b If the sum of the first n terms is n, find the value of n.

[5]

4 The diagram shows the area enclosed by $y = 2^x$, the lines $y = 2$ and $y = 4$ and the y-axis.



Find the volume when the shaded area is rotated 2π radians around the y-axis.

[5]

5 The function f(*x*) is defined as

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & x < 0 \\ 1 + bx & x > 0 \end{cases}$$

This function is continuous at $x = 0$. Use L'Hôpital's rule to find the value of b which makes the function differentiable at $x = 0$.

[6]

6 An air traffic control tower is situated at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

At 12 noon, the controller observes the trajectories of two planes.

Plane A follows the path $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Plane B follows the path $\mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 22 \end{pmatrix} + t\begin{pmatrix} 1.5 \\ 1.5 \\ -2 \end{pmatrix}$.

Where t is the time in minutes and the units are in kilometres.

Assume that the two planes maintain the same trajectory.

a Find the speed of plane A.

b Show that the two planes both pass through the same point.

c Show that the two planes do not crash.

d If the two planes will come within 5 km of each other, the controller issues an alert. Determine if the controller should issue an alert.

[9]

7 **a** Given that $x > 0$, solve $\ln x > \dfrac{1}{x}$.

b Given that $x > 0$ and $k > 0$ solve $\ln x + \ln k > \dfrac{1}{kx}$ giving your answer for x in terms of k.

[5]

8 A football team contains five girls and six boys. They stand in a straight line for a photo. The photographer requires that no two people of the same gender can stand next to each other.

a How many different possible arrangements are there?

b If one of the arrangements from part **a** is selected at random, find the probability that

i Alessia (a girl) is in the middle spot

ii Daniel (a boy) and Theo (a boy) are at the ends of the line.

[7]

9 If A, B and C are angles in a non-right-angled triangle, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

[7]

Section B

10 *[17 marks]*

Two schools, Jomer Tree and St Atistics, compete in a race.

Within each school, the times taken to complete the race are thought to be normally distributed.

30% of students from Jomer Tree take less than 10 minutes to complete the race. 40% of students from Jomer Tree take more than 12 minutes.

a Use these results to estimate the mean and standard deviation of the times for students from Jomer Tree.

[4]

Students who take less than 10 minutes score 5 points. Students who take between 10 minutes and 12 minutes score 3 points. Students who take more than 12 minutes score 0 points.

b Find the expectation and standard deviation in the number of points scored by students from Jomer Tree.

[4]

Jomer Tree enters two students in the race. The outcome for each student is independent of the outcome of any other student.

c Find the probability that Jomer Tree scores less than 6 points in total.

[2]

The times for students from St Atistics are also thought to be normally distributed. The mean is 11 minutes and the standard deviation is 3 minutes. In the race, St Atistics enters one student.

d If a student scores 0 points, find the probability that the student comes from Jomer Tree.

[3]

e The team with most points wins. Find the probability that Jomer Tree wins.

[4]

11 *[18 marks]*

a For the differential equation $(1+x)\dfrac{dy}{dx} = e^x - y$,

i show that an integrating factor is $(1+x)$

ii find the particular solution of the equation given that $y = 2$ when $x = 0$.

[6]

b Now consider the differential equation $(1+x)\dfrac{dy}{dx} = e^x - y^2$ with $y = 2$ when $x = 0$.

i Show that $(1+x)\dfrac{d^2y}{dx^2} = e^x - (1+2y)\dfrac{dy}{dx}$ and find a similar expression for $(1+x)\dfrac{d^3y}{dx^3}$.

ii Find the values of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3x}{dx^3}$ when $x = 0$.

iii Hence write the solution of the equation as a Maclaurin series, up to an including the term in x^3.

[8]

c For the differential equation $(1+x)^2\dfrac{dy}{dx} = e^x - y^2$, use Euler's method with step length 0.1 to estimate the value of y when $x = 0.3$, given that $y = 2$ when $x = 0$. Give your answer to two decimal places.

[4]

12 *[20 marks]*

a Let $z = \cos\theta + \mathrm{i}\sin\theta$.

i Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

ii Hence show that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

iii Find a similar expression for $32\sin^6\theta$.

[9]

b i Show that the first three non-zero terms in the Maclaurin expansion of $\cos^6 x$ are $1 - 3x^2 + 4x^4$.

ii Hence find the first three non-zero terms in the Maclaurin expansion of $\mathrm{e}^{1-\cos^6 x}$.

[7]

c Find the exact value of $\displaystyle\int_0^{n\pi} (\sin^6 x + \cos^6 x)\,\mathrm{d}x$.

[4]

Guidance for Paper 3

Paper 3 is a new type of examination to the IB. It will include long, problem-solving questions. Do not be intimidated by these questions – they look unfamiliar, but they will be structured to guide you through the process. Although each question will look very different, it might help you to think about how these questions are likely to work:

- There might be some 'data collection', which will likely involve working with your calculator or simple cases to generate some ideas.
- There might be a conjecturing phase where you reflect on your data and suggest a general rule.
- There might be a learning phase where you practise a technique on a specific case.
- There might be a proving phase where you try to form a proof. It is likely that the method for this proof will be related to the techniques suggest earlier in the question.
- There might be an extension phase where you apply the introduced ideas to a slightly different situation.

All of these phases have their own challenges, so it is not always the case that questions get harder as you go on (although there might be a trend in that direction). Do not assume that just because you could not do one part you should give up – there might be later parts that you can still do. Some parts might look unfamiliar, and it is easy to panic and think that you have just not been taught about something. However, one of the skills being tested is mathematical comprehension so it is possible that a new idea is being introduced. Stay calm, read the information carefully and be confident that you do have the tools required to answer the question, it might just be hidden in a new context.

You are likely to have a lot of data so be very systematic in how you record it. This will help you to spot patterns. Then when you are suggesting general rules, always go back to the specific cases and check that your suggestion works for them.

These questions are meant to be interlinked, so if you are stuck on one part try to look back for inspiration. This might be looking at the answers you have found, or it might be trying to reuse a method suggested in an earlier part. Similarly, even more than in other examinations, it is vital in Paper 3 that you read the whole question. Sometimes later parts will clarify how far you need to go in earlier parts, or give you ideas about what types of method are useful in the question.

These questions are meant to model the thinking process of mathematicians. Perhaps the best way to get better at them is to imitate the mathematical process at every opportunity. So the next time you do a question, see if you can spot underlying patterns, generalize them and then prove your conjecture. The more you do this, the better you will become.

Analysis and approaches HL:

Practice Paper 3

Calculator.
1 hours, maximum mark for the paper [55 marks].

1 *[30 marks]*

This question is about finding the number of solutions to equations, without necessarily finding what those solutions are.

a i Sketch the graph $y = x^2 - 2x + 1$.

ii Add the line $y = 3x$ to your sketch. How many solutions are there to the equation $x^2 - 2x + 1 = 3x$?

iii Use a discriminant method to find the values of a for which to the equation $x^2 - 2x + 1 = ax$ has zero, one or two distinct solutions.

iv In the two situations in part **iii** where there is only one solution, sketch the graphs of $y = x^2 - 2x + 1$ and $y = ax$. What is the geometric relationship between the two graphs in each sketch?

[9]

b Consider the curve with equation $y = x^3 - bx + 1$. Sketch the curve in the situation when

i $b = -1$

ii $b = 1$

iii $b = 3$.

iv Show that the curve only has a local minimum point if b is positive and find the coordinates of the local minimum point in that case.

v Find the exact value of b that results in the equation $x^3 - bx + 1 = 0$ having two distinct solutions.

[9]

c Sketch $y = \ln x$ and $y = cx$ in the situation

i $c = -1$

ii $c = 1$

iii $c = 0.2$.

iv Find the equation of the tangent to the curve $y = \ln x$ at the point $x = p$ for $p > 0$.

v Find the value of p such that the tangent passes through the origin.

vi Hence find the values of k such that the equation $\ln x = kx$ has exactly one solution.

[8]

d Find, accurate to four decimal places, the value of d such that the equation $\sin x = dx$ has exactly five solutions.

[4]

2 *[25 marks]*

Cauchy wants to write a computer program to solve a mathematics game. He needs to find all possible numbers that can be made using 4 numbers and basic arithmetic operations.

This question is about investigating two sequences involving expressions with paired parentheses and using links between them to generate a formula to see how feasible this program might be.

a A_n is the number of possible expressions with n pairs of parentheses around $n + 1$ consecutive letters being summed, such that each addition sign is nested in one set of parentheses.

For example, when $n = 1$ there is only possible expression: $(a + b)$.

So $A_1 = 1$.

When $n = 2$ one possible expression is $(a + (b + c))$.

The expression $((a + b + c))$ would not be allowed because there are parentheses containing two addition signs.

The expression $a + ((b + c))$would not be allowed because there are two sets of parentheses around one addition sign.

i Show that $A_2 = 2$. **ii** Find the value of A_3.

[4]

b B_n is the number of correct expressions with n pairs of parentheses around $n + 1$ letters being summed, such that each addition sign is nested in one set of parentheses. However, in this situation, the letters do not need to be in order.

For example, when $n = 1$ the possible expressions are $(a + b)$ and $(b + a)$, so $B_1 = 2$.

i Show that $B_2 = 12$.

ii Find and justify a relationship between A_n and B_n. Hence find B_3.

[6]

c **i** Consider an expression of the form $(X + Y)$.

Show that there are four ways to add a term, Z, to this expression whilst following the rules for expressions defined in part **b** and keeping the new term within the original set of brackets.

ii Hence explain why $B_{n+1} = (4n + 2)B_n$.

[4]

d Prove by induction that $A_n = \frac{1}{n+1}\,{}^{2n}\mathrm{C}_n$.

[8]

e A computer program wants to find all the possible calculations that can be done with the numbers 1, 2, 3 and 4 and the operations $+$, $-$, $\div$ and $\times$.

For example, $(1 \times (2 + (4 + 3))$ would count as one calculation. $(1 \times ((4 + 3) + 2)$ would count as a different calculation for the purpose of this automated process.

How many different calculations are possible?

[3]

You are the Researcher

The sequence A_n in question 2 is called the Catalan numbers and it has a huge number of applications. This question focused on the 'paired parentheses' interpretation suggested by the Belgian mathematician Eugene Catalan (1814–1894). However, it is possible to reinterpret this as a voting problem, cutting polygons into triangles, constructing mountain ranges, walking around a constrained grid and many others. It has another beautiful recursion relation which arises naturally in some of these situations:

$$A_{n+1} = \sum_{k=0}^{k=n} A_k A_{n-k}$$

The idea of two different looking problems actually having the same underlying structure is called an isomorphism, which is a fundamental technique in advanced mathematics leading, in particular, to an area called group theory.

Answers

Chapter 1 Prior Knowledge

1 a 120 b 35

2 a $\frac{1}{12}$ b $\frac{7}{12}$

Exercise 1A

1 a 40 b 28

2 a 13 b 11

3 a 280 b 20

4 a 362 880 b 720

5 a 2880 b 2 177 280

6 a 384 b 47 520

7 a 126 b 56

8 a 525 b 5880

9 a 550 b 266

10 a 15 120 b 336

11 a 25 200 b 101 606 400

12 a 9240 b 157 920

13 55

14 28

15 a 1.05×10^{10} b 223 781 030

16 a 240 b 88 c 118

17 86 400

18 a 5040 b 720

19 a 40 320 b 1440

20 210

21 a 35 b 15

22 3024

23 336

24 272

25 a 216 b 180

26 a 120 b 72

27 29 610 360

28 20 160

29 1.37×10^{26}

30 a 61 880 b 21 216

31 84

32 611 520

33 228 009 600

34 a 3003 b i 1001 ii 2982 c 0.993

35 21

36 12

38 a 32 b 81

39 6.40×10^{15}

40 a 120 b 210

41 31

Exercise 1B

1 725 760

2 30 240

3 600

4 4920

5 460

6 4896

7 a 560 b 2180

8 3720

9 186

10 4 263 402

11 87 091 200

12 3 592 512 000

13 $\frac{1}{126}$

14 a 48 b 240 c $\frac{2}{3}$

15 a $\frac{33}{66640}$ b $\frac{253}{9996}$ c $\frac{2062}{2499}$

16 10 800

17 4802

18 270 200

Chapter 1 Mixed Practice

1 336

2 180 835 200

3 75 075

4 $\frac{1}{120}$

5 241 920

6 a 27 132 b $\frac{5}{57}$

7 729

8 44 100

9 210

10 2400

11 95 680

12 50 232

13 a 5005
b i 3003 ii 4165
c 0.832

14 240

15 a 5040 b 720
c 1440

16 a 48 b 72
c 42

17 a $\frac{1}{15890700}$ b $\frac{347}{19740}$

18 20

19 a 144 b 144

20 a 34 b 81

21 15 120

22 a 151 200
b 33 600

23 a 462
b 5775

Chapter 2 Prior Knowledge

1 $16-96x+216x^2-216x^3+81x^4$

2 $(5x-2)(x+3)$

3 $\frac{3x+5}{x^2+2x-3}$

4 $x=5,\ y=-2$

Exercise 2A

1 a $1-2x+3x^2+\ldots,|x|<1$
b $1-3x+6x^2+\ldots,|x|<1$

2 a $1+\frac{1}{3}x-\frac{1}{9}x^2+\ldots,|x|<1$
b $1+\frac{1}{4}x-\frac{3}{32}x^2+\ldots,|x|<1$

3 a $1+\frac{1}{4}x+\frac{1}{16}x^2+\ldots,|x|<4$
b $1-2x+\frac{5}{2}x^2+\ldots,|x|<2$

4 a $1-x+\frac{3}{2}x^2+\ldots,|x|<\frac{1}{2}$
b $1+2x+5x^2+\ldots,|x|<\frac{1}{3}$

5 a $\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2+\ldots,|x|<3$
b $\frac{1}{25}-\frac{2}{125}x+\frac{3}{625}x^2+\ldots,|x|<5$

6 a $2+\frac{1}{12}x-\frac{1}{288}x^2+\ldots,|x|<8$
b $3+\frac{1}{6}x-\frac{1}{216}x^2+\ldots,|x|<9$

7 a $\frac{1}{8}+\frac{9}{16}x+\frac{27}{16}x^2+\ldots,|x|<\frac{2}{3}$
b $\frac{1}{3}-\frac{4}{9}x+\frac{16}{27}x^2+\ldots,|x|<\frac{3}{4}$

8 a $128+7x+\frac{21}{256}x^2+\ldots,|x|<32$
b $8-x+\frac{1}{48}x^2+\ldots,|x|<12$

9 $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3+\ldots$

10 $1+\frac{3}{4}x+\frac{3}{8}x^2+\frac{5}{32}x^3+\ldots$

11 $2-\frac{1}{12}x+\frac{1}{144}x^2+\ldots$

12 $\frac{1}{2}+\frac{5}{4}x+\frac{25}{8}x^2+\ldots$

13 a $3+\frac{1}{6}x-\frac{1}{216}x^2+\ldots$
b $|x|<9$
c 3.162 04

14 a $2-\frac{1}{4}x-\frac{1}{32}x^2+\dots$

b $|x|<\frac{8}{3}$

c 1.718 75

15 a $x-6x^2+27x^3+\dots$

b $|x|<\frac{1}{3}$

16 $1+2x+2x^2+\dots$

17 $-\frac{21}{4}$

18 5

19 a $1-2x-2x^2-4x^3+\dots$

b $|x|<\frac{1}{4}$

c 4.795 84

20 4

21 a $1+x+\frac{2}{3}x^3+\dots$

b 2.080 09

22 –15

23 $\frac{8}{3}$

24 126

Exercise 2B

1 a $\frac{1}{x+1}+\frac{2}{x+2}$ b $\frac{2}{x+3}-\frac{1}{x-2}$

2 a $\frac{3}{2(x+5)}-\frac{1}{2(x-1)}$ b $\frac{2}{3(x-2)}+\frac{4}{3(x-5)}$

3 a $\frac{2}{x}+\frac{5}{x-3}$ b $\frac{3}{x}-\frac{2}{x+4}$

4 a $\frac{1}{2x-1}+\frac{1}{2x+3}$ b $\frac{2}{3x-2}-\frac{1}{2x+5}$

5 $A=2, B=3$

6 $A=3, B=-4$

7 $\frac{1}{4(x+8)}+\frac{3}{4(x-4)}$

8 $\frac{3}{x+2}-\frac{5}{2x}$

9 $\frac{2}{x+3}-\frac{6}{4x+5}$

10 $\frac{2}{x-3}+\frac{5}{x+6}$

11 $\frac{1}{2(3x-1)}-\frac{1}{2(3x+1)}$

12 $\frac{2}{3x}-\frac{1}{x+4}$

13 $\frac{2}{x}+\frac{1}{x-a}$

14 $\frac{2}{\sqrt{x}+1}-\frac{2}{\sqrt{x}+3}$

15 $\frac{1}{3(x^2+2)}-\frac{1}{3(x^2+5)}$

16 a $A=3, B=-2$ b $2-\frac{7}{2}x+\frac{11}{4}x^2+\dots$

c $|x|<1$

17 a $\frac{1}{1-x}-\frac{4}{2+3x}$ b $-1+4x-\frac{7}{2}x^2+\dots$

c $|x|<\frac{2}{3}$

18 a $1-3x+7x^2+\dots$ b $|x|<\frac{1}{2}$

19 a $3-x+11x^2+\dots$ b $|x|<\frac{1}{3}$

20 $\frac{1}{x-2a}-\frac{1}{x-a}$

Exercise 2C

1 a $x=5, y=-2$ b $x=\frac{34}{23}, y=-\frac{26}{23}$

2 a $x=-\frac{5}{19}, y=-\frac{65}{38}$ b $x=-3, y=4$

3 a $x=\frac{17}{37}, y=\frac{16}{37}, z=\frac{39}{37}$

b $x=-11, y=64, z=88$

4 a $x=-2, y=1, z=0$

b $x=\frac{2}{5}, y=-\frac{1}{7}, z=\frac{52}{35}$

5 a $x=3, y=2, z=2$ b $x=1, y,=1, z=4$

6 a $x=\frac{1}{6}, y=\frac{1}{6}, z=\frac{7}{6}$

b $x=\frac{1}{2}, y=4, z=\frac{1}{2}$

7 a No solution b No solution

8 a $x=2-\lambda, y=\lambda+1, z=\lambda$

b $x=2\lambda+1, y=\lambda-1, z=\lambda$

9 a $x=\lambda+3, y=\lambda, z=1$

b $x=2\lambda+1, y=\lambda, z=0$

10 a $x=2\mu-\lambda+5, y=\mu, z=\lambda$

b $x=4\mu+3\lambda+2, y=\mu, z=\lambda$

11 $x=2, y=-1, z=3$

12 $x=5-2\lambda, y=\lambda-2, z=\lambda$

13 a $\begin{cases} a+b+c=4 \\ 9a+3b+c=15 \\ 16a+4b+c=25 \end{cases}$

b $a=2, b=-3, c=5$

14 a $\begin{cases} -a+b-c=7 \\ 8a+4b+2c=4 \\ 27a+9b+3c=3 \end{cases}$

b $a=-1,\ b=4,\ c=-2$

15 a $k=-3$

b $x=2,\ y=-1,\ z=4$

16 a Proof

b $x=2.1,\ y=-1.7,\ z=1.8$

17 a $a=-2$

b $x=\lambda+0.4,\ y=\lambda,\ z=-0.8$

18 $k=2$ or -1

19 5

20 $k=2$ or 7

21 a $k\neq 1$ b $k=1$ and $c=\frac{4}{7}$

c $k=1$ and $c\neq\frac{4}{7}$

22 754

Chapter 2 Mixed Practice

1 $1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3+\dots$

2 a $\frac{1}{3}-\frac{2}{27}x+\frac{2}{81}x^2+\dots$

b $|x|<\frac{9}{4}$

3 a $a=3,\ n=-2$

b $1-6x+27x^2-108x^3+\dots$

4 $A=-1,\ B=4$

5 $\frac{3}{2(3x-4)}-\frac{1}{2(x+2)}$

6 $\frac{2}{x-5}-\frac{3}{x+6}$

7 $x=-3,\ y=2,\ z=4$

8 $x=2\lambda-8,\ y=\lambda-4,\ z=\lambda$

9 $x=-1-4\lambda,\ y=1,\ z=\lambda$

10 a $\begin{cases} 4a-2b+c=12 \\ a-b+c=1 \\ a+b+c=-3 \end{cases}$

b $a=3,\ b=-2,\ c=-4$

11 a $1-x^2+2x^3+\dots$

b $|x|<1$

12 $a=4$

13 a $4+2x-\frac{1}{4}x^2+\dots$ b $|x|<\frac{4}{3}$

c 4.639

14 a $\frac{3}{1+3x}+\frac{4}{2-5x}$

b $5-4x+\frac{79}{2}x^2+\dots$

c $|x|<\frac{1}{3}$

15 a $k=9$

b $x=6-\lambda,\ y=1+4\lambda,\ z=7\lambda$

16 $a=-3,\ b=-4$

17 $1-x+x^3+\dots$

18 $-1+\frac{4}{3}x+\frac{34}{9}x^2+\dots$

19 a $1-\frac{7}{2}x+\frac{287}{8}x^2+\dots$ b $|x|<\frac{1}{12}$

c 3.87

20 $b=-\frac{35}{2}$

21 -270

22 a i $\alpha=2,\ \beta\neq 0$

ii $\alpha\neq 1$

iii $\alpha=2,\ \beta=0$

b $\frac{x+2}{-2}=\frac{y-4}{-2}=z$

Chapter 3 Prior Knowledge

1 $\frac{\sqrt{3}}{2}$

2 $\frac{7}{9}$

3 $\frac{-2\pm\sqrt{7}}{3}$

4 a $\frac{3\pi}{4},\ \frac{7\pi}{4}$ b $0,\ 2\pi$

c $\frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \frac{11\pi}{9},\ \frac{13\pi}{9},\ \frac{17\pi}{9}$

Exercise 3A

1 a i -1.36 ii 7.09

b i -0.642 ii -2.40

c i 1.05 ii 0.577

2 a i 1 ii $\sqrt{2}$

b i 1 ii $\frac{2\sqrt{3}}{3}$

c i 1 ii $\frac{\sqrt{3}}{3}$

d i -1 ii 0

3 a i 1.16, 5.12 b i 3.48, 5.94

ii 1.32, 4.97 ii 0.340, 2.80

c i 0.464, 3.61 ii 1.19, 4.33

d i 0.421, 2.72, 3.56, 5.86

ii 0.101, 1.47, 3.24, 4.61

4 a i $\frac{\pi}{6}, \frac{5\pi}{6}$ ii $\frac{\pi}{2}$

b i $\frac{\pi}{3}, \frac{4\pi}{3}$ ii $\frac{\pi}{4}, \frac{5\pi}{4}$

c i $0, 2\pi$ ii $\frac{\pi}{6}, \frac{11\pi}{6}$

d i $\frac{\pi}{2}, \frac{3}{2}\pi$ ii $\frac{3\pi}{4}, \frac{7\pi}{4}$

5 a i $\frac{5}{4}$ ii $\frac{\sqrt{26}}{5}$

b i $-\sqrt{15}$ ii $-\sqrt{48}$

c i $\frac{1}{\sqrt{5}}$ ii $\frac{-1}{\sqrt{10}}$

d i $\pm\frac{1}{\sqrt{3}}$ ii $\pm\frac{1}{\sqrt{17}}$

6 a i 0.927 ii −0.644

b i 2.42 ii 1.56

c i 1.26 ii 1.47

7 a i $\frac{\pi}{6}$ ii 0

b i $\frac{\pi}{6}$ ii $\frac{\pi}{2}$

c i $\frac{\pi}{3}$ ii $\frac{\pi}{4}$

d i $\frac{2\pi}{3}$ ii $-\frac{\pi}{6}$

8 $\operatorname{cosec} A = \frac{5}{3}$, $\sec B = \frac{2}{\sqrt{3}}$

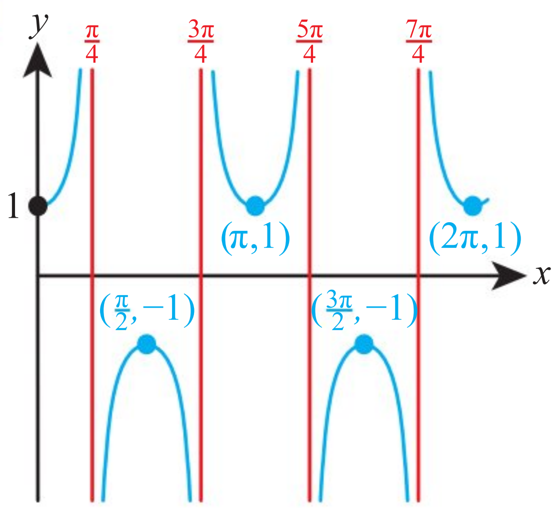
10 0.644

11 (0.715, 2.39)

$f(x) \geqslant 2.39$

14 0

15



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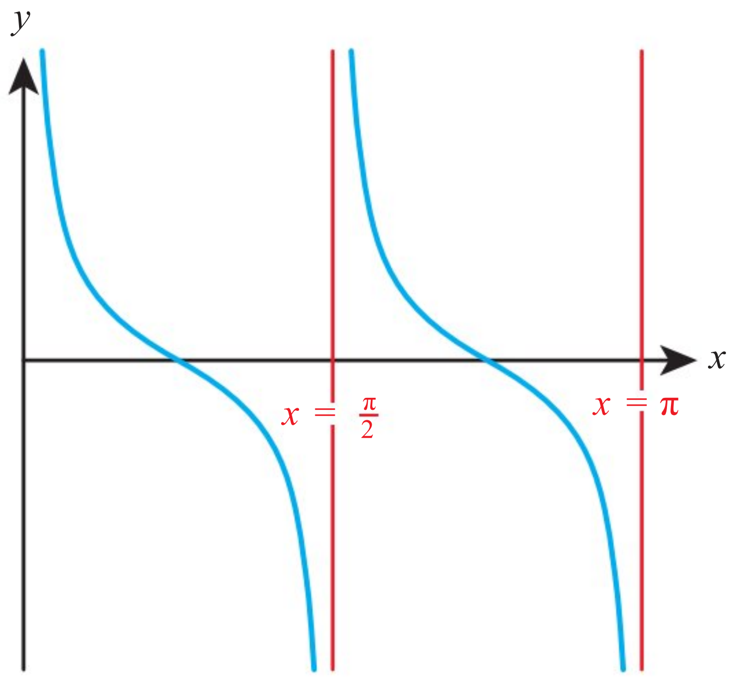

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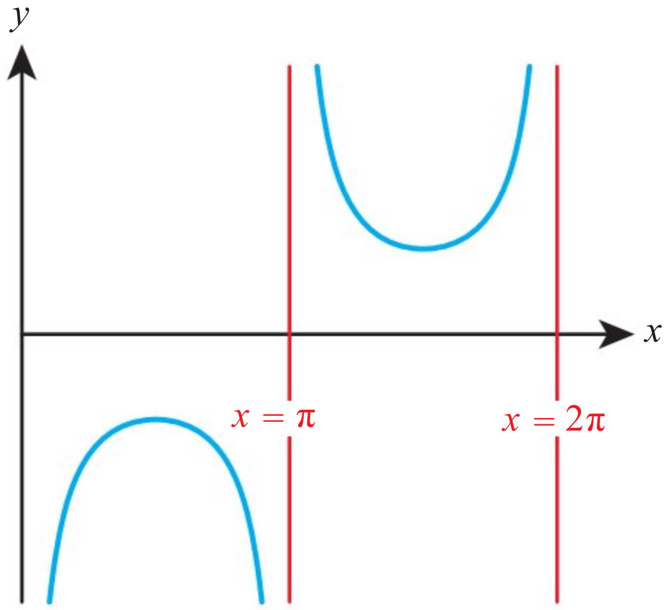


21 $\pm\frac{2\pi}{3}$

22 1

23 $\pi - \arccos x$

24 b 1, 2 c $\frac{\pi}{4}, \frac{5\pi}{4}$, 1.11, 4.25

27 $\arccos\left(\frac{1}{x}\right)$

28 a $x \in \mathbb{R}, -1 \leqslant f(x) \leqslant 1$

b i $x - 2\pi$ ii $2\pi - x$

iii $-x$

29 isin $x = \pi - \arcsin x$

Exercise 3B

1 a $\frac{\sqrt{6}+\sqrt{2}}{4}$ b $\frac{\sqrt{6}-\sqrt{2}}{4}$

2 a $\frac{\sqrt{6}+\sqrt{2}}{4}$ b $\frac{\sqrt{6}-\sqrt{2}}{4}$

3 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ b $\frac{-\sqrt{6}+\sqrt{2}}{4}$

4 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ b $\frac{\sqrt{6}+\sqrt{2}}{4}$

5 a $2-\sqrt{3}$ b $-2+\sqrt{3}$

6 a $-\frac{1}{\sqrt{3}}$ b 1

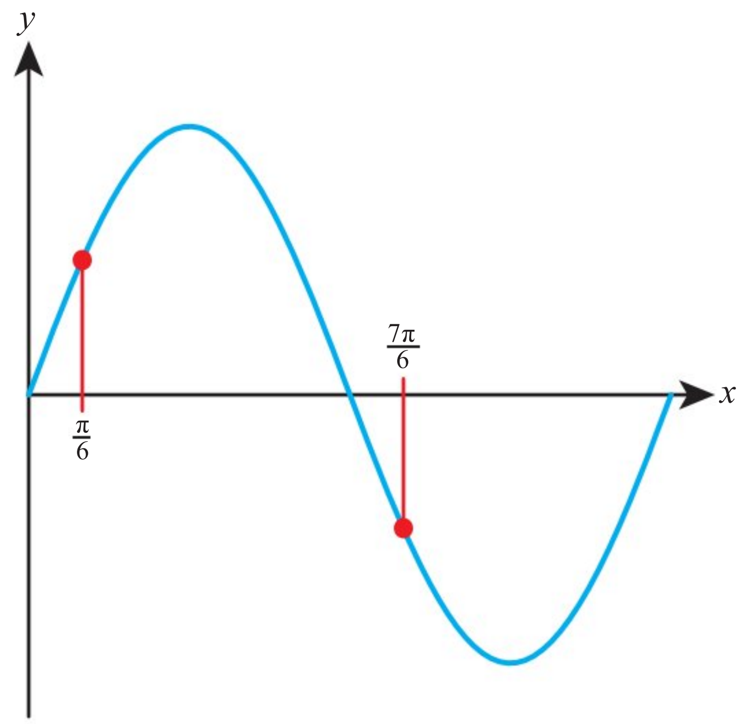
7 a $-2-\sqrt{3}$ b $2+\sqrt{3}$

8 a $-\sin x$ b $\cos x$

9 a $\cos x$ b $\sin x$

10 a $-\tan x$ b $\tan x$

11 b



14 a $\sqrt{2}\cos x$ b $60°, 300°$

17 a $\frac{3}{4}$ b $\frac{13}{9}$

18 3

19 a $\frac{2\sqrt{2}}{3}$ b $\frac{6\sqrt{2}-4}{15}$

20 $\frac{16}{65}$

21 $4-\sqrt{3}$

22 $\tan y = \frac{\tan x - 2}{1 + 2\tan x}$

23 $3, -\frac{1}{3}$

24 $\frac{\sqrt{3}}{2}$

25 $\sqrt{2}$

27 a $2\sin x\cos x$

b $0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

28 $\sqrt{2}-1$

29 b $-\frac{1}{2}, -\frac{1}{3}$

30 b $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

32 a $\sqrt{58}$ b $\frac{3}{10-\sqrt{58}}$

33 a $2\sqrt{3}\sin\left(\theta+\frac{\pi}{6}\right)$ b $\frac{3-\sqrt{3}}{12}$ for $x=\frac{\pi}{6}$

34 $\frac{3}{11}$

Chapter 3 Mixed Practice

1 $-\frac{7}{9}$

2 b $\frac{\pi}{4}$

3 b $\frac{\pi}{3}, \frac{4\pi}{3}$

4 a $-\frac{3}{4}$ b $\sqrt{10}$

5 $\frac{1\pm 2\sqrt{6}}{6}$

6 $\pm\frac{\sqrt{3}}{2}$

7 $-2-\sqrt{3}$

8 $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

9 $0°, 60°, 120°, 180°, 240°, 300°$

10 a $\frac{1}{7}$ b $\frac{1}{\sqrt{50}}$

11 b $-\sqrt{2}-1$

12 $\frac{\pi}{3}$

15 b $1+\sqrt{2}$

16 b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

17 a $2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$ b $\frac{\pi}{6}, \frac{\pi}{2}$

18 $a=6, b=1$

19 a $\frac{n\pi}{2}, n \in \mathbb{Z}$ b 2

20 b $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

21 a $\frac{1}{7}$ b $\frac{4}{3}$

22 $\frac{1}{3}$

23 a $\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ b $-\frac{\pi}{4}, 0, \frac{\pi}{4}$

24 b $\frac{\pi}{10}, \frac{\pi}{2}$

$1 - 3\sin x - 2\sin^2 2x + 4\sin^3 x$

d $(s-1)\left(2s + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(2s + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

e $\frac{-1+\sqrt{5}}{4}$

25 b 21

c e.g. As n increases, the compactness of a polygon with n sides gets closer to that of a circle ($c = 1$).

26 a $p = 3$ b $\frac{\pi}{4}$

Chapter 4 Prior Knowledge

1 $\frac{3 \pm \sqrt{3}}{3}$

2 $5 - 2\sqrt{5}$

3 a $\frac{\sqrt{2}}{2}$ b $-\frac{1}{2}$

4 a $\sin\frac{3\pi}{10}$ b $\frac{5\pi}{24}$

5 $e^{(\ln 4)x}$

6 $a(x-3)(x+5)$

7 $\frac{1}{1-x}$

Exercise 4A

1 a i b −1

2 a 4i b 5

3 a −9 b −8i

4 a 1 b −1

5 a $x = \pm 3i$ b $x = \pm 6i$

6 a $x = \pm 2\sqrt{2}i$ b $x = \pm 5\sqrt{3}i$

7 a $x = 1 \pm 2i$ b $x = 2 \pm 3i$

8 a $x = -1 \pm \frac{\sqrt{2}}{2}i$ b $x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}i$

9 a 11 + 4i b −4 + 4i

10 a 1 − 2i b −6 + 10i

11 a 8 − i b 16 + 2i

12 a 8 + 6i b 7 − 24i

13 a 4 − 2i b −3 + 3i

14 a 4 + 3i b 1 − i

15 a $\frac{1}{2} + \frac{1}{2}i$ b $-\frac{2}{13} - \frac{23}{13}i$

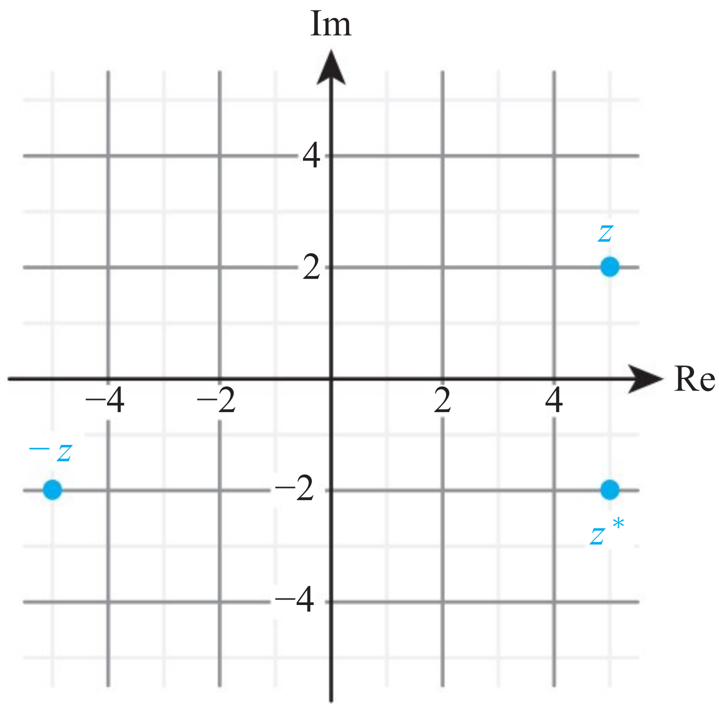
16 a $a = 5, b = 7$ b $a = -3, b = 9$

17 a $a = -\frac{3}{2}, b = \frac{31}{2}$ b $a = \frac{1}{2}, b = -8$

18 a $z = -4 - i$ b $z = 1 + 2i$

19 a $z = \frac{2}{3} + 7i$ b $z = -\frac{5}{3} + \frac{1}{3}i$

20 a



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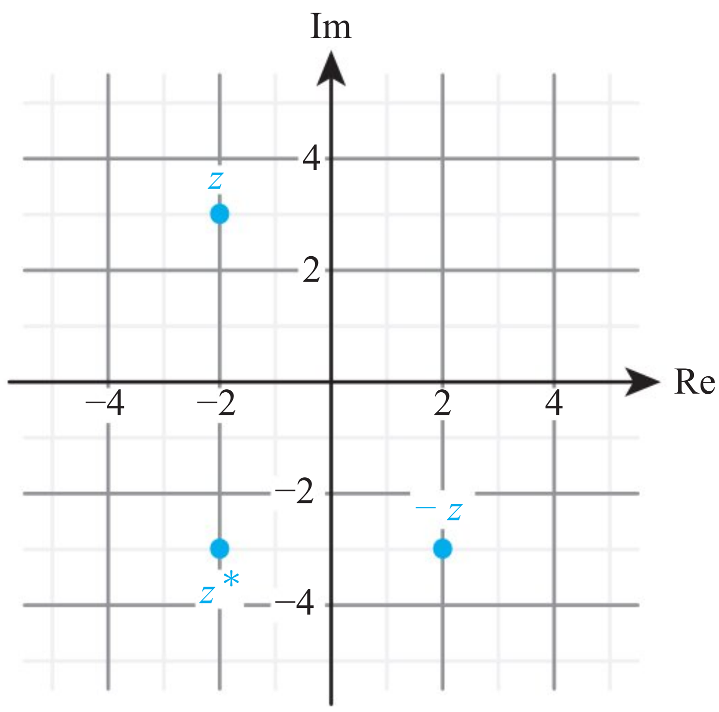
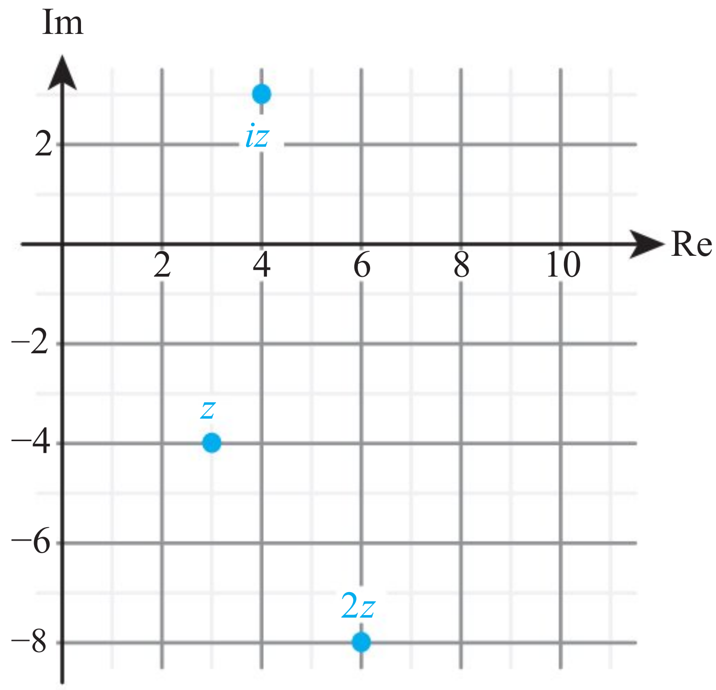


21 a



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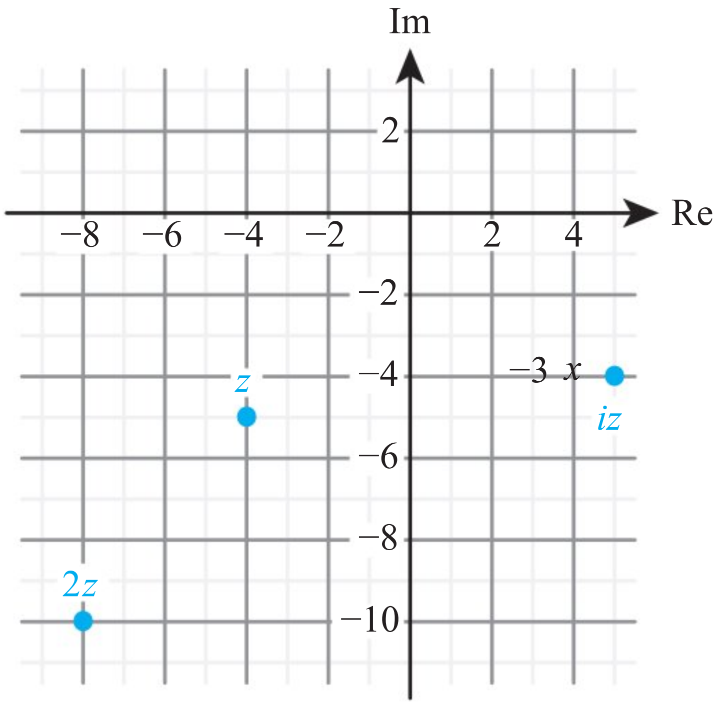
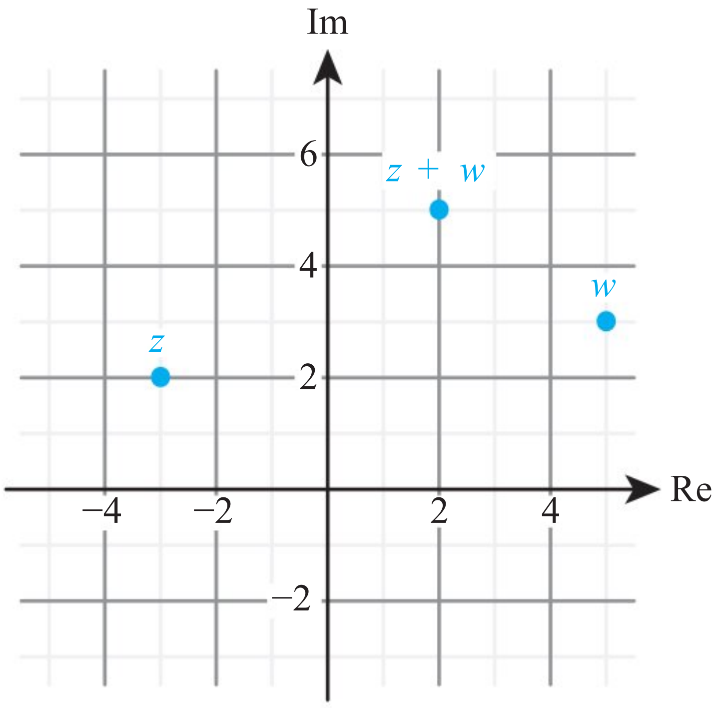


22 a



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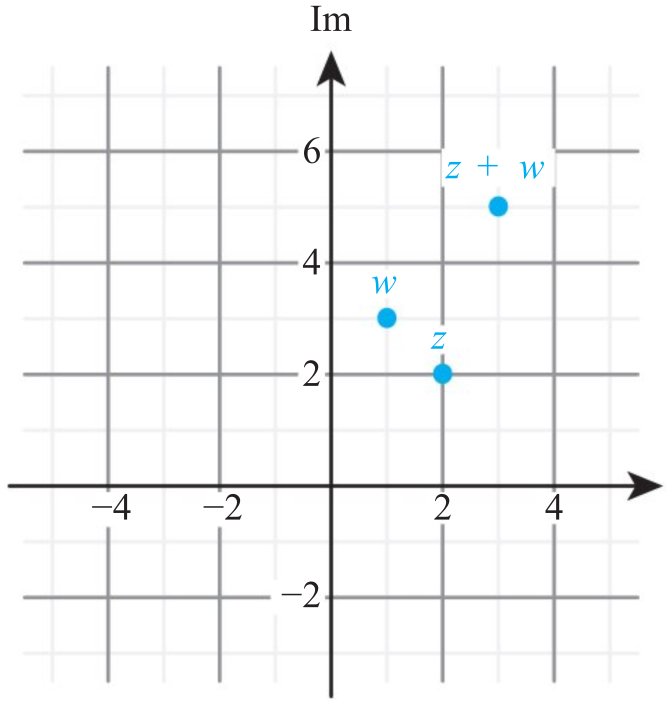
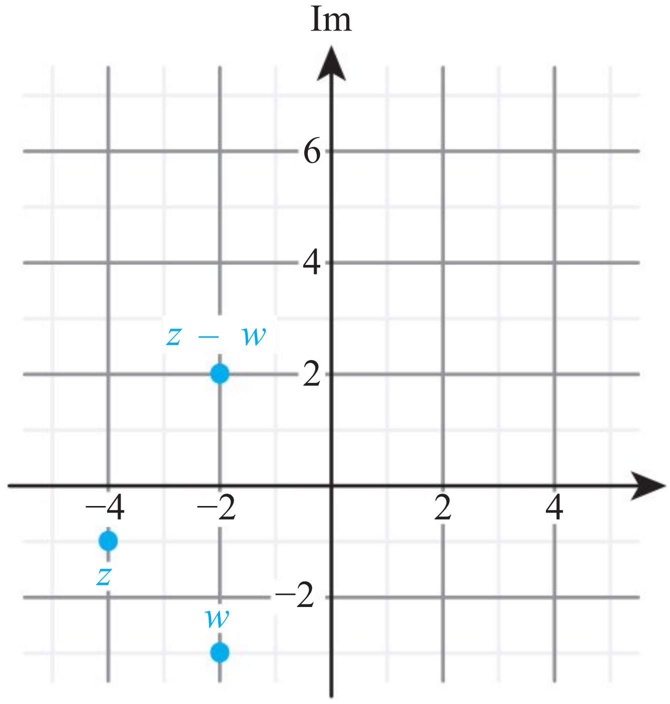


23 a



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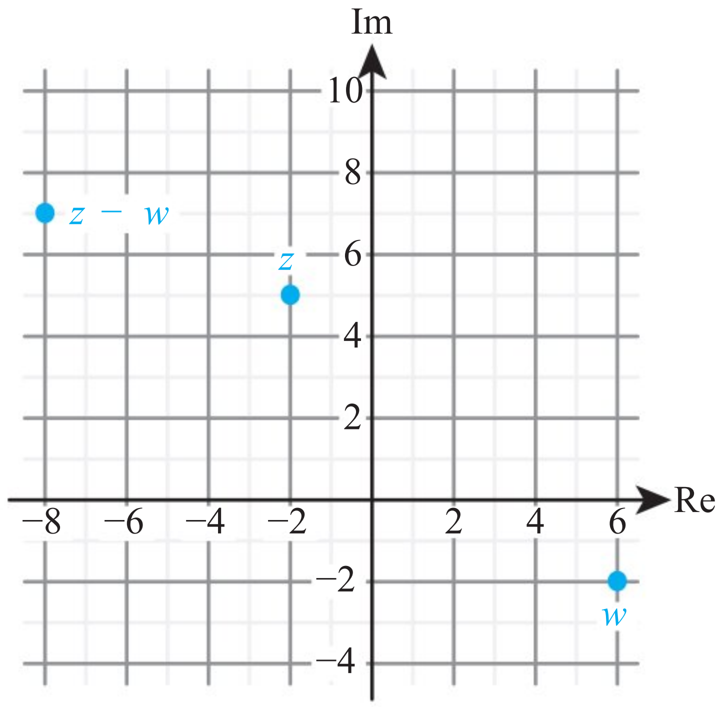
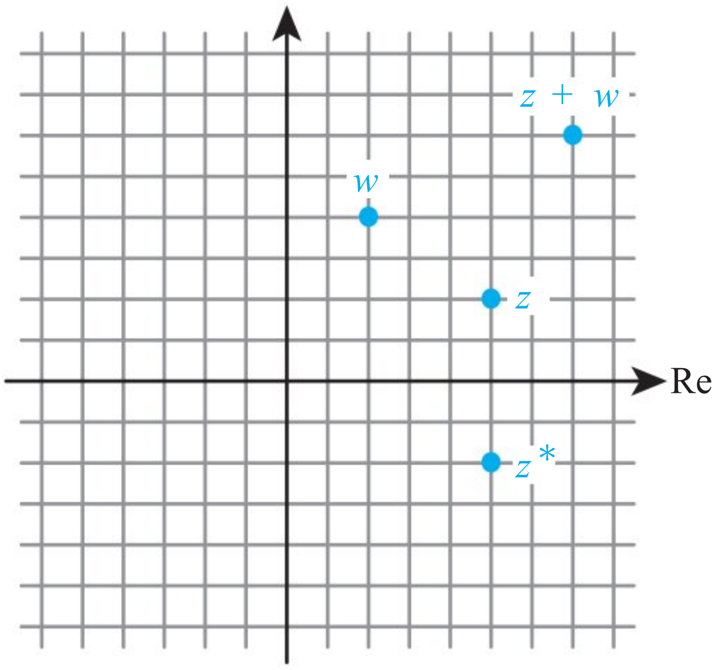


24 a



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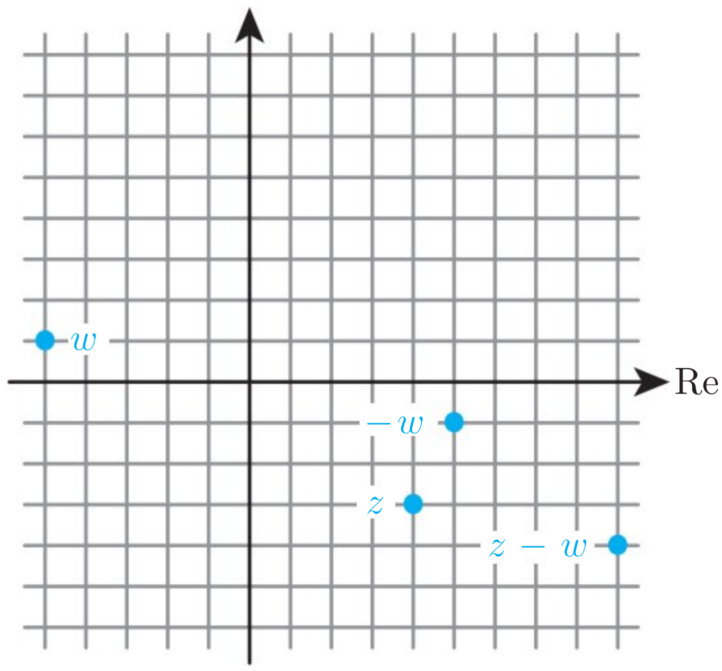

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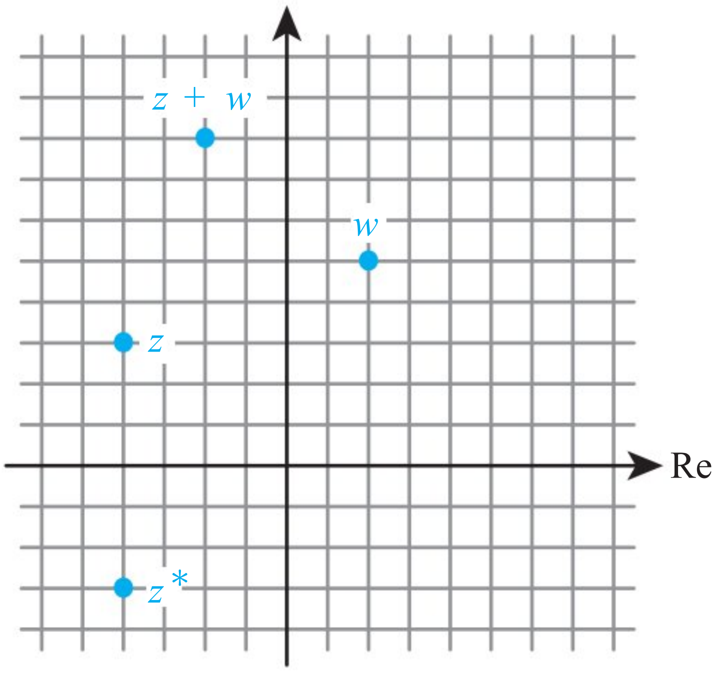
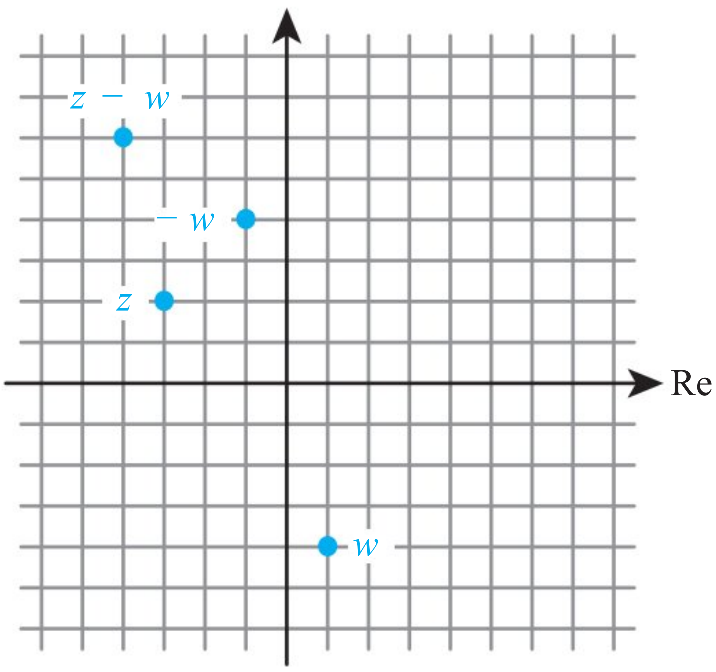


25 a



26 $x = -\frac{3}{5} \pm \frac{4}{5}\text{i}$

27 $z^* = 5 + \text{i}$

28 $z = 2 + 3\text{i}$

29 $z = -1 + 2\text{i}$

30 $a = \pm 3$

32 $z = 2 - 5\text{i},\ w = 4 + \text{i}$

33 $z = \frac{3}{2} + \frac{1}{2}\text{i},\ w = 2\text{i}$

34 $a = 8,\ \ b = 1$

$a = -1,\ \ b = 10$

35 $a = 4,\ -\frac{3}{2}$

36 $z = 1 - 2\text{i}$ or $-1 + 2\text{i}$

37 $z = 3 - \text{i}$ or $-3 + \text{i}$

38 $0,\ 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$

Exercise 4B

1 **a** $4\text{cis}\,\pi$ **b** $5\text{cis}\,0$

2 **a** $3\text{cis}\frac{\pi}{2}$ **b** $2\text{cis}\left(-\frac{\pi}{2}\right)$

3 **a** $2\text{cis}\frac{\pi}{6}$ **b** $4\text{cis}\left(-\frac{\pi}{6}\right)$

4 **a** $\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$ **b** $3\sqrt{2}\text{cis}\frac{3\pi}{4}$

5 a $5\text{cis}\frac{3\pi}{2}$ **b** $7\text{cis}\frac{3\pi}{2}$

6 a $6\text{cis}\frac{5\pi}{3}$ **b** $4\sqrt{2}\text{cis}\frac{7\pi}{4}$

7 a $2\sqrt{2}\text{cis}\frac{5\pi}{4}$ **b** $2\text{cis}\frac{4\pi}{3}$

8 a $z=-10\text{i}$ **b** $z=8\text{i}$

9 a $z=2+2\sqrt{3}\text{i}$ **b** $z=1+\text{i}$

10 a $z=-2\sqrt{6}+2\sqrt{6}\text{i}$ **b** $z=-1+\sqrt{3}\text{i}$

11 a $z=4\sqrt{3}-4\text{i}$ **b** $z=\sqrt{2}-\sqrt{2}\text{i}$

12 a $7\text{cis}\left(-\frac{\pi}{8}\right)$ **b** $5\text{cis}\left(-\frac{\pi}{9}\right)$

13 a $8\text{cis}\frac{2\pi}{7}$ **b** $2\text{cis}\frac{3\pi}{8}$

14 a $3\text{cis}\left(-\frac{6\pi}{7}\right)$ **b** $4\text{cis}\left(-\frac{4\pi}{5}\right)$

15 a $12\text{cis}\frac{8\pi}{15}$ **b** $5\text{cis}\frac{\pi}{8}$

16 a $3\text{cis}\frac{35\pi}{18}$ **b** $\frac{1}{3}\text{cis}\frac{4\pi}{7}$

17 a 6i **b** $2+2\sqrt{3}\text{i}$

18 a $-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$ **b** $\sqrt{2}+\sqrt{2}\text{i}$

19 a $3\text{e}^{\text{i}\pi}$ **b** $2\text{e}^{0\text{i}}$

20 a $1\text{e}^{\text{i}\frac{\pi}{2}}$ **b** $\sqrt{2}\text{e}^{\frac{\text{i}\pi}{2}}$

21 a $\sqrt{2}\text{e}^{\frac{\text{i}\pi}{4}}$ **b** $2\text{e}^{\frac{\text{i}\pi}{6}}$

22 a $\text{e}^{0.4\text{i}}$ **b** $\text{e}^{1.8\text{i}}$

23 a $4\text{e}^{\frac{\pi}{5}\text{i}}$ **b** $7\text{e}^{\frac{\pi}{10}\text{i}}$

24 a -1 **b** $-\text{i}$

25 a $-1+\text{i}$ **b** $-\sqrt{3}+\text{i}$

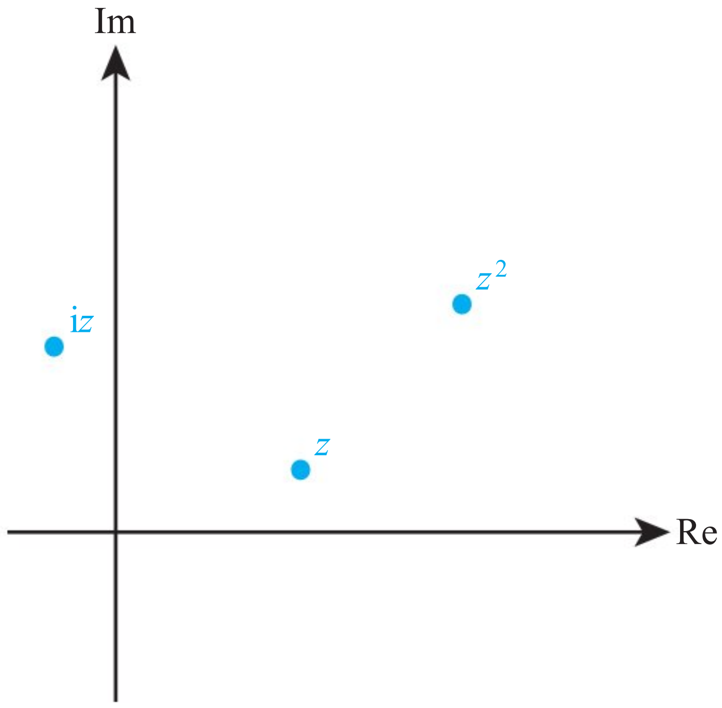
26 a $15\text{e}^{-0.1\text{i}}$ **b** $2\text{e}^{2\text{i}}$

27 a $2\text{e}^{\frac{3\pi\text{i}}{4}}$ **b** $4\text{e}^{-\frac{\pi\text{i}}{12}}$

28 a $\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$ **b** $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\text{i}$

29 a i
b $-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}$

30



31 a $2\sqrt{2}$ **b** $\frac{3\pi}{4}$
c $8, \frac{3\pi}{2}$ **d** -8i

32 $\text{cis}\,1$

33 a $\frac{\pi}{3}$
b $\frac{7\pi}{12}$

34 a i
b $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)(1+\text{i})$

35 $\cos\left(\frac{13\pi}{12}\right)+\text{i}\sin\left(\frac{13\pi}{12}\right)$

36 $\cos\left(\frac{7\pi}{20}\right)+\text{i}\sin\left(\frac{7\pi}{20}\right)$

37 2π

38 $\pi+\arctan\left(\frac{b}{a}\right)$

39 $\text{cis}\left(\theta+\frac{\pi}{2}\right)$

40 $\sec\theta\ \text{cis}\theta$

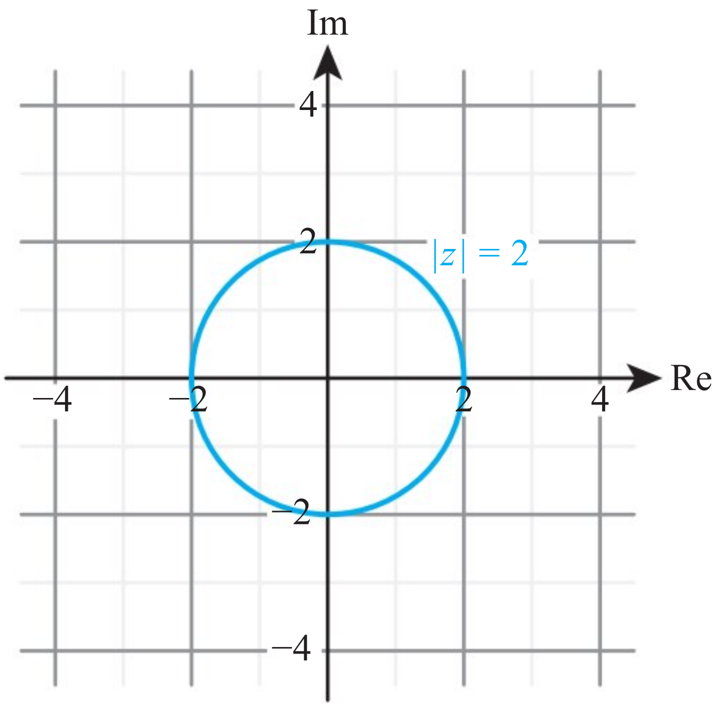
41 e.g. 5, −1

42 $|z|=4, \arg z=\frac{13\pi}{24}$
$|w|=2, \arg w=\frac{5\pi}{24}$

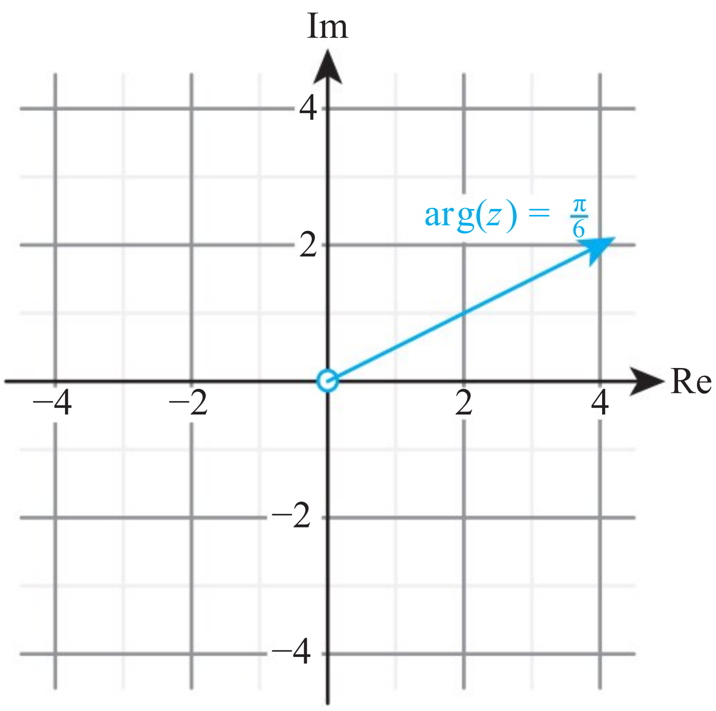
43 $3+4\text{i}$

44 b $2\text{Re}(z)$

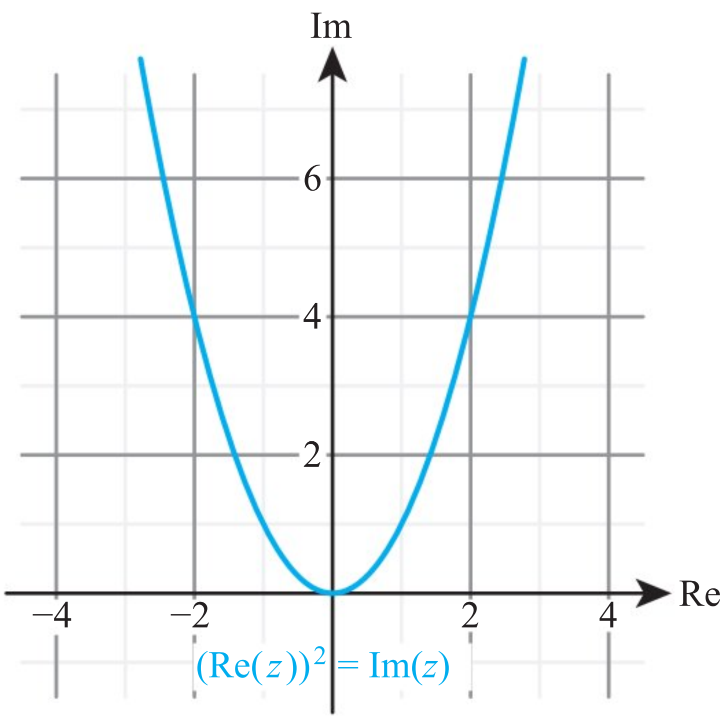
45 a



b



c



46 2

48 a i 1 ii $\frac{\pi}{4}$ iii 2 iv $\frac{\pi}{3}$

b $2\,\text{cis}\,\frac{\pi}{12}$

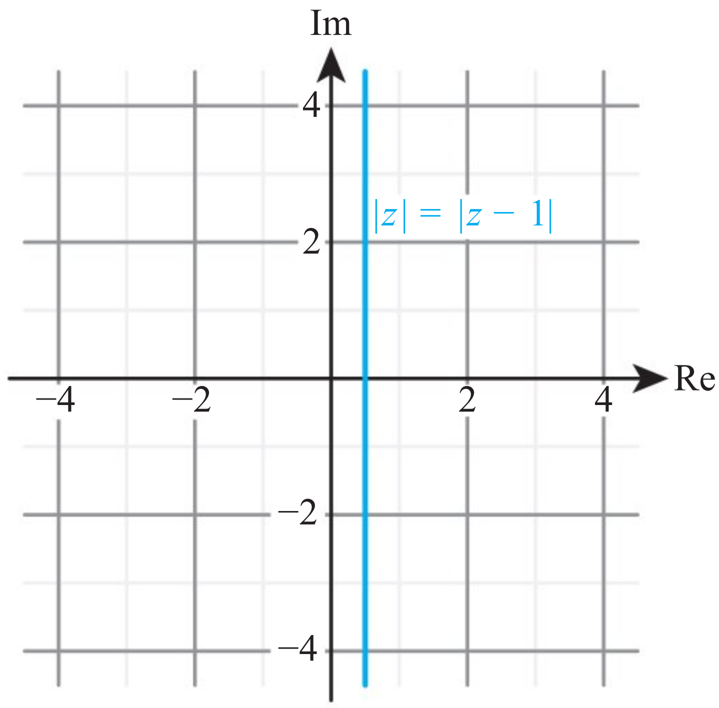
c $\frac{\sqrt{6}+\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{2}}{2}\text{i}$

d $\frac{\sqrt{6}+\sqrt{2}}{4}$

49 $z = 3 + 4\text{i}$

50 a $\text{Re}(z) = \frac{1}{2}$

b



51 a $\omega e^{\frac{2\text{i}\pi}{3}}, \omega e^{\frac{4\text{i}\pi}{3}}$

b $\left|\omega\left(1-e^{\frac{2\text{i}\pi}{3}}\right)\right| = \sqrt{3}|\omega|$

52 $1\,\text{cis}(\ln 3)$

53 b 0.2

54 a $2e^{\text{i}\pi}$

b $\ln 2 + \text{i}\pi$

55 a $e^{\frac{\text{i}\pi}{2}}$

b $\frac{\text{i}\pi}{2}$

c Could be $\frac{\text{i}\pi}{2} + 2k\pi\text{i}$ where $k \in \mathbb{Z}$.

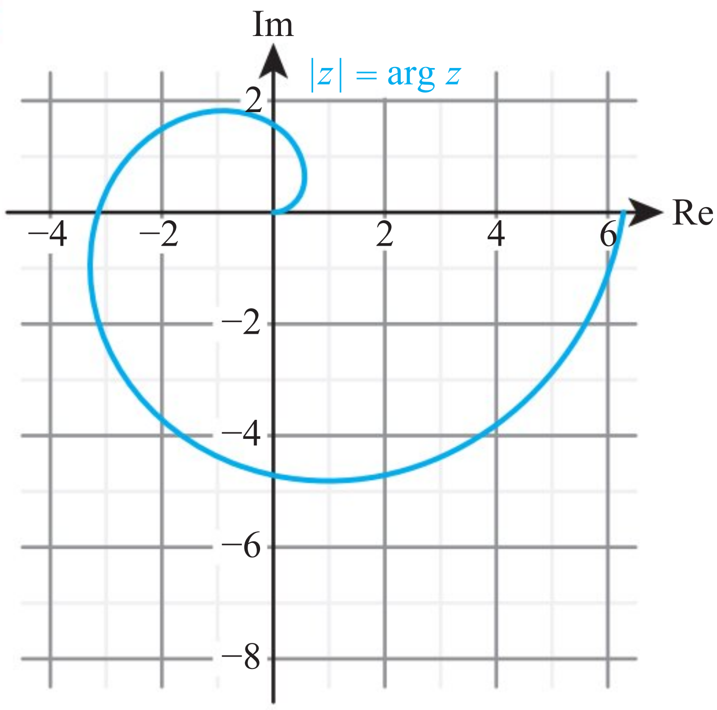
56 a $e^x \cos x$

b $\frac{e^x}{2}(\sin x - \cos x)$

57 a $e^{\text{i}\theta}$

58 a $z = \frac{4 + \text{i}w}{w}$

59



Exercise 4C

1 a $(x-2i)(x+2i)$ b $(x-5i)(x+5i)$

2 a $(x-2\sqrt{3}i)(x+2\sqrt{3}i)$ b $(x-3\sqrt{2}i)(x+3\sqrt{2}i)$

3 a $(2x-7i)(2x+7i)$ b $(3x-8i)(3x+8i)$

4 a $(x-1-i)(x-1+i)$

b $(x+3-4i)(x+3+4i)$

5 a $2\left(x-\left(\frac{3+\sqrt{5}i}{2}\right)\right)\left(x-\left(\frac{3-\sqrt{5}i}{2}\right)\right)$

b $3\left(x-\left(\frac{1+\sqrt{2}i}{3}\right)\right)\left(x-\left(\frac{1-\sqrt{2}i}{3}\right)\right)$

6 a $(x-2)(x^2+4x+7)$ b $(x+1)(x^2+2x+5)$

7 a $(x+2)(2x^2-4x+3)$ b $(x-1)(3x^2+2x+2)$

8 a $(2x-1)(2x^2-3x+4)$ b $(3x+1)(2x^2+x+3)$

9 a $x=5,\ 3\pm 2i$ b $x=-3,\ 2\pm i$

10 a $x=1,\ 1\pm 2i$ b $x=-4,\ 3\pm i$

11 a $x=-4,\ 1,\ 1\pm i$ b $x=-1,\ 2,\ 4\pm i$

12 a $x=1\pm 3i,\ \pm 2i$ b $x=3\pm i,\ \pm i$

13 a $(x-2)(x^2-6x+10)$ b $x=2,\ 3\pm i$

14 a $x=5\pm 2i$

b $(x+2)(x-5-2i)(x-5+2i)$

15 b $x=\frac{1}{2},\ -2\pm\sqrt{2}i$

16 a $x=1+4i$

b $x=-3$

17 b $x=-4,\ \pm 3i$

18 $(x+1)(x-2)(x^2+4x+5)$

$x=-1,\ 2,\ -2\pm i$

19 a $x=1\pm\sqrt{3}i$

b $(x+2)(x-3)(x-1-\sqrt{3}i)(x-1+\sqrt{3}i)$

20 a $x=2-5i$ b $x=\pm i$

21 a $-2i,\ 4+i$ b $(x^2+4)(x^2-8x+17)$

22 $x^3-11x^2+41x-51$

23 $b=-5,\ c=12,\ d=18$

24 $x^4-4x^3+29x^2-64x+208$

25 $b=-4,\ c=14,\ d=-36,\ e=45$

26 $x=\pm\sqrt{5}i,\ \pm 2\sqrt{2}i$

27 e.g. $(z-i)^2(z+i)=0$

Exercise 4D

1 a $32+32\sqrt{3}i$ b $243i$

2 a $32\sqrt{2}+32\sqrt{2}i$ b $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

3 a $18-18\sqrt{3}i$ b $-4\sqrt{2}+4\sqrt{2}i$

4 a $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ b $\frac{-3\sqrt{2}}{4}-\frac{3\sqrt{2}i}{4}$

5 a $z=1,\ \text{cis}\frac{2\pi}{3},\ \text{cis}\frac{4\pi}{3}$

b $z=1,\ \text{cis}\frac{\pi}{2},\ \text{cis}\pi,\ \text{cis}\frac{3\pi}{2}$

6 a $z=\sqrt{3}\text{cis}\frac{\pi}{6},\ \sqrt{3}\text{cis}\frac{\pi}{2},\ \sqrt{3}\text{cis}\frac{5\pi}{6},$

$\sqrt{3}\text{cis}\frac{7\pi}{6},\ \sqrt{3}\text{cis}\frac{3\pi}{2},\ \sqrt{3}\text{cis}\frac{11\pi}{6}$

b $z=3,\ 3\text{cis}\frac{2\pi}{5},\ 3\text{cis}\frac{4\pi}{5},\ 3\text{cis}\frac{6\pi}{5},\ 3\text{cis}\frac{8\pi}{5}$

7 a $z=2\text{cis}\frac{\pi}{10},\ 2\text{cis}\frac{\pi}{2},\ 2\text{cis}\frac{9\pi}{10},\ 2\text{cis}\frac{13\pi}{10},\ 2\text{cis}\frac{17\pi}{10}$

b $z=4\text{cis}\frac{\pi}{2},\ 4\text{cis}\frac{7\pi}{6},\ 4\text{cis}\frac{11\pi}{6}$

8 a $z=\sqrt{2}\text{cis}\frac{\pi}{16},\ \sqrt{2}\text{cis}\frac{9\pi}{16},\ \sqrt{2}\text{cis}\frac{17\pi}{16},\ \sqrt{2}\text{cis}\frac{25\pi}{16}$

b $z=\sqrt{2}\text{cis}\frac{7\pi}{24},\ \sqrt{2}\text{cis}\frac{15\pi}{24},\ \sqrt{2}\text{cis}\frac{23\pi}{24},$

$\sqrt{2}\text{cis}\frac{31\pi}{24},\ \sqrt{2}\text{cis}\frac{39\pi}{24},\ \sqrt{2}\text{cis}\frac{47\pi}{16}$

9 a $z=2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

b $-128+128i$

10 a $z = 2\text{cis}\frac{\pi}{6}$

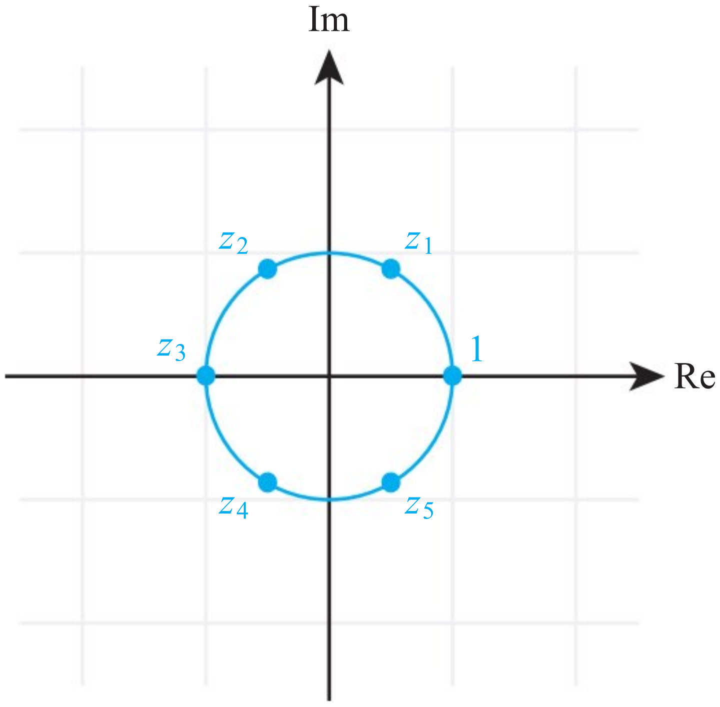
b $-\frac{1}{8}\text{i}$

11 a $w = 2\text{cis}\left(-\frac{3\pi}{4}\right)$

b 64i

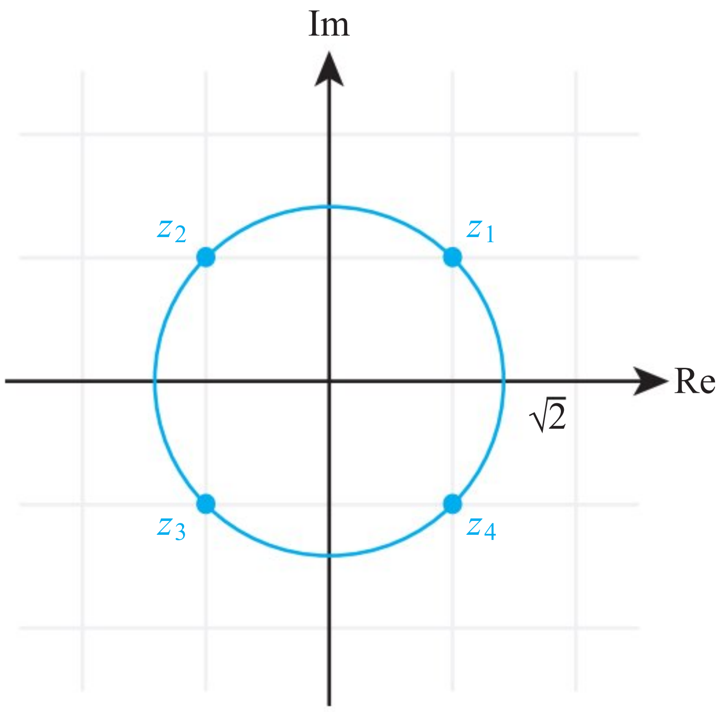
12 a $z = \pm 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}$

b



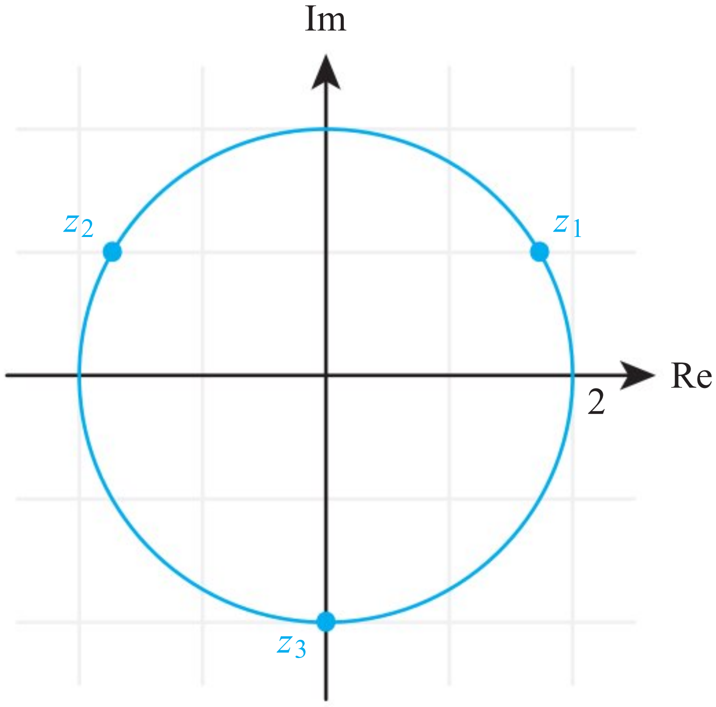
13 a $z = \sqrt{2} \pm \sqrt{2}\text{i}, -\sqrt{2} \pm \sqrt{2}\text{i}$

b



14 a $z = 2e^{i\frac{\pi}{6}}, 2e^{i\frac{5\pi}{6}}, 2e^{-i\frac{\pi}{2}}$

b



15 $n = 5, w = -9\sqrt{3}\text{i}$

16 $\frac{1}{8} - \frac{\sqrt{3}}{8}\text{i}$

17 $-162\sqrt{2} - 162\sqrt{2}\text{i}$

18 $n = 24$

19 $n = 9$

20 a 1 (or ω^0), $\omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$

b Does not exist – consider $\omega^7 = 1$, or an Argand diagram.

c 3 d 5

21 b -1

22 a $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}$ b $\frac{-1 \pm \sqrt{3}\text{i}}{2}$

23 $z = \sqrt{2}e^{i\frac{\pi}{12}}, \sqrt{2}e^{i\frac{7\pi}{12}}, \sqrt{2}e^{-i\frac{5\pi}{12}}, \sqrt{2}e^{-i\frac{11\pi}{12}}$

24 $z = 2e^{i\frac{\pi}{3}}, 2e^{i\frac{11\pi}{15}}, 2e^{i\frac{17\pi}{15}}, 2e^{i\frac{23\pi}{15}}, 2e^{i\frac{29\pi}{15}}$

25 $z = \sqrt{2}\text{cis}\left(-\frac{\pi}{24}\right), \sqrt{2}\text{cis}\left(-\frac{9\pi}{24}\right), \sqrt{2}\text{cis}\left(-\frac{17\pi}{24}\right)$

$\sqrt{2}\text{cis}\frac{7\pi}{24}, \sqrt{2}\text{cis}\frac{15\pi}{24}, \sqrt{2}\text{cis}\frac{23\pi}{24}$

26 $n = 4, w = -324$

27 c $\frac{-1 + \sqrt{5}}{4}$

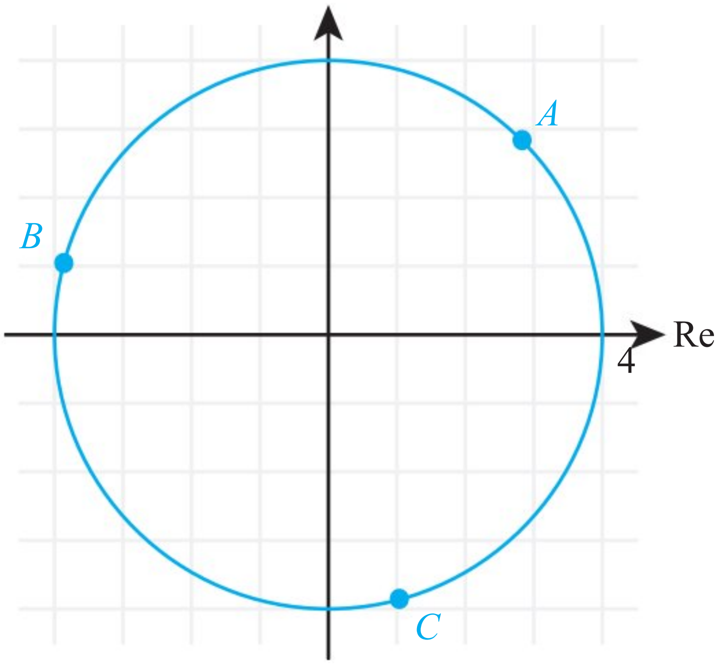
28 a $-1, \text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right)$ b $x^3 + 6x^2 + 12x + 8$

c $-3, -\frac{3}{2} + \frac{\sqrt{3}}{2}\text{i}, -\frac{3}{2} - \frac{\sqrt{3}}{2}\text{i}$

29 a $z = 1 \pm \text{i}, -1 \pm \text{i}$ b $z = 1 \pm \text{i}, \frac{3 \pm \text{i}}{5}$

30 a $z = 4\text{cis}\frac{\pi}{4}, 4\text{cis}\frac{11\pi}{12}, 4\text{cis}\frac{19\pi}{12}$

b



c $4\sqrt{2} - 4\sqrt{2}\text{i}$

Exercise 4E

1 a $\cos^3\theta - 3\cos\theta\sin^2\theta$

b $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

2 a $4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

b $-1, -0.309, 0.809$

4 a $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

b $\sin\theta = 0, \pm\frac{1}{\sqrt{2}}$

5 c $\frac{3\pi + 8}{32}$

7 a $A = 6, B = 15, C = 10$

8 b $-1, -0.309, 0.809$

9 a $(z + z^{-1})^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$

$(z - z^{-1})^6 = z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}$

10 a $\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x + \text{i}(\sin^5 x - 10\sin^3 x\cos^2 x + 5\sin x\cos^4 x)$

c 5

11 b $\cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$

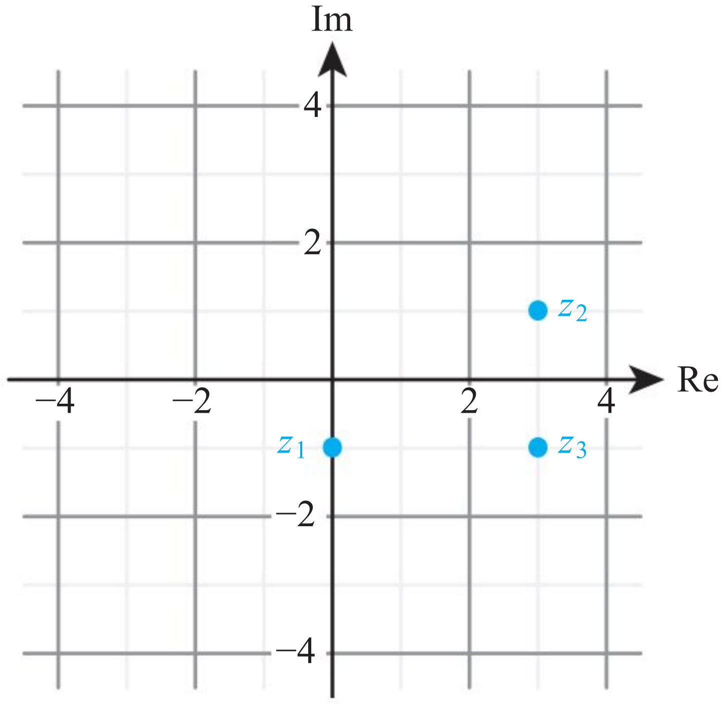
12 a Re: $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

Im: $4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

c $x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$

Chapter 4 Mixed Practice

1



2 $1 \pm \text{i}$

3 $3 \pm \sqrt{3}\text{i}$

4 $\frac{3}{5} - \frac{1}{5}\text{i}$

5 $b = -2, c = 5$

6 $-0.5 - \text{i}$

7 $-\frac{\text{i}}{3}$

8 $z = 2 - 2\text{i}$

9 a $|z| = \sqrt{2}, \arg z = \frac{\pi}{4}$

b 8i

10 $16\sqrt{2}, \frac{3\pi}{4}$

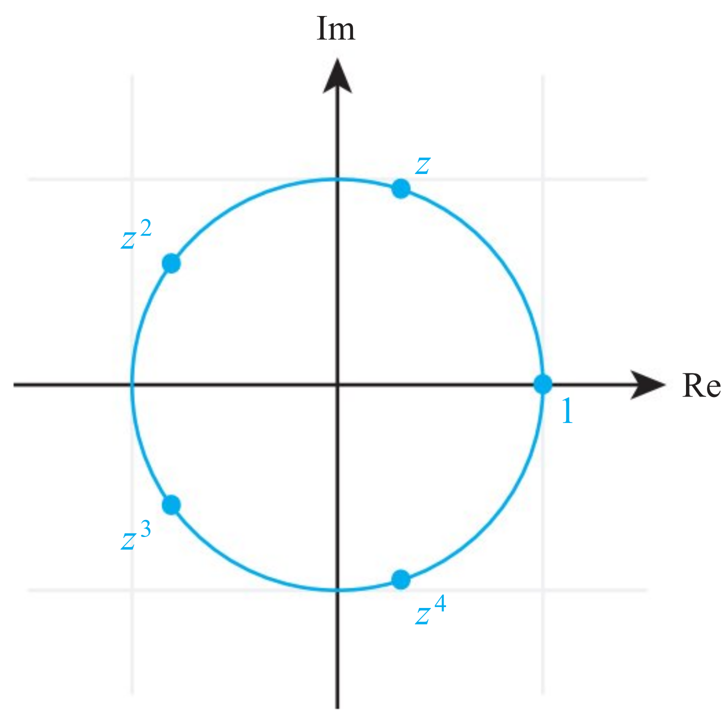
11 a $p = 3, q = 0.5$

b $p = \frac{1}{3} - \frac{2}{3}\text{i}, q = \frac{1}{3} + \frac{2}{3}\text{i}$

12 $\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}, \frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}, -1$

13 a $1, \text{e}^{\frac{2\pi\text{i}}{5}}, \text{e}^{\frac{4\pi\text{i}}{5}}, \text{e}^{\frac{6\pi\text{i}}{5}}, \text{e}^{\frac{8\pi\text{i}}{5}}$

b



14 **a** $\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$ **b** $-\frac{\sqrt{3}}{2}+\frac{1}{2}\text{i}$

15 $\frac{a}{a^2+1}-\frac{1}{a^2+1}\text{i}$

16 $x=-\frac{\text{i}}{3}$

17 $z=4+3\text{i}$

18 $1, \pm\text{i}$

19 $-1,\ 3\pm 2\text{i}$

20 $1, \frac{\pi}{6}$

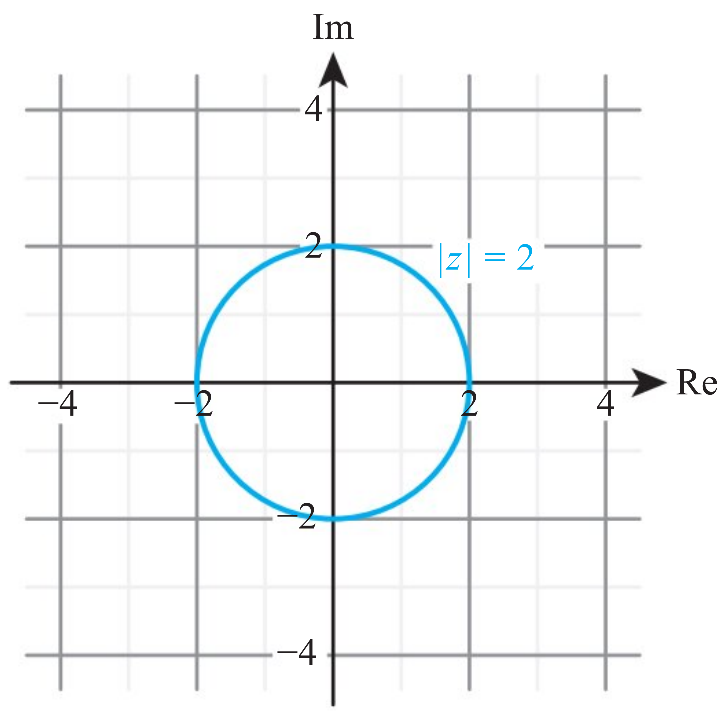
21 $\pm\sqrt{\frac{3}{2}}$

22 $2\pm\text{i}$

23 $5+12\text{i}$

24 **a** 2

b



25 $5,\ 3+2\text{i}$

26 **a** e.g. $z=\text{i}$ **b** 0

27 **a** **i** $2\text{cis}\,\pi,\ 2\text{cis}\frac{\pi}{3},\ 2\text{cis}\frac{5\pi}{3}$

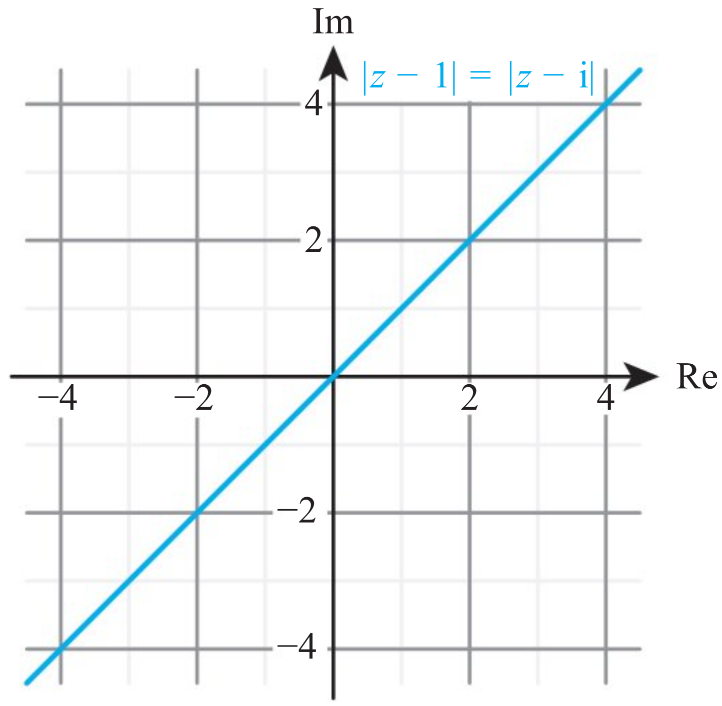
ii $-2,\ 1+\sqrt{3}\text{i},\ 1-\sqrt{3}$

b $3\sqrt{3}$

28 $x^2+y^2-2x=3$

29 $\frac{3}{\sqrt{5}}$

30



31 **b** $\frac{\pi}{2}, \frac{3\pi}{2}$

32 **a** $(1-\sqrt{3})+(1+\sqrt{3})\text{i}$

b $z=\sqrt{2}\left(\cos\frac{\pi}{4}+\text{i}\sin\frac{\pi}{4}\right),\ w=2\left(\cos\frac{\pi}{3}+\text{i}\sin\frac{\pi}{3}\right),$

$|zw|=2\sqrt{2}, \arg(zw)=\frac{7\pi}{12}$

c $\frac{\sqrt{6}+\sqrt{2}}{4}$

33 **a** $2, \frac{\pi}{6}$ **b** $-128\sqrt{3}$

34 **a** $2,\ -1+\text{i}\sqrt{3},\ -1-\text{i}\sqrt{3}$

b $2,\ -\frac{4}{7}+\frac{2\sqrt{3}}{7}\text{i},\ -\frac{4}{7}-\frac{2\sqrt{3}}{7}\text{i}$

35 **b** $x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

c $a=\frac{1}{8},\ b=\frac{1}{2},\ c=\frac{3}{8}$

d $\frac{3\pi}{8}$

36 **a** $2\sqrt{2-\sqrt{3}}$ **b** $\frac{\pi}{12}$

37 **a** 0 **b** x^2+y^2-xy

38 **b** z^*w

40 $-3, \frac{3}{2}+\frac{3\sqrt{3}}{2}\text{i}, \frac{3}{2}-\frac{3\sqrt{3}}{2}\text{i}$

42 $1, 1\pm\sqrt{3}$

44 **b** **ii** $-\frac{1}{2}$

45 **a** **i** $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right), z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right), z_3 = 2\text{cis}\left(\frac{3\pi}{2}\right)$

b **i** $\text{cis}\left(\frac{2k\pi}{7}\right)$ for $k = 0, 1, \ldots, 6$

ii $\frac{\pi}{7}$

iii $z^2 - 2z\cos\left(\frac{4\pi}{7}\right)+1$ and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right)+1$

46 **a** **iii** $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

b $r = 1, \alpha = 72°$ **d** $\frac{\sqrt{10+2\sqrt{5}}}{4}$

47 **a** $(\cos^3\theta - 3\cos\theta\sin^2\theta) + \text{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$

d $\pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}$ **e** $-\sqrt{\frac{(5-\sqrt{5})}{8}}$

Chapter 5 Prior Knowledge

1 **a** 2, 5, 10, 17 **b** 3, 5, 9, 17

2 15

5 **a** $\frac{2\sqrt{2}}{3}$ **b** $\frac{4\sqrt{2}}{9}$ **c** $\frac{7}{9}$ **d** $\frac{23}{27}$

6 $6\text{ cis}\frac{7\pi}{12}$

7 $(3x+1)e^{3x}$

Chapter 6 Prior Knowledge

1 $\left(\frac{2}{3}, 0\right), (-1, 0)$

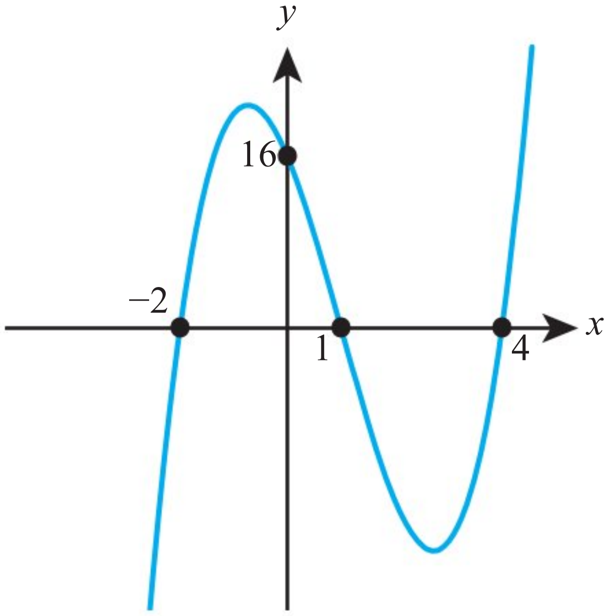
2 $y = -2x^2 - 2x + 4$

3 $3 \pm \text{i}$

Exercise 6A

1	**a** **i** A	**ii** C	**iii** B
	b **i** B	**ii** C	**iii** B
2	**a** **i** A	**ii** B	**iii** C
	b **i** C	**ii** B	**iii** A
3	**a** **i** B	**ii** A	**iii** C
	b **i** C	**ii** A	**iii** B

4 **a**



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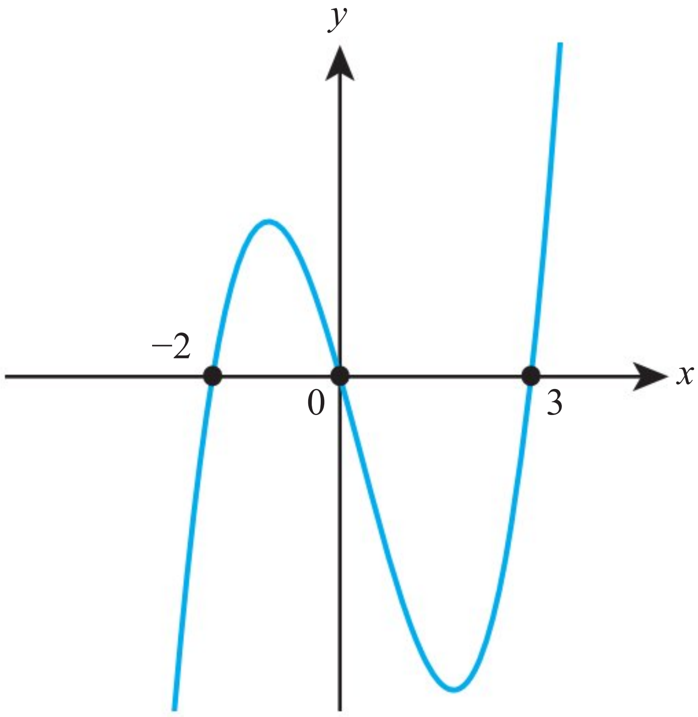
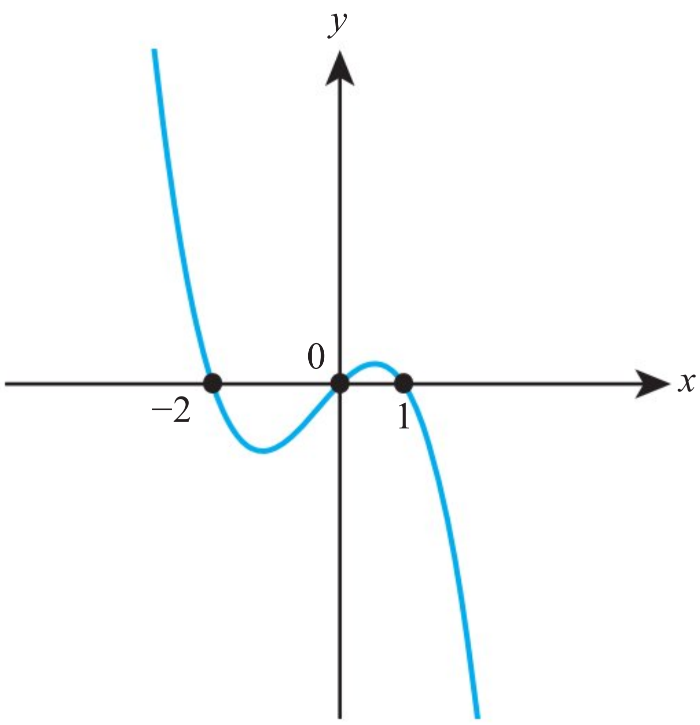


5 **a**



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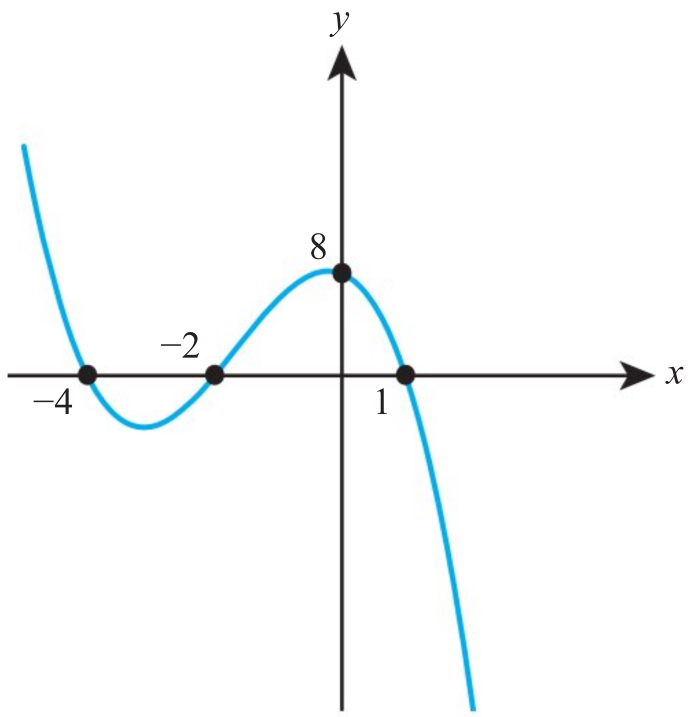
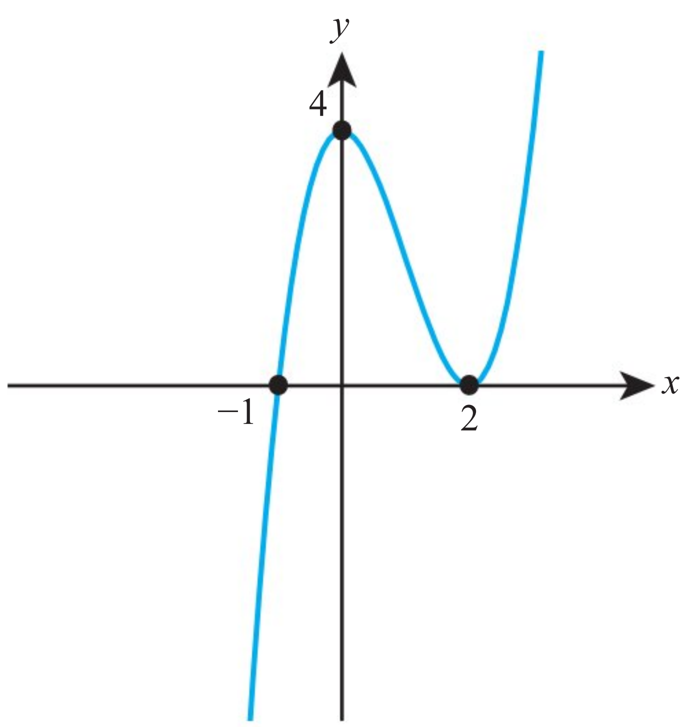


6 a



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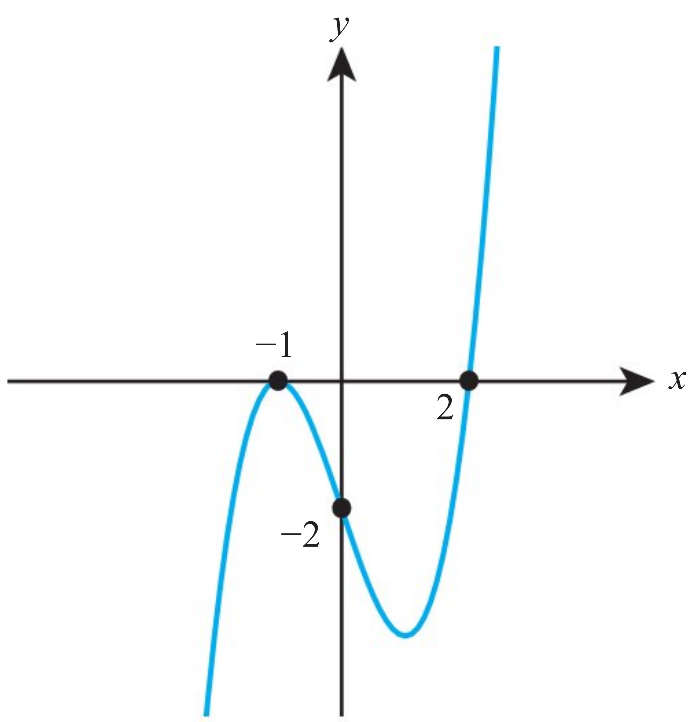

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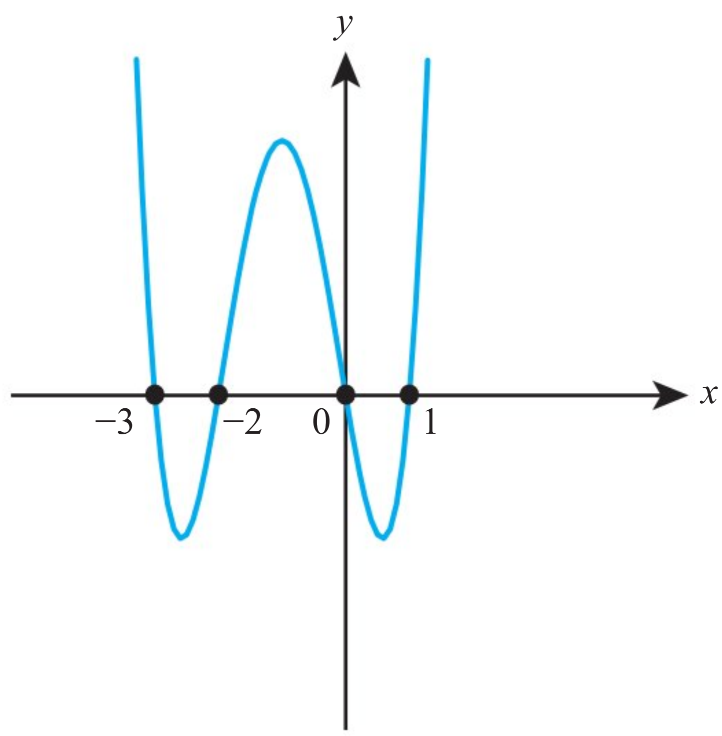

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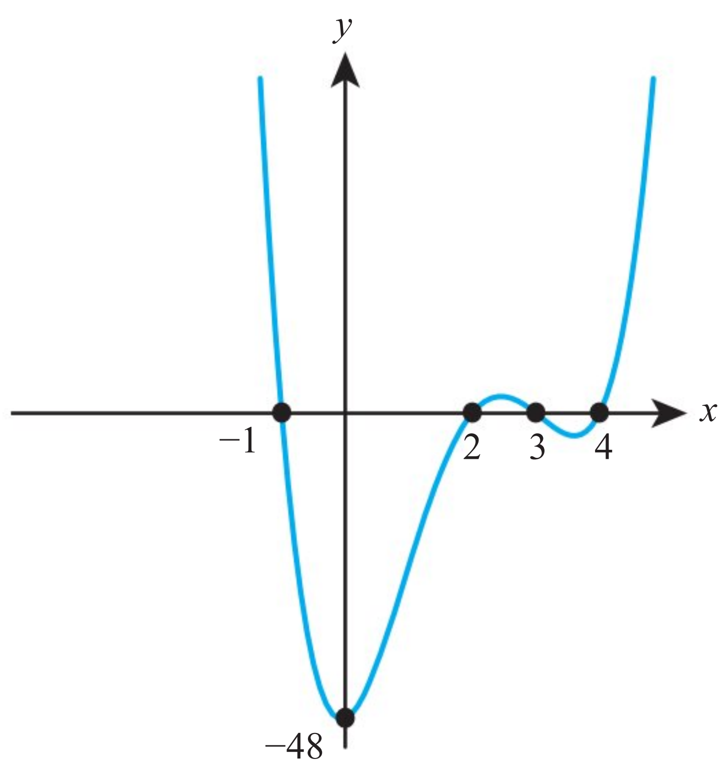

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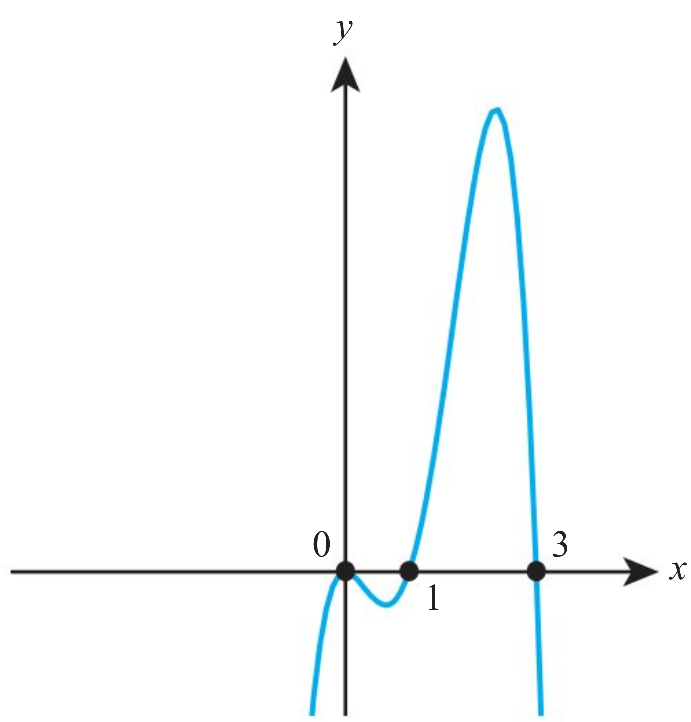

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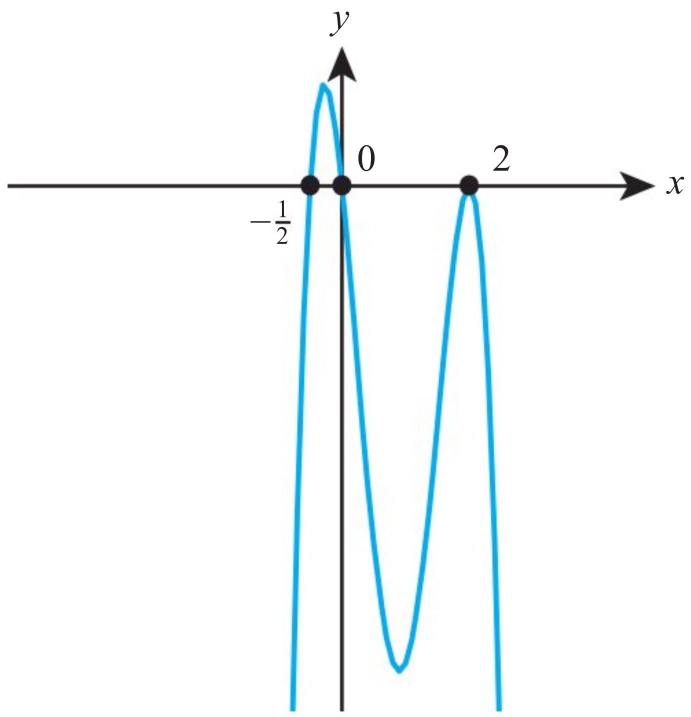
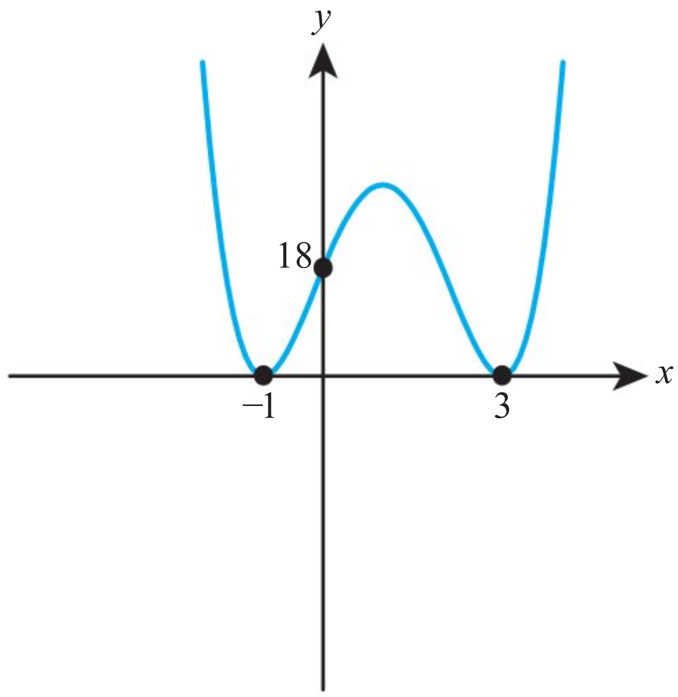


9 a



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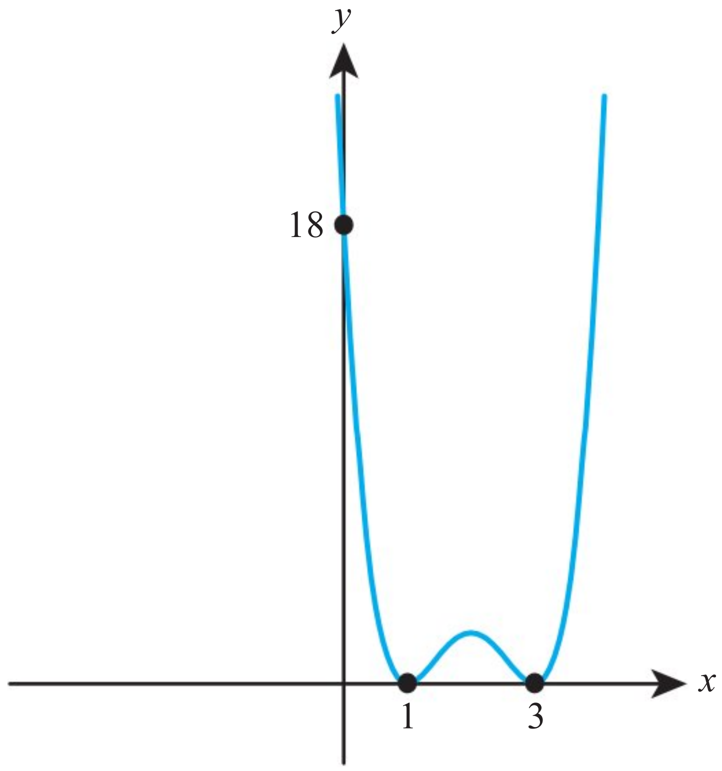
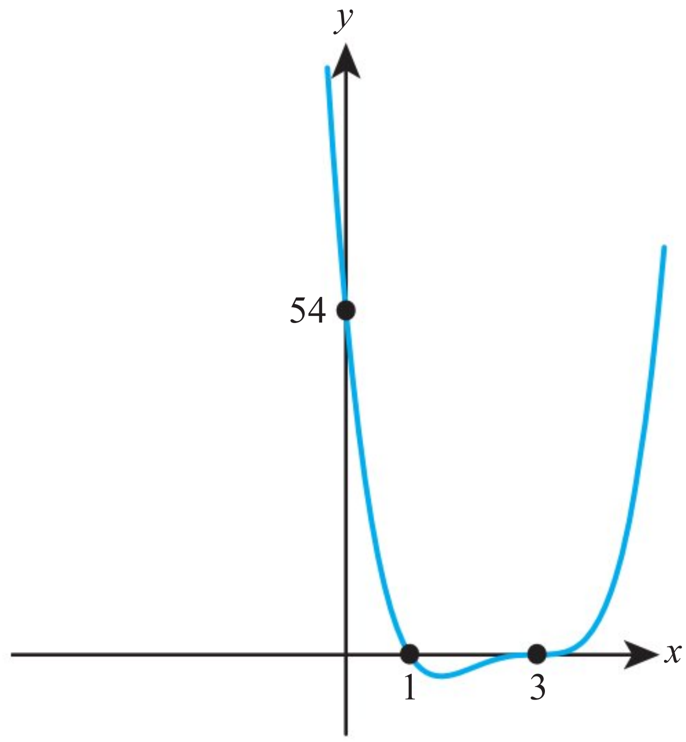
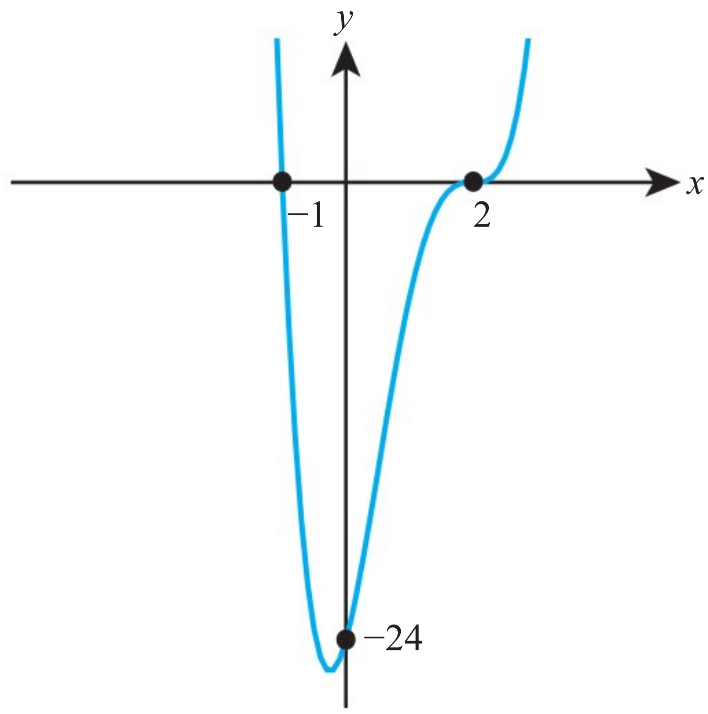


10 a



b



11 a $y = 2(x-2)(x-3)(x-4)$

b $y = 7(x-5)(x+1)(x-3)$

12 a $y = -4(x-5)(x-3)^2$

b $y = -2(x-1)(x-2)(x-3)$

13 a $y = -x(x-4)^2$

b $y = (x-2)^2(x+2)$

14 a $y = (x+2)(x+3)(x-2)(x-3)$

b $y = -4(x-3)(x-2)(x+1)(x+3)$

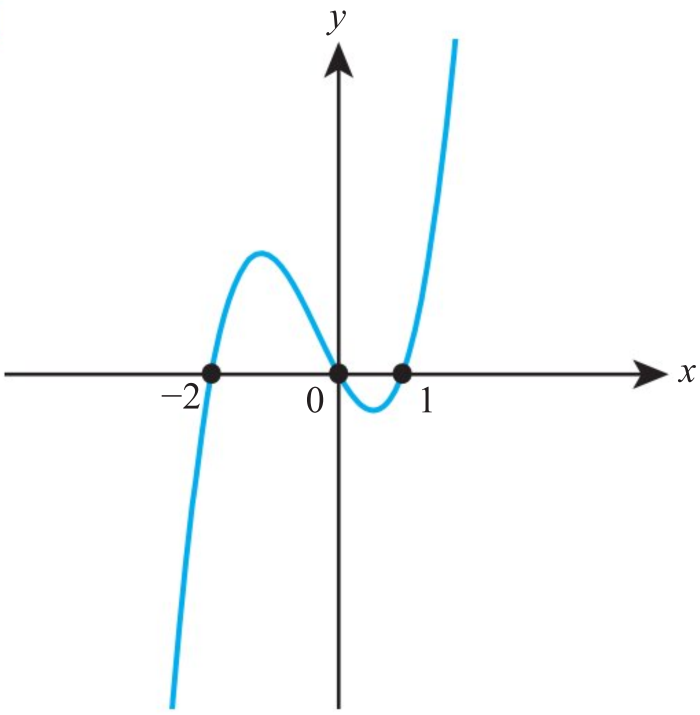
15 a $y = (x-3)(x-2)^2(x-4)$

b $y = -x^2(x-1)(x+2)$

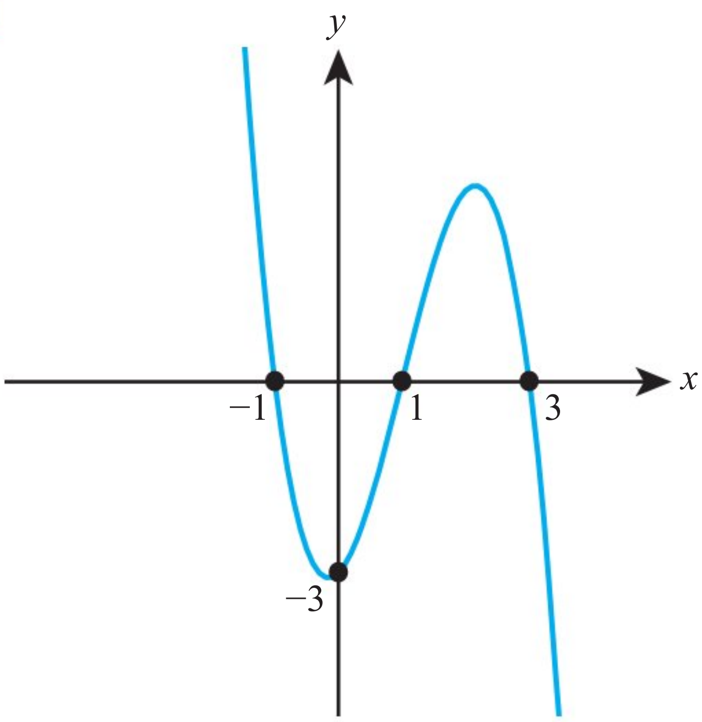
16 a $y = 2(x+1)^3(x-3)$

b $y = -x^3(x-4)$

17

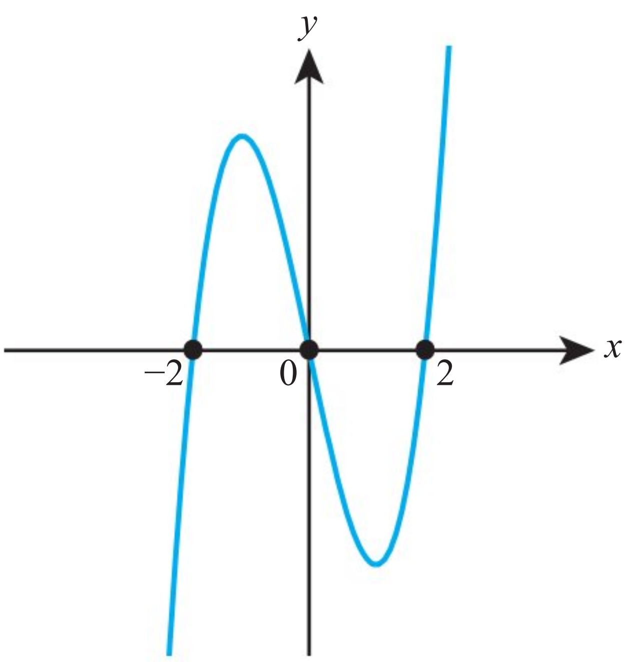


18



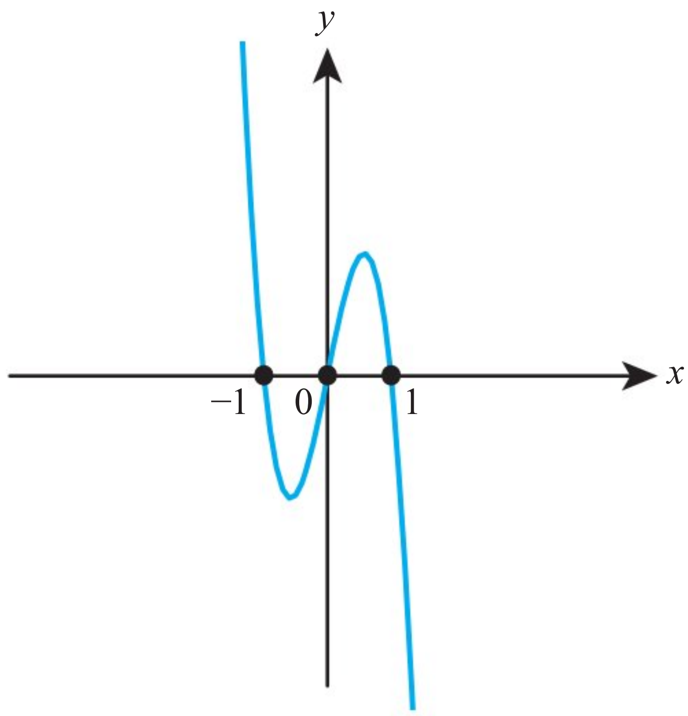
19 **a** $3x(x-2)(x+2)$

b

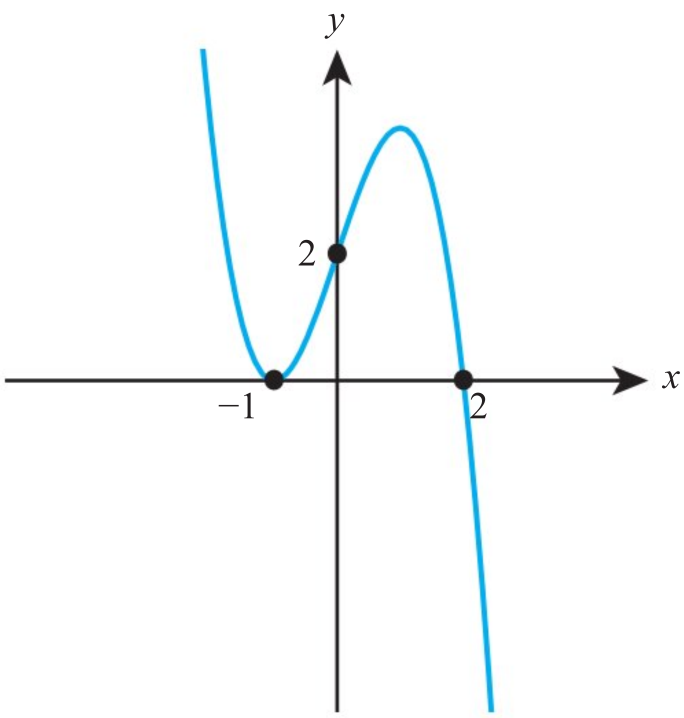


20 **a** $-5x(x-1)(x+1)$

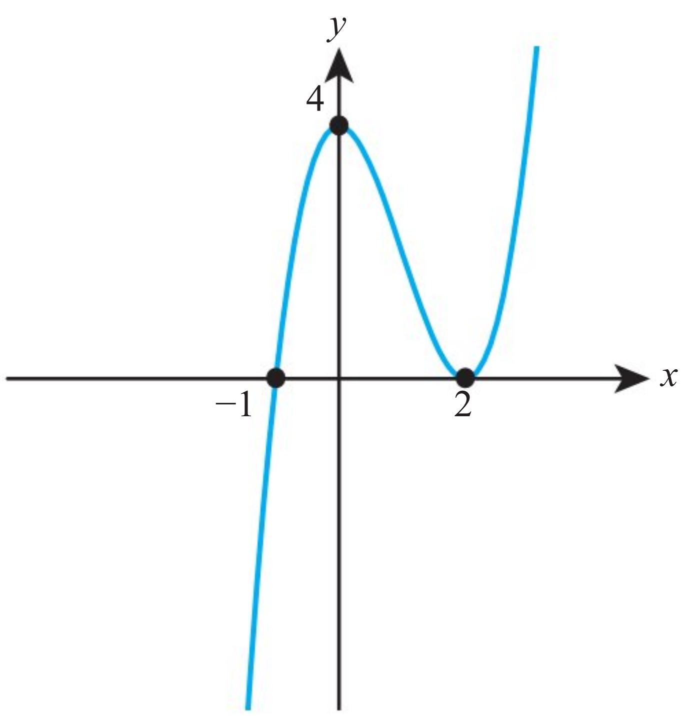
b



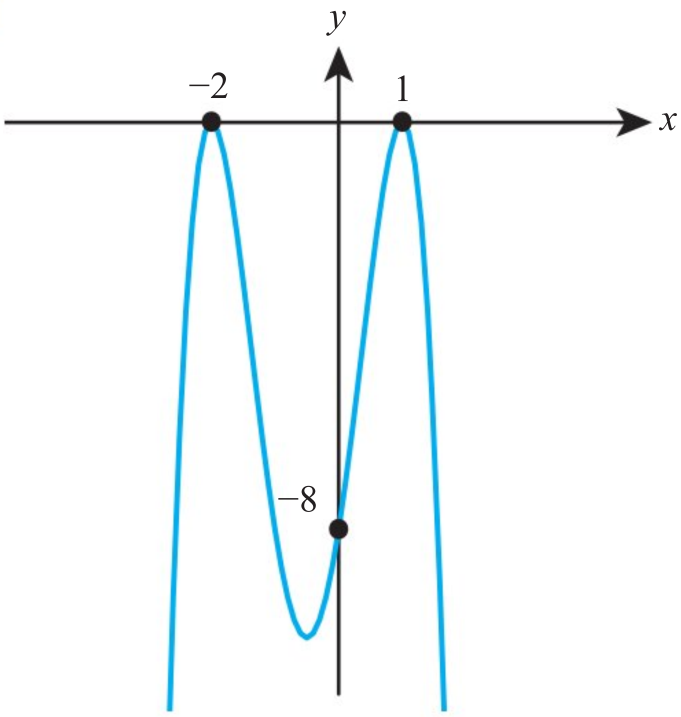
21 **a**



b

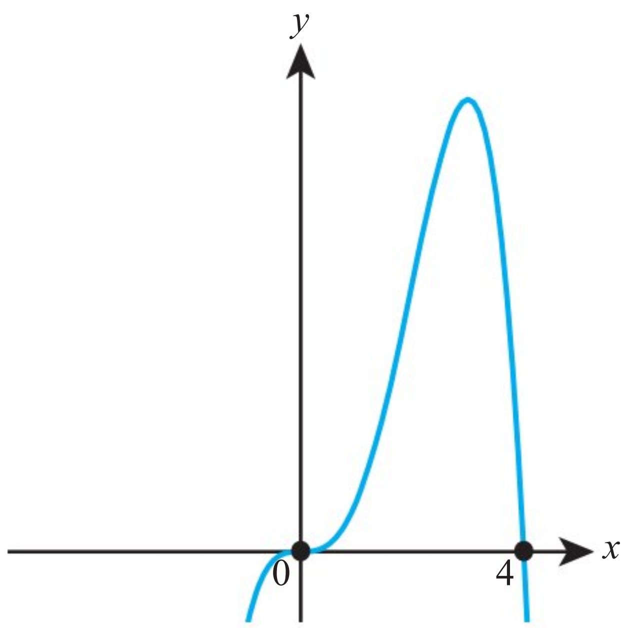


22



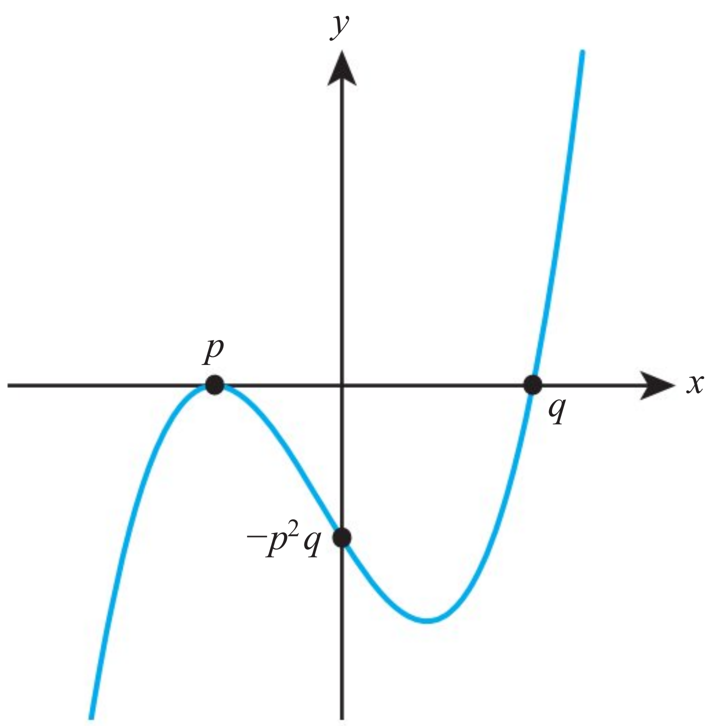
23 a $-x^3(x-4)$

b



24 a 2, −8, −6, 36 b −1, 3, 0, 0

25 a



b 1

Exercise 6B

1 a $x+2$, 3 b $x-1$, −2

2 a x^2-3x-5, −7 b x^2-6x+5, 8

3 a $3x^3-x^2-x+4$, −2

b $4x^3-2x^2+3x+1$, −4

4 a x^2-2x+4, −8

b $3x^3+3x^2+3x+3$, 3

5 a 17 b 127

6 a 28 b 2

7 a $-\frac{25}{27}$ b $-\frac{3}{8}$

8 a $-\frac{81}{8}$ b $\frac{142}{27}$

9 a −14 b −132

10 a −5 b 3

11 a $-\frac{56}{27}$ b $\frac{11}{8}$

12 a $\frac{8}{27}$ b 144

13 a $(x-1)(x+1)(x+2)$, three

b $(x-2)(x+1)(x+2)$, three

14 a $(x+3)(x-2)^2$, two

b $(x+1)(x-3)^2$, two

15 a $(x-1)(x^2-2x+10)$, one

b $(x-3)(x^2+x+5)$, one

16 a $(3x-1)(x-1)(2x-1)$, three

b $(3x-5)(x+2)(4x+3)$, three

17 2

18 17

19 0, 4

20 $a=1$, $b=-18$

21 $a=-44$, $b=48$

22 $k=-\frac{1}{2}$

23 3, 7, −8

24 0

25 b 4, 1, −3

26 b $2, \frac{3\pm\sqrt{5}}{2}$

27 a 1

28 b Three

29 b $p, -p, \frac{p}{2}$

31 $b = \pm\sqrt{\dfrac{a^3}{a-1}}$

32 $-\dfrac{1}{2}$, 2, 3i, −3i

33 14

34 $a = 37$, $b = -30$

Exercise 6C

1 a $\dfrac{23}{5}$ b $\dfrac{17}{4}$

2 a 6 b 5

3 a 9 b $\dfrac{23}{5}$

4 a −4 b $-\dfrac{3}{2}$

5 a $\dfrac{55}{9}$ b $\dfrac{41}{4}$

6 a 3 b 6

7 a $x^2 + x - 6 = 0$ b $x^2 + 4x - 5 = 0$

8 a $6x^2 - 7x + 2 = 0$ b $20x^2 - 23x + 6 = 0$

9 a $x^2 - 8x + 25 = 0$ b $x^2 - 4x + 29 = 0$

10 a $36x^2 - 36x + 25 = 0$

b $144x^2 - 216x + 97 = 0$

11 a $2x^2 - 11x + 20 = 0$ b $2x^2 + 9x + 15$

12 a $2x^2 - 15x + 150 = 0$

b $4x^2 - 3x + 3 = 0$

13 a $3x^2 - 3x + 4 = 0$ b $6x^2 - 15x + 50 = 0$

14 a $a = 6$, $k = -10$ b $a = 3$, $k = -12$

15 a $a = 2$, $k = -6$ b $a = 3$, $k = -6$

16 a $a = -9$, $k = -5$ b $a = -3$, $k = 5$

17 a $a = -4$, $k = 12$ b $a = -1$, $k = -18$

18 a $a = 3$, $k = 28$ b $a = 5$, $k = -9$

19 a $a = 1$, $k = 2$ b $a = 1$, $k = -4$

20 a $a = 0$, $k = 16$ b $a = 0$, $k = 36$

21 $\dfrac{39}{2}$

22 a $15a$ b a^2

23 a $\dfrac{3}{k} + 2$ b $\dfrac{3}{5k}$

24 $\dfrac{9}{a^2} + 4a$

25 $\dfrac{3}{2}$

26 $-\dfrac{8}{3}$

27 $-\dfrac{5}{3}$

28 a $\dfrac{56}{5}$ b $b = -33$, $c = 56$

29 a −3i, 3 + i b $a = 6$, $d = 90$

30 a $\dfrac{a}{3}$ b 2

31 $\dfrac{11}{4}$

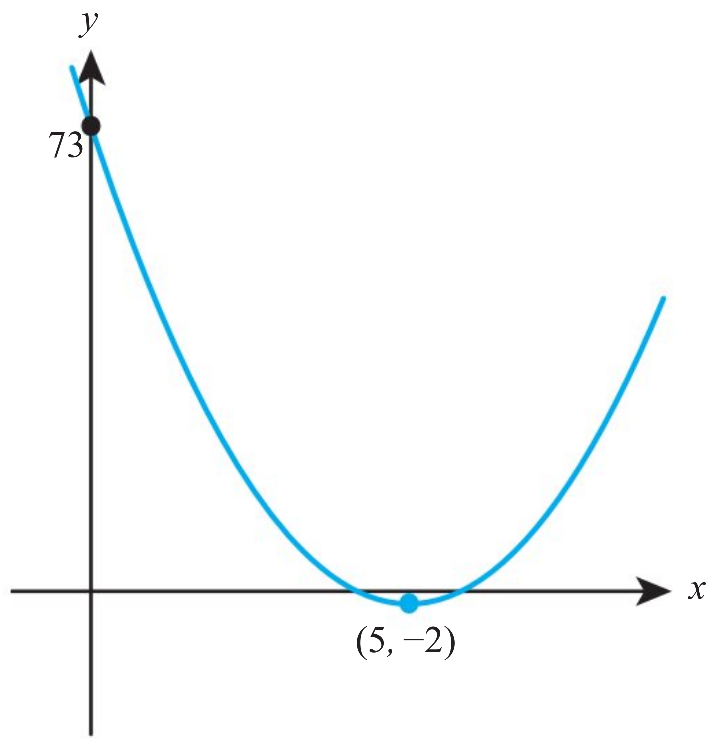
33 a $\dfrac{2}{5}, -\dfrac{11}{25}$ b $25x^2 + 11x + 4 = 0$

34 a $\dfrac{49}{9}$ b $9x^2 - 49x + 64 = 0$

35 b $-\dfrac{23}{64}$ c $64x^2 + 23x + 8 = 0$

36 a $\alpha + \beta + 2\sqrt{\alpha\beta}$ b $3(x-5)^2 - 2$

c



d $\sqrt{10 + 2\sqrt{\dfrac{73}{3}}}$

38 $15x^2 + 26x + 15 = 0$

39 b $-2a^2$

c Sum of squares is negative

40 b i $-\dfrac{b}{a}, -\dfrac{d}{a}$

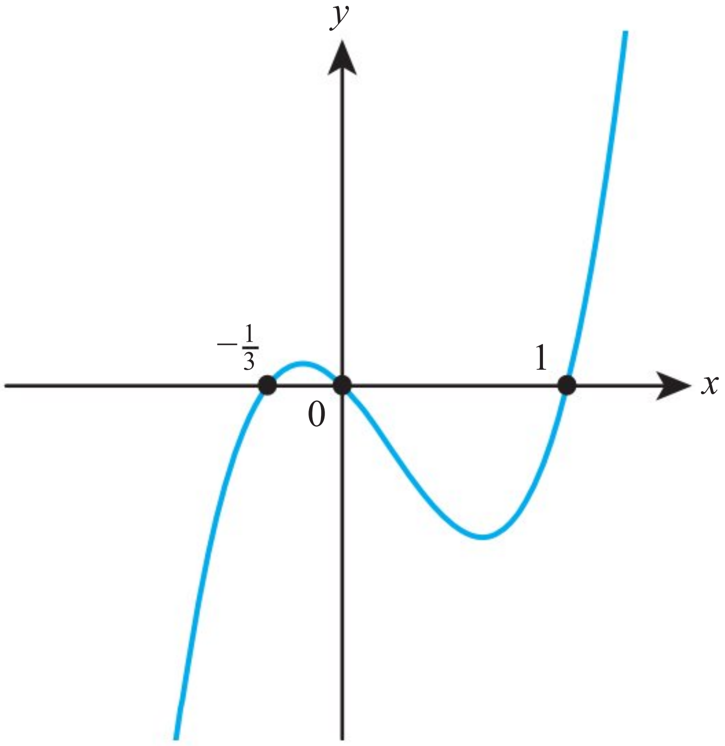
c ii $\dfrac{49}{4}$, 4

iii $4x^3 - 28x^2 + 49x - 16 = 0$

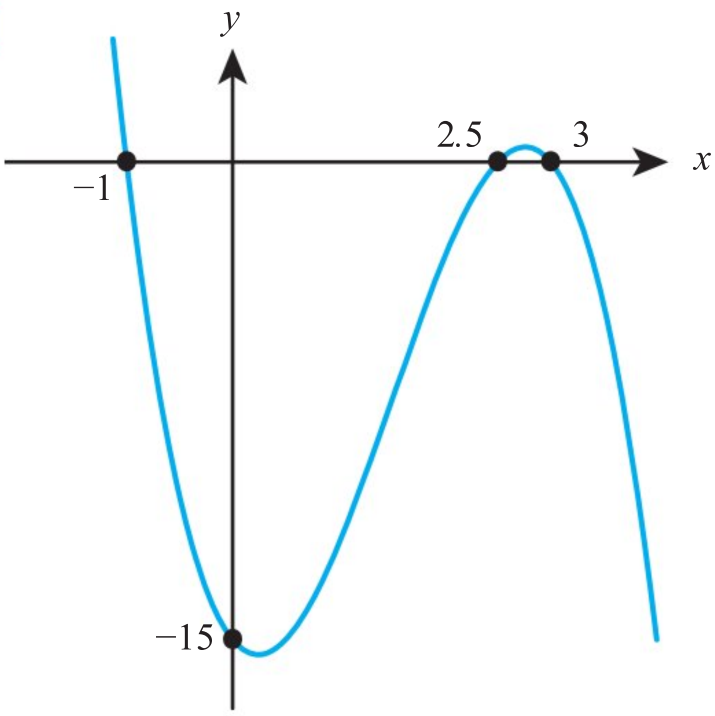
Chapter 6 Mixed Practice

1 a $x(x-1)(3x+1)$

b



2



3 $3(x+1)(x+2)(x-1)$

4 $p=2,\ k=3$

5 $a=1,\ \ b=0$

6 $a=-2,\ b=-3$

7 $p=-3,\ q=1$

8 $b=-40,\ c=325$

9 6

10 4

11 ± 2

12 a $-\frac{5}{3}$ b $\frac{2}{5}$

13 −4, 5

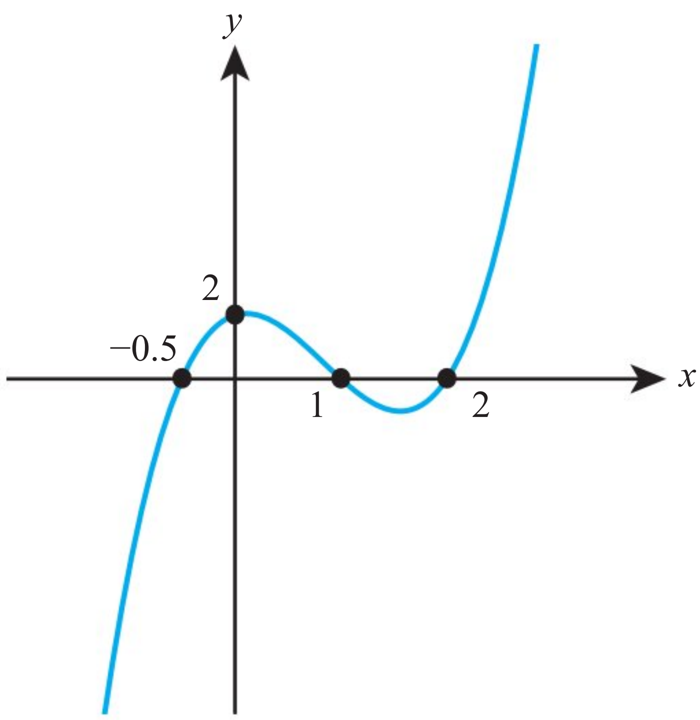
14 $a=1,\ b=2,\ c=-12, d=-18, e=27$

15 $-x^4+2x^3-2x+1$

16 $\frac{1}{81}x^4-\frac{4}{27}x^3$

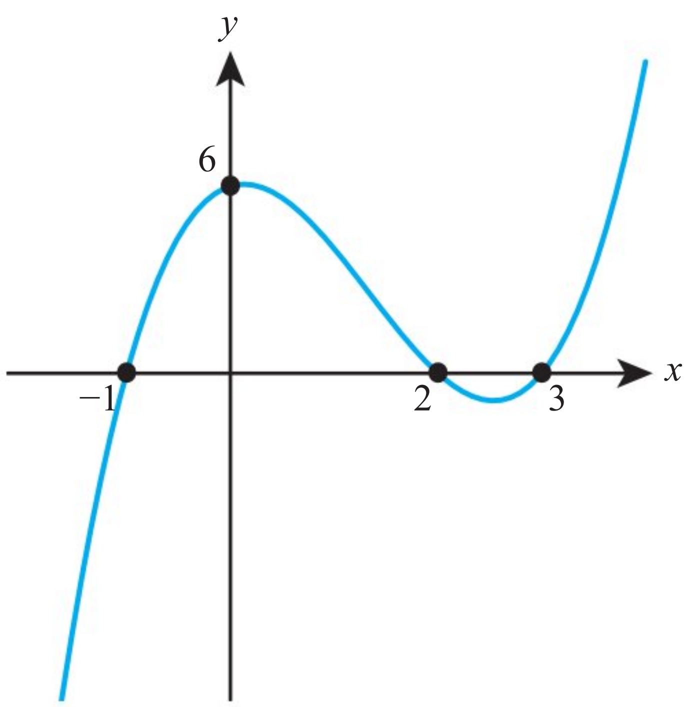
17 b $(x-2)(x-1)(2x+1)$

c

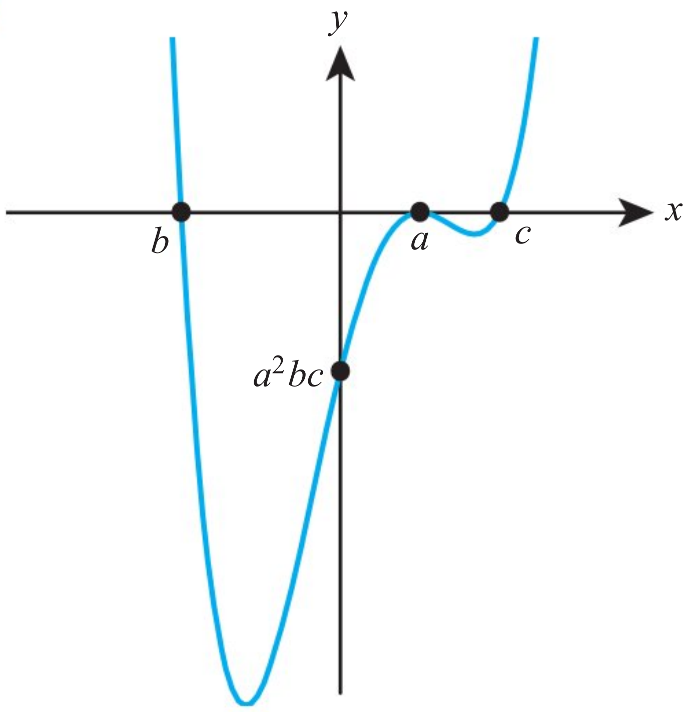


18 b $(x-2)(x+1)(x-3)$

c



19



20 $-3,\ 2 \pm \text{i}$

21 $a = 3,\ b = -42$

22 $\dfrac{3 \pm \text{i}\sqrt{3}}{3}$

24 $(a,\ b) = \pm\left(\dfrac{5}{3},\ -\dfrac{4}{3}\right),\ \pm\left(-\dfrac{1}{3},\ \dfrac{8}{3}\right)$

25 $p - 2,\ q = 45$

26 a $-\dfrac{26}{9}$

b $9x^2 + 26x + 49 = 0$

27 a 2

b 22

28 a $-\dfrac{2}{5}, \dfrac{3}{5}$

b $\dfrac{1}{135}$

29 a -2

30 50

31 a $2 + \text{i},\ 2 - \text{i}$

b $a = -12,\ b = 15$

32 $a = -10,\ b = -18$

33 b $b = -8,\ e = 12$

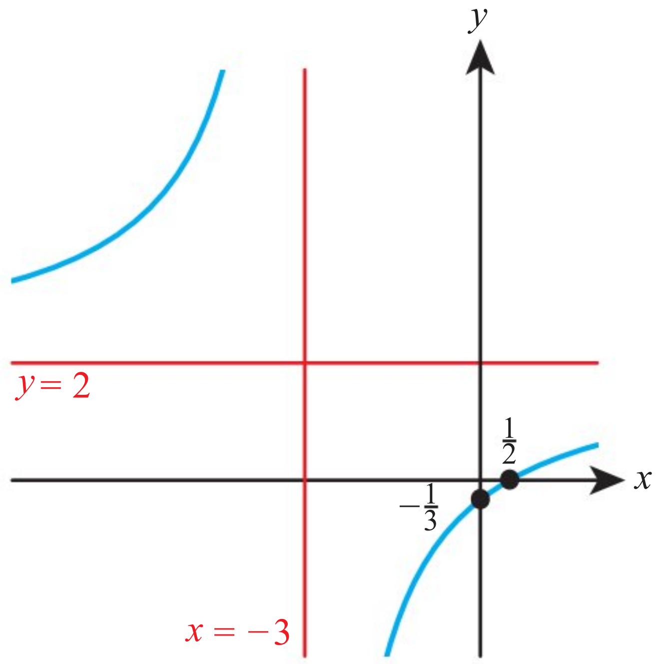
34 $c = 19,\ d = -6$

35 b ii -20

c -27

Chapter 7 Prior Knowledge

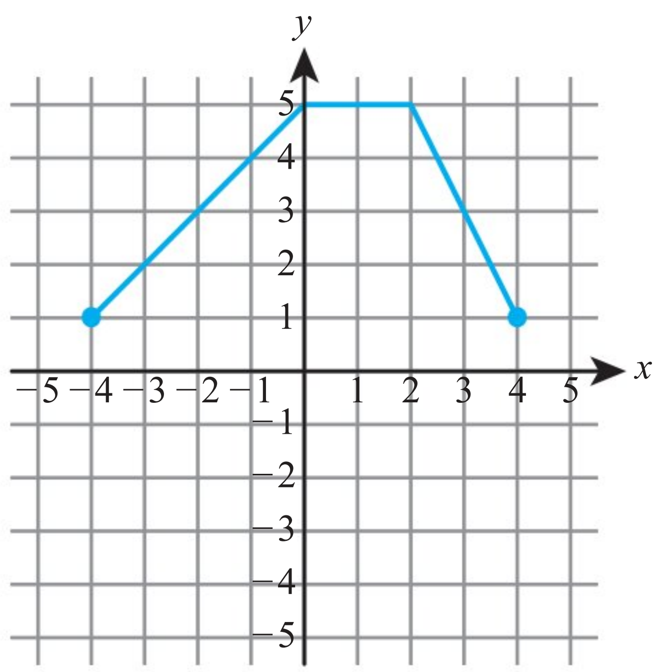
1



2 $-4 < x < 2$

3 1.40

4 a



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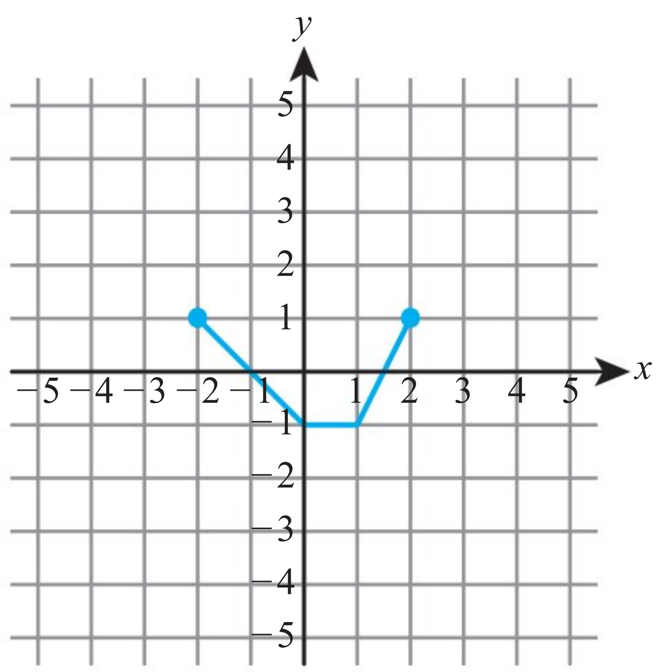
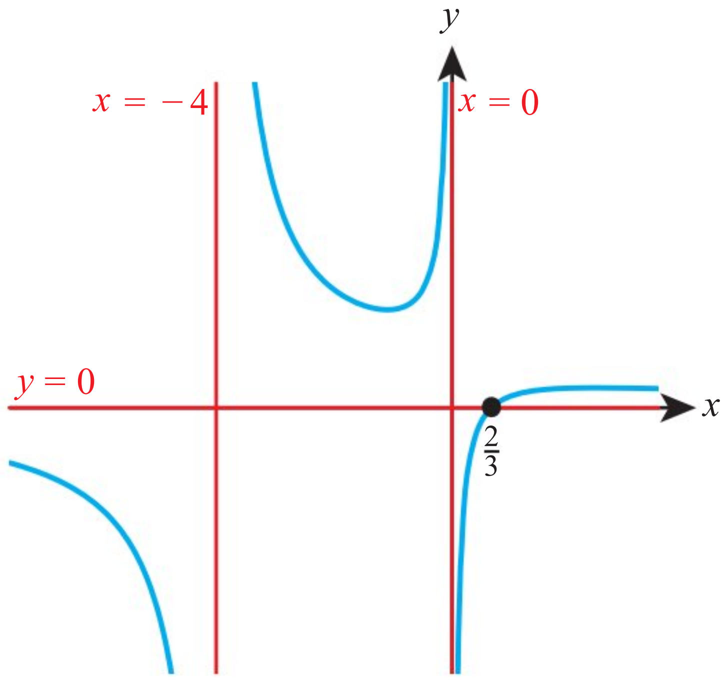


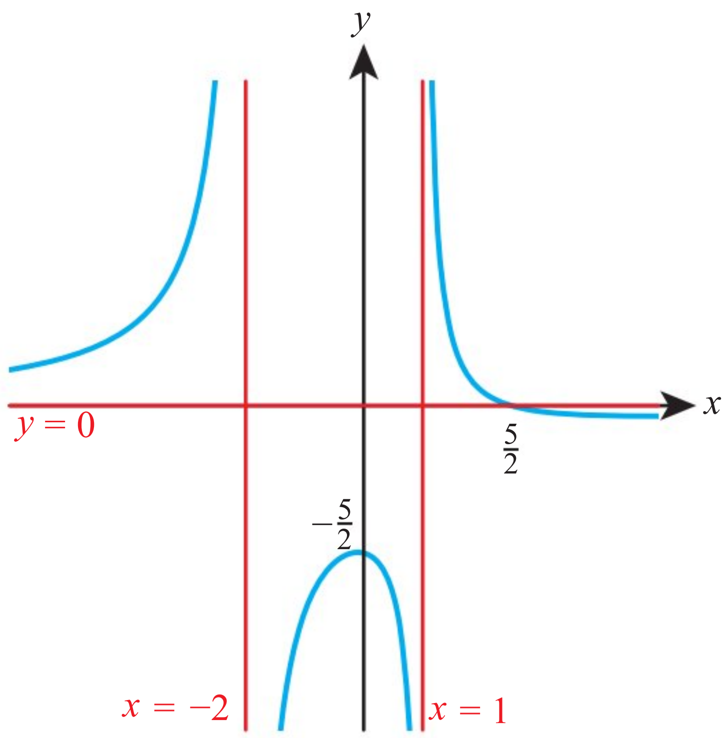
5 $\dfrac{3x + 1}{2 - x},\ x \neq 2$

Exercise 7A

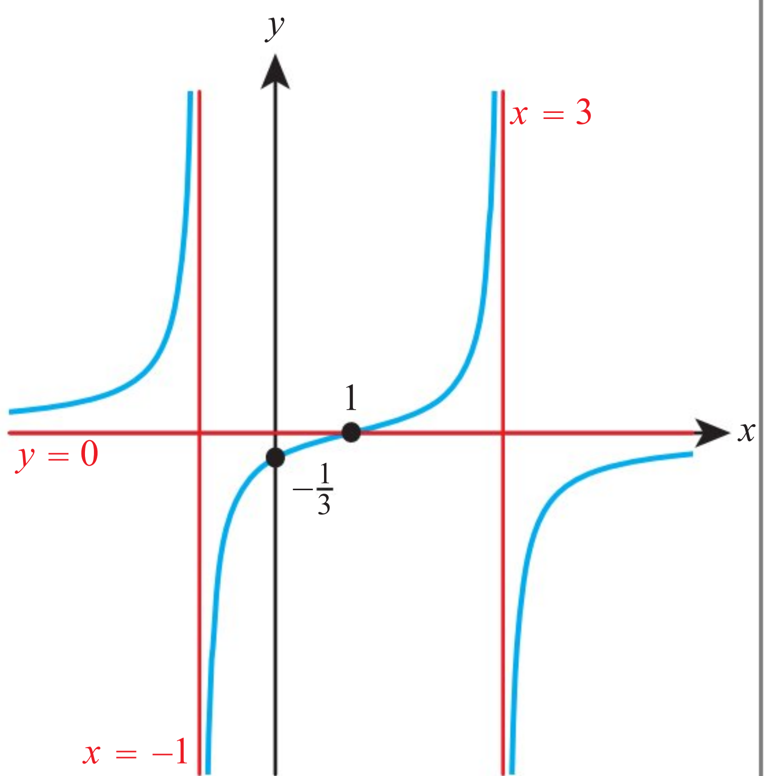
1 a



b



2 a



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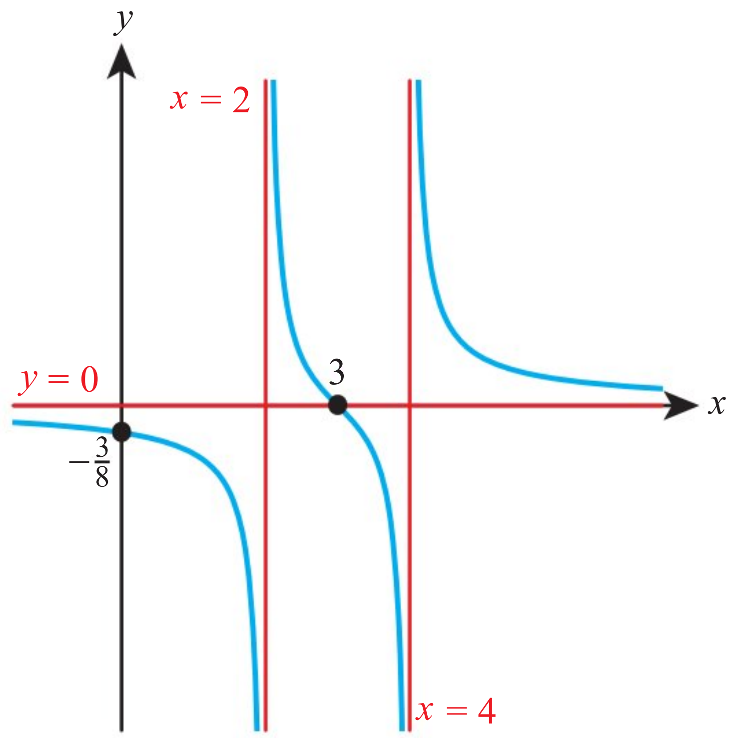
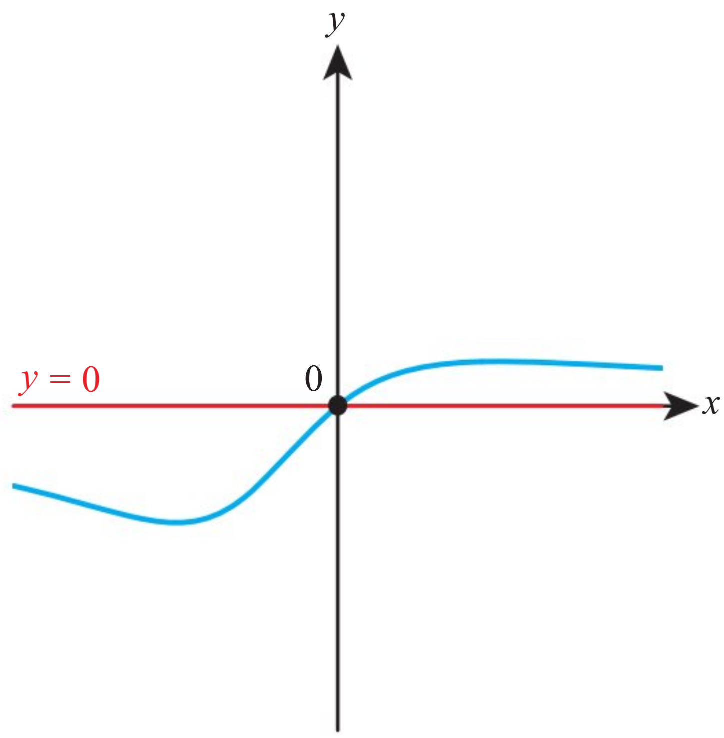
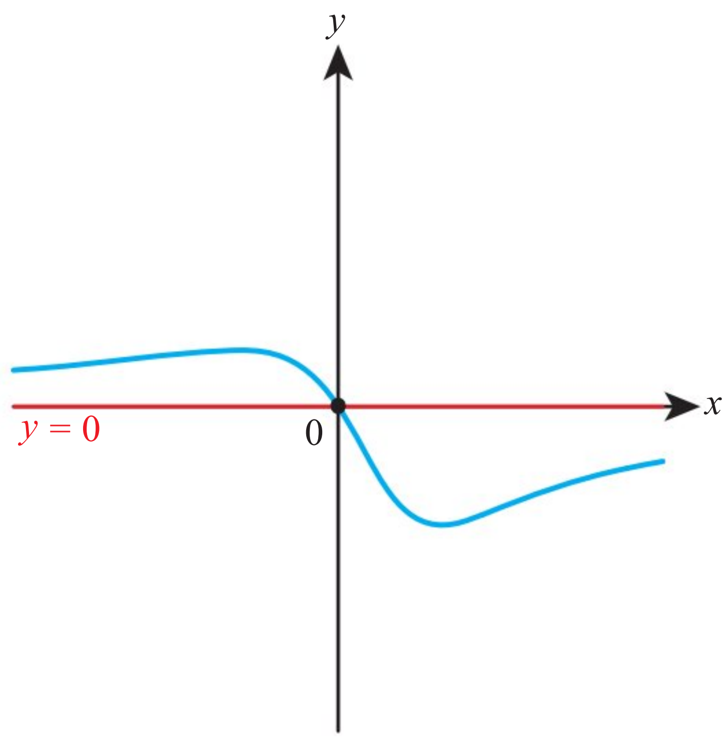


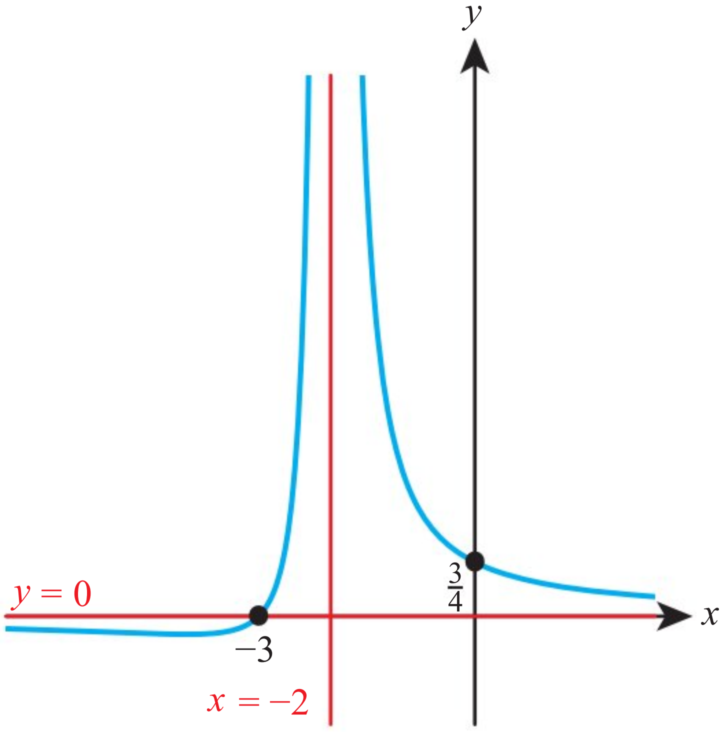
3 a



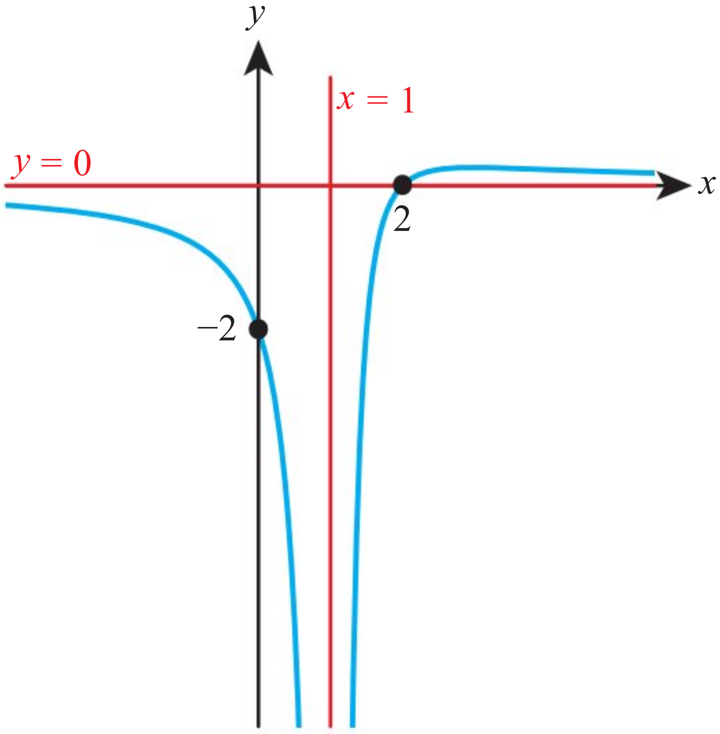
b



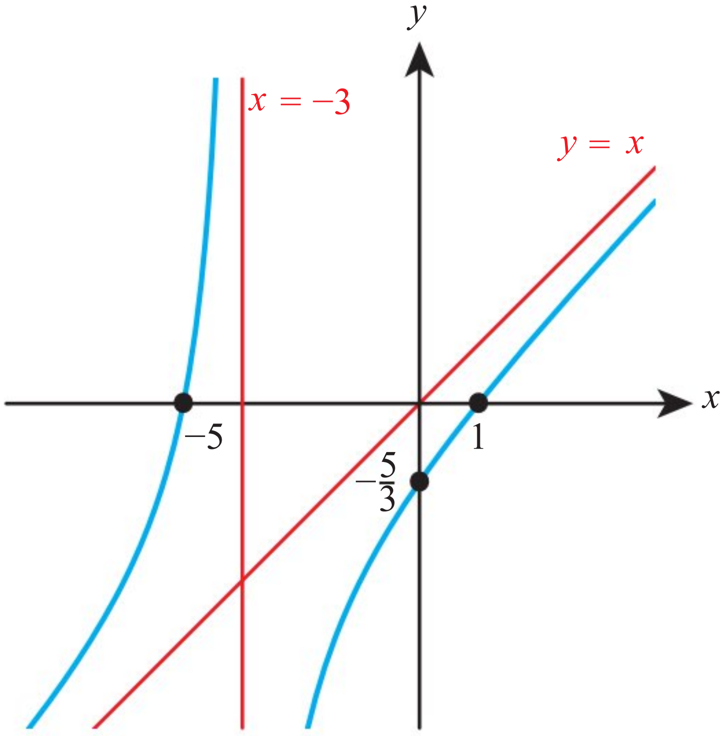
4 a


y
y = 0
−3
x = −2
3/4
x

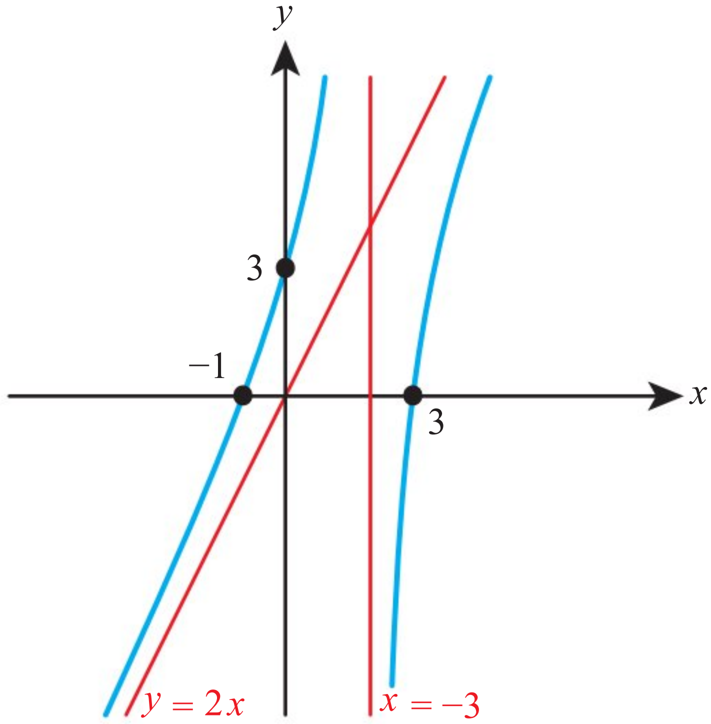
b


y
x = 1
y = 0
2
−2
x

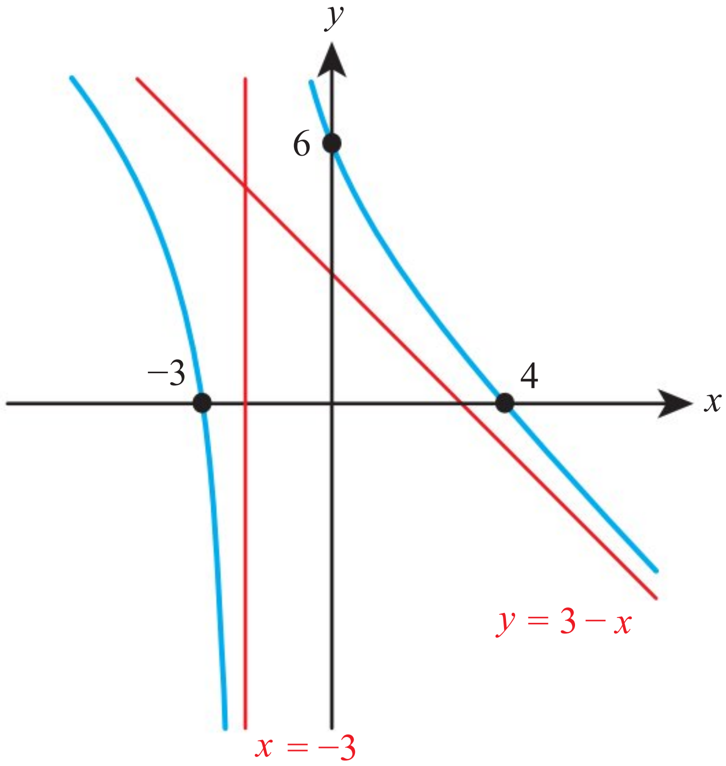
5 a


y
x = −3
y = x
−5
1
−5/3
x

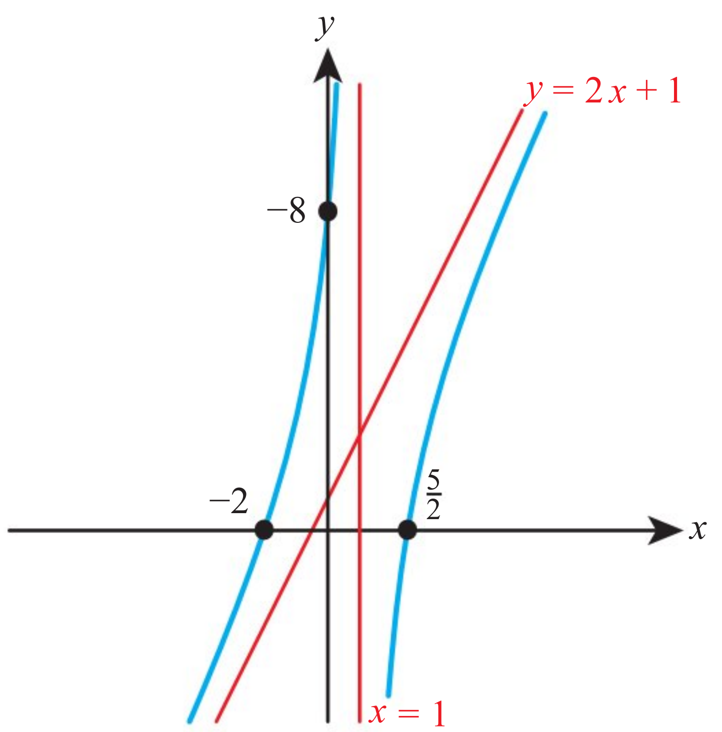
b


y
3
−1
3
x
y = 2x
x = −3

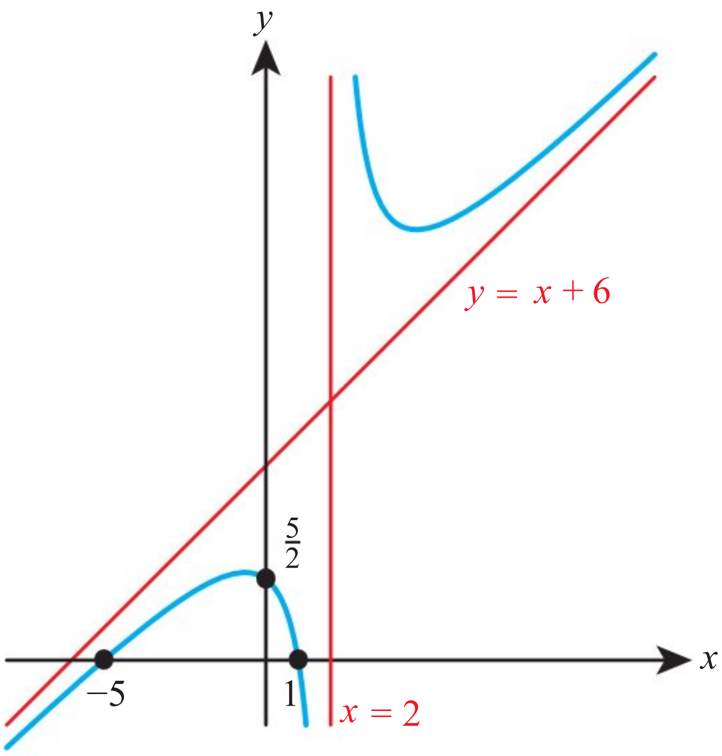
6 a


y
6
−3
4
x
y = 3 − x
x = −3

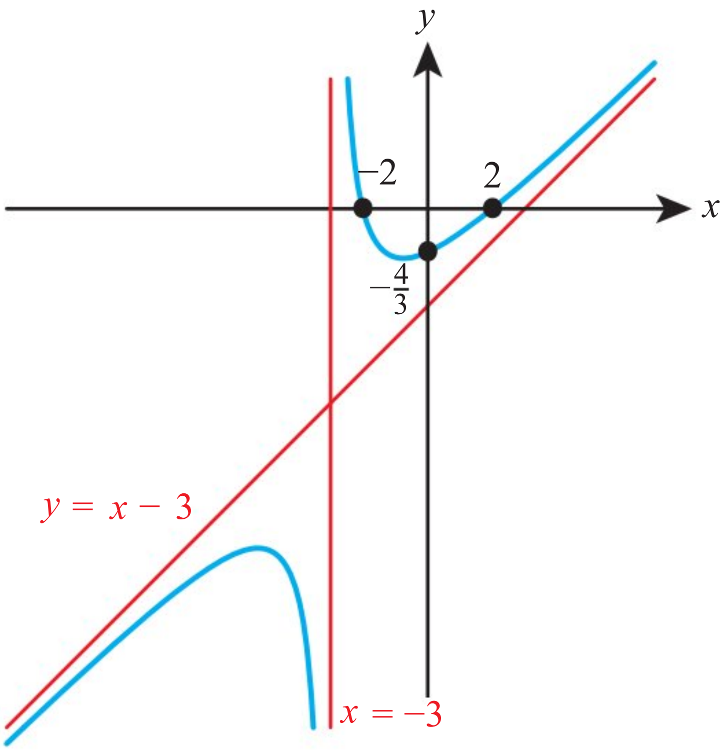
b


y
y = 2x + 1
−8
−2
5/2
x
x = 1

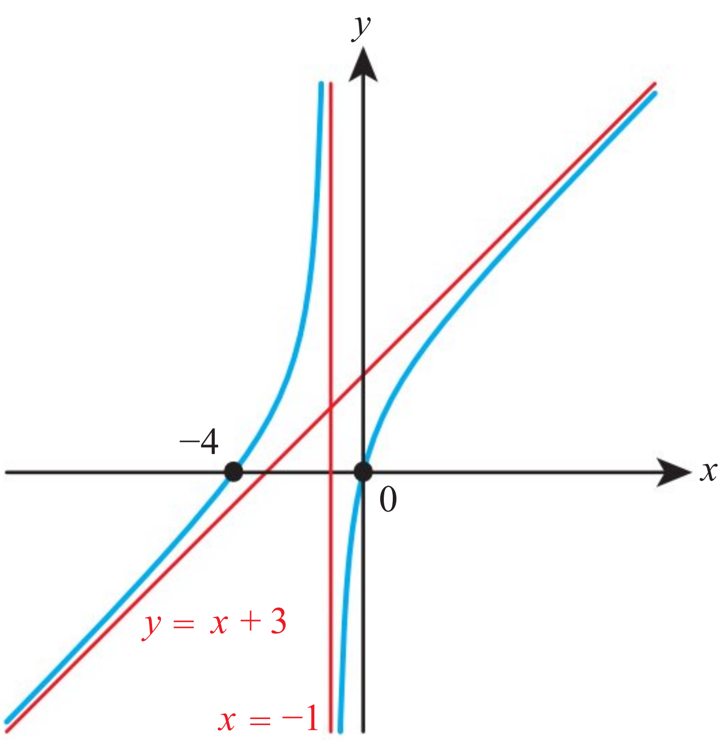
7 a



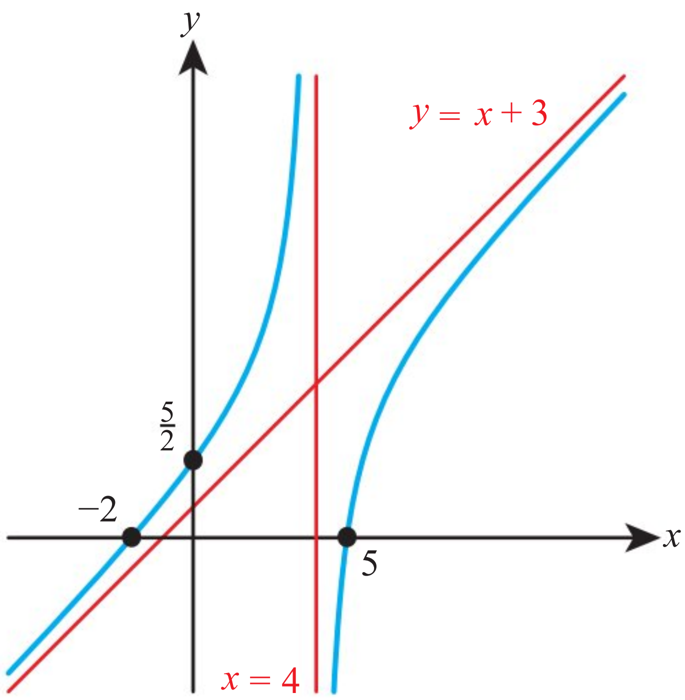
b



8 a

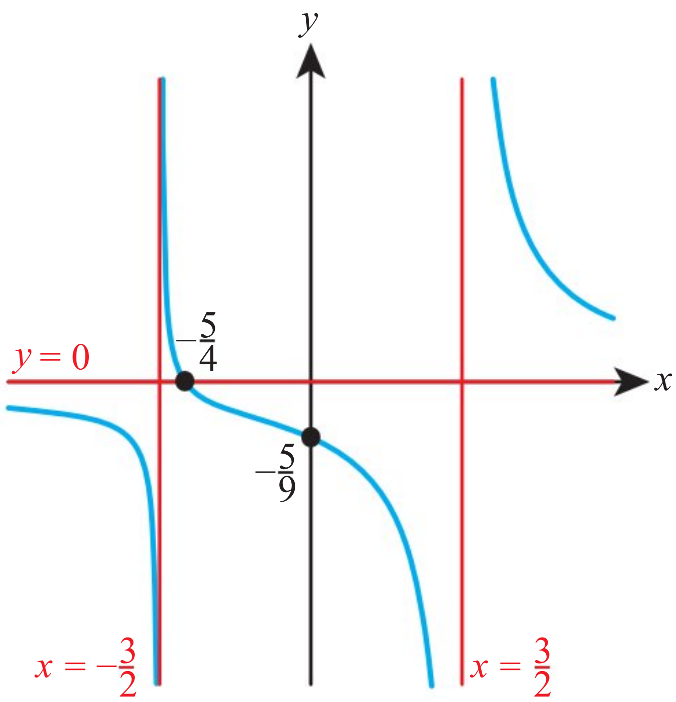


b



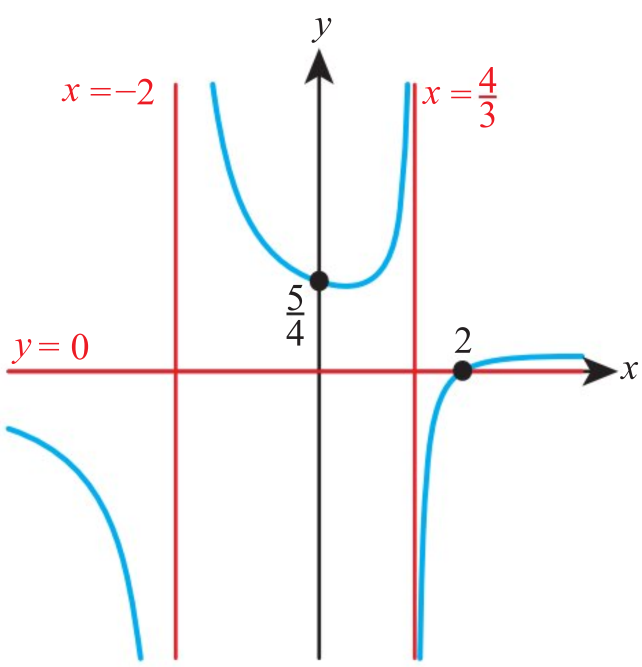
9 a $x = \pm\frac{3}{2}$

b



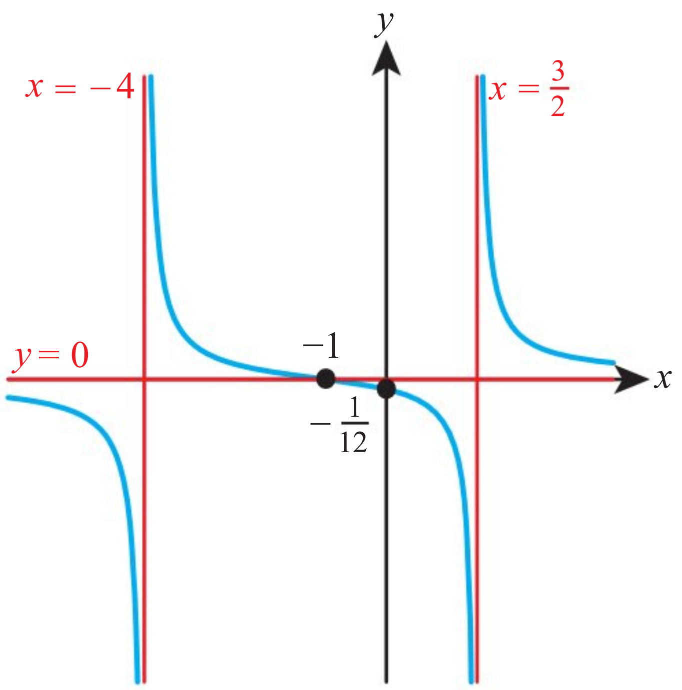
10 a $x = \frac{4}{3}, x = -2$

b

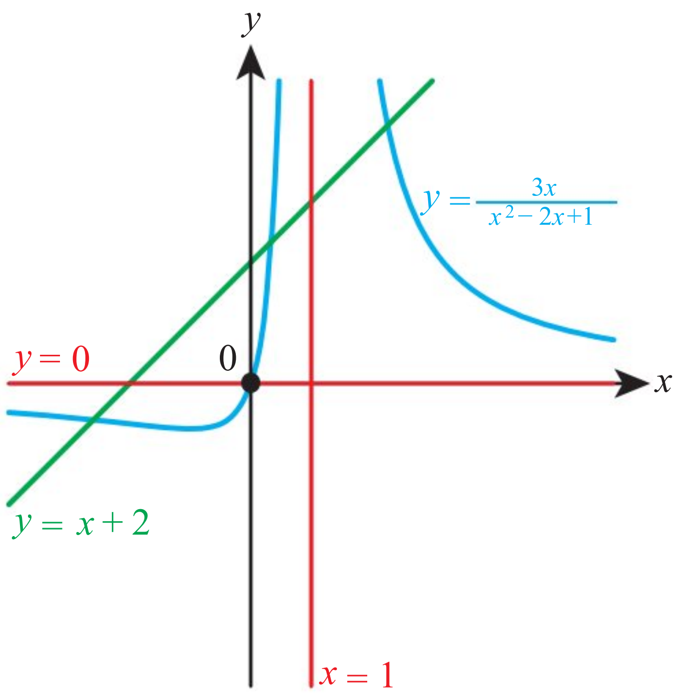


11 a $k = 5$ b $x = \frac{3}{2}$

c

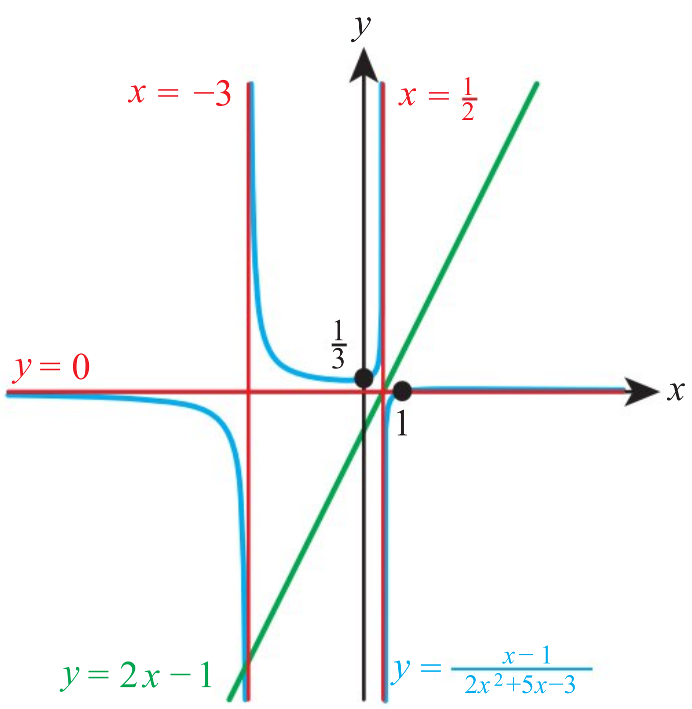


12 a



b Three

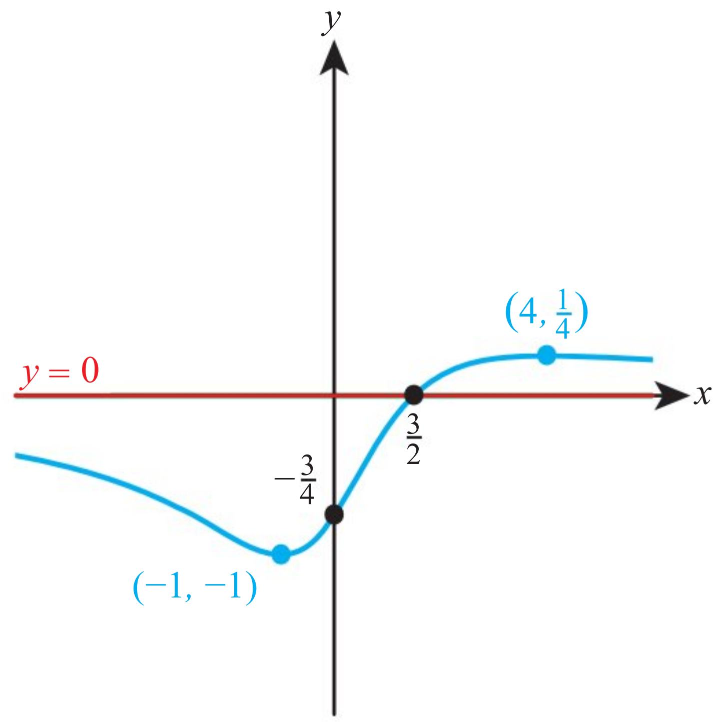
13 a



b One

14 a ii $(-1, -1), \left(4, \frac{1}{4}\right)$

b

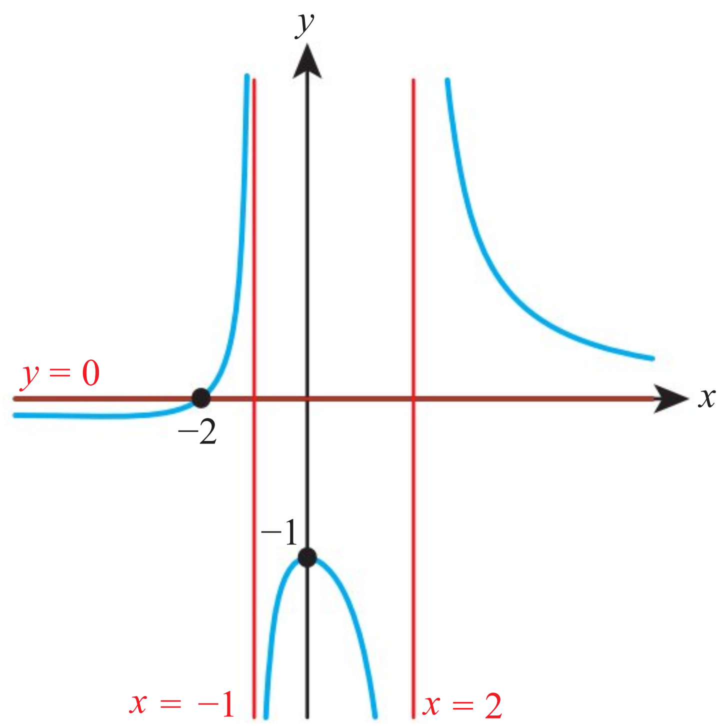


15 a i $k \geqslant -\frac{1}{9}$ or $k \leqslant -1$

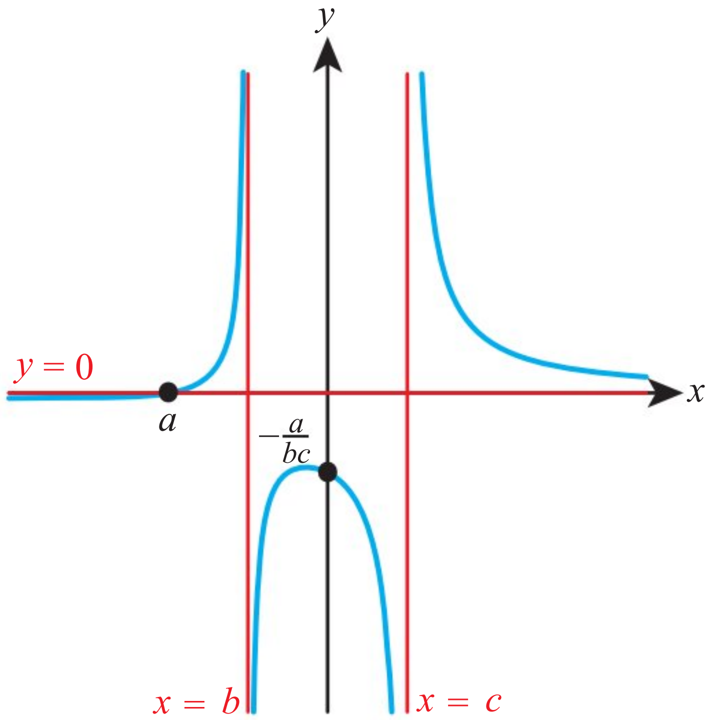
ii $f(x) \geqslant -\frac{1}{9}$ or $f(x) \leqslant -1$

b $x = 2$, $x = -1$, $(-2, 0)$, $(0, -1)$

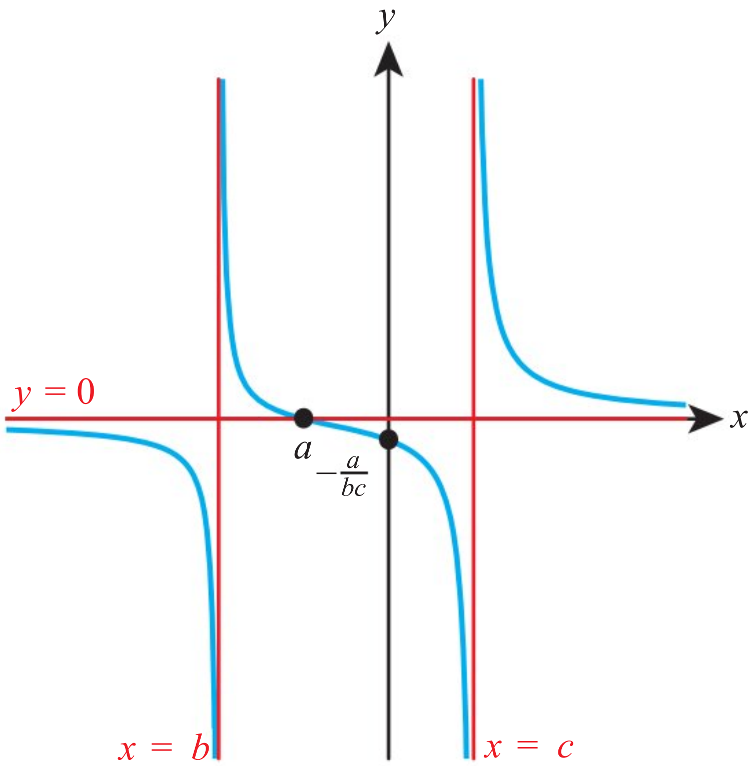
c



16 a



b

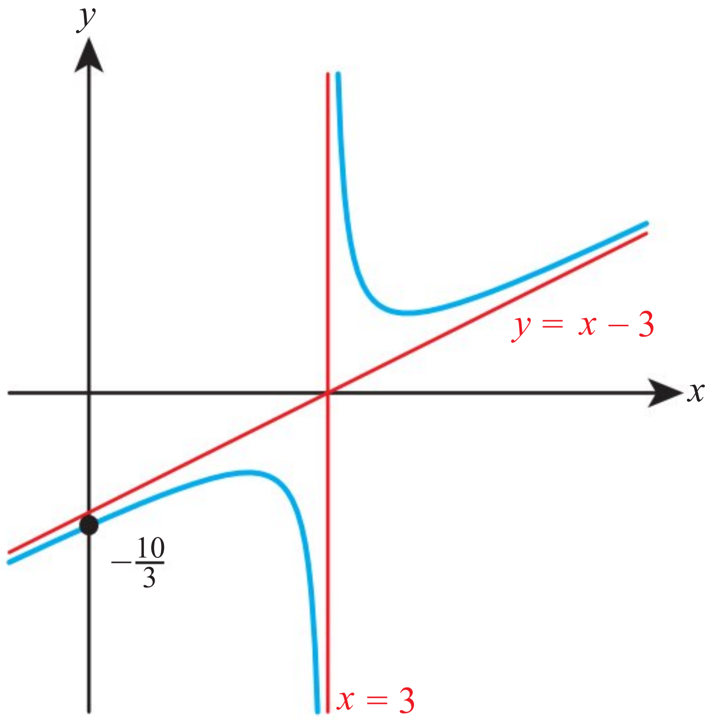


17 a $A = 1,\ B = -3$

b $(4, 2), (2, -2)$

c $\left(0, -\frac{10}{3}\right),\ x = 3$

d



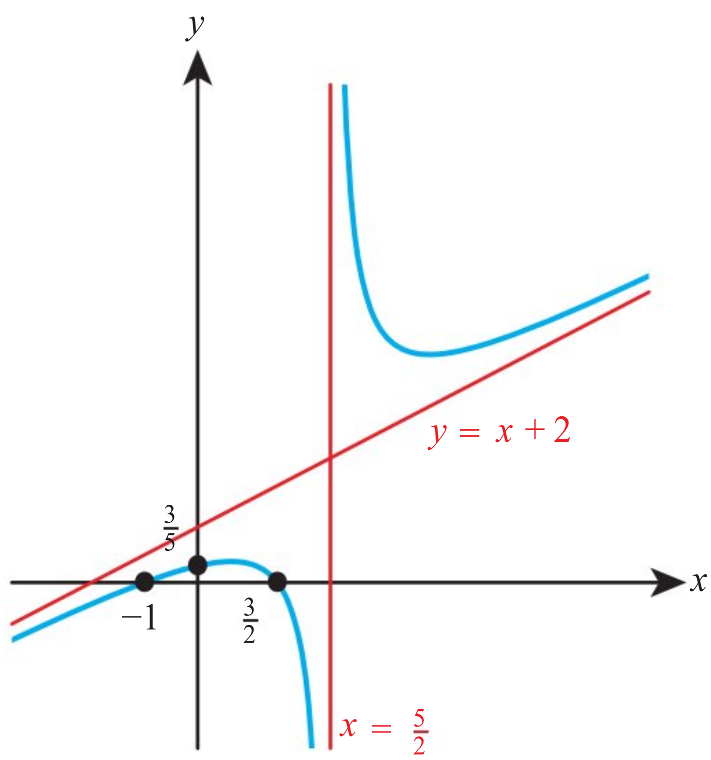
18 a $A = 1,\ B = 2,\ C = 7$

b $y = x + 2$

c $f(x) \geqslant \frac{9 + 2\sqrt{14}}{2}$ or $f(x) \leqslant \frac{9 - 2\sqrt{14}}{2}$

d $\left(0, \frac{3}{5}\right), \left(\frac{3}{2}, 0\right), (-1, 0),\ x = \frac{5}{2}$

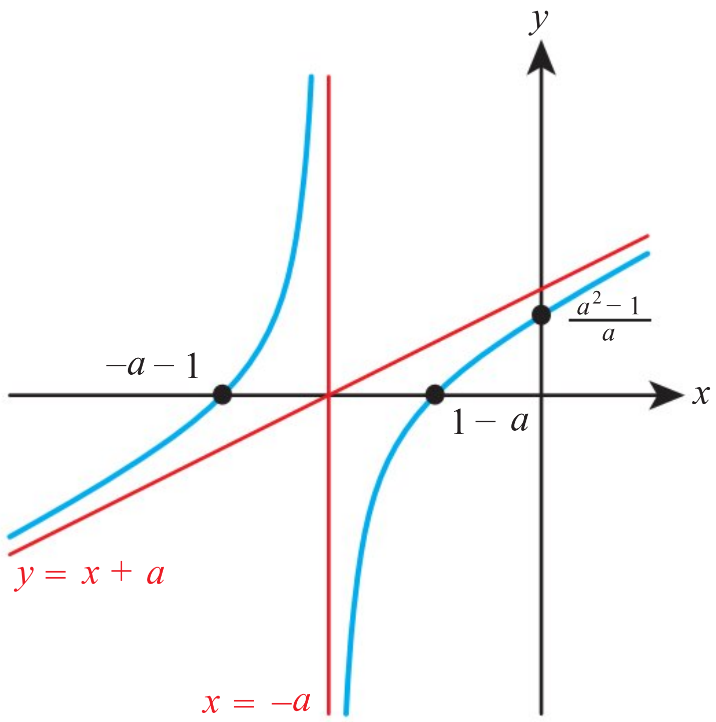
e



19 $-2 < c < 0$

20 a $y = x + a$

c



Exercise 7B

1 a $0 < x < 1$ or $x > 2$

b $-1 < x < 0$ or $x > 5$

2 a $x < 0$ or $2 < x < 4$

b $x < 0$ or $3 < x < 6$

3 a $x \leqslant -3$ or $2 \leqslant x \leqslant 5$

b $x \leqslant 1$ or $3 \leqslant x \leqslant 4$

4 a $x \leqslant -2$ or $3 \leqslant x \leqslant 4$

b $x \leqslant 1$ or $2 \leqslant x \leqslant 8$

5 a $-1 < x < 2$ or $x > 2$

b $x > 4$

6 a $x \leqslant 0.820$ b $x \leqslant -1$

7 a $x \leqslant 1.82$ b $x \leqslant 1.32$

8 a $0.728 < x < 11.9$ b $1.06 < x < 2.79$

9 a $-2.50 < x < -1.22$ or $0.220 < x < 1.50$

b $-3.98 < x < -2.06$

10 a $-0.901 \leqslant x \leqslant -0.468$ or $x \geqslant 0.081$

b $x \leqslant 1.22$ or $2.10 \leqslant x \leqslant 2.91$

11 $-2 < x < 0$ or $x > \frac{3}{2}$

12 b $x \leqslant -\frac{3}{2}$ or $-1 \leqslant x \leqslant 2$

13 b $x > \frac{1}{2}$

14 $a < x < b$ or $x > c$

15 $x < a$

16 $x \in [-2.27, -0.251]$

17 $-0.933 \leqslant x \leqslant -0.377$ or $0.371 \leqslant x \leqslant 1.76$ or $x \geqslant 2.18$

18 $x \in (-0.727, 1.48) \cup (13.7, \infty)$

19 $b = -5, c = 7, d = -1$

20 $a = -2, b = -7, c = 7, d = 15$

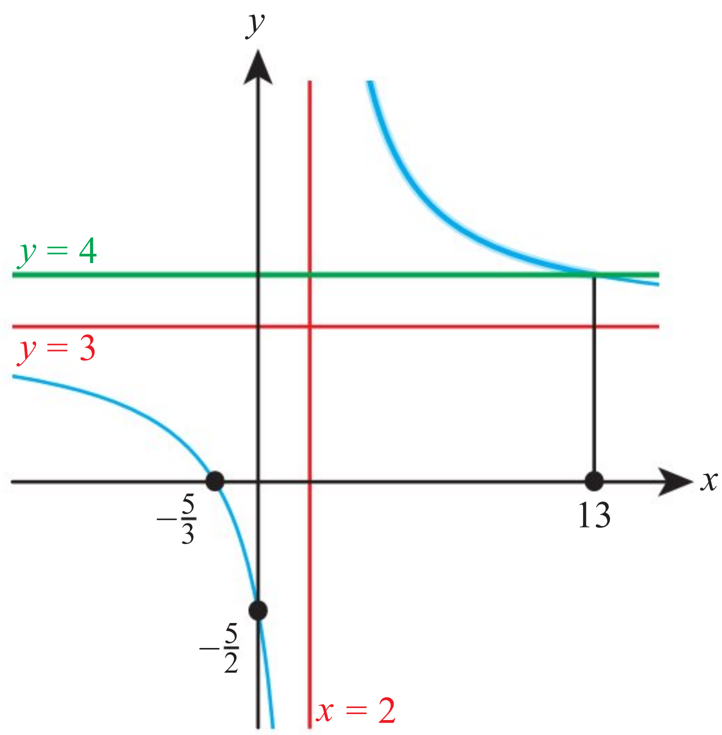
21 $-3 < x \leqslant -1$ or $2 < x \leqslant 2.27$

22 $-3.26 \leqslant x < -2$ or $-1.54 \leqslant x \leqslant 1.29$ or $x > \frac{4}{3}$

23 $x \in (6, \infty)$

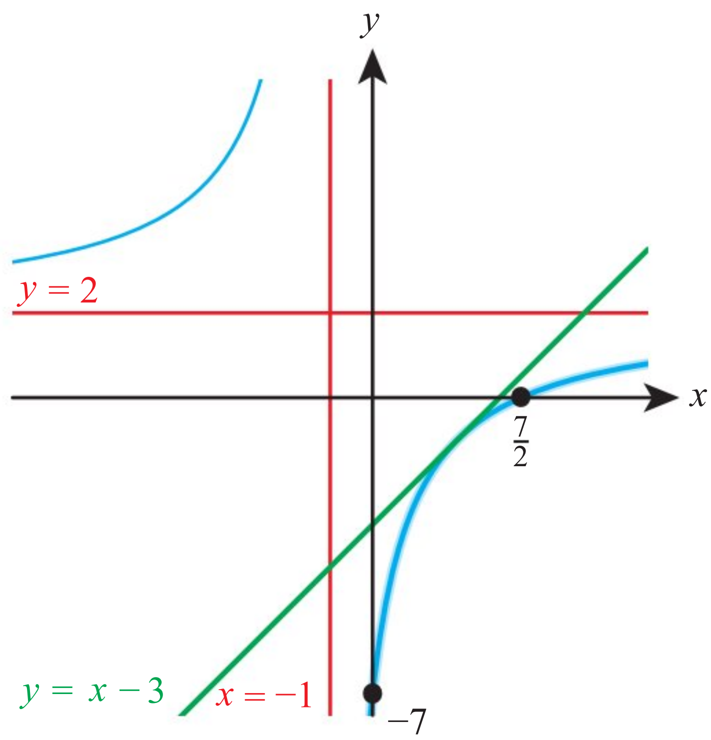
24 $x \in (-1, 1) \cup (3, \infty)$

25 a



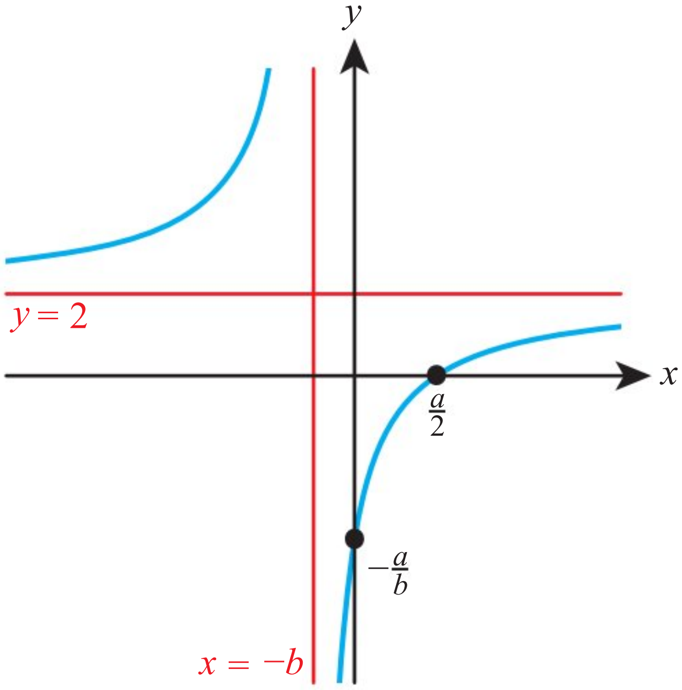
b $2 < x \leq 13$

26 a



b $x > -1, x \neq 2$

27 **a**

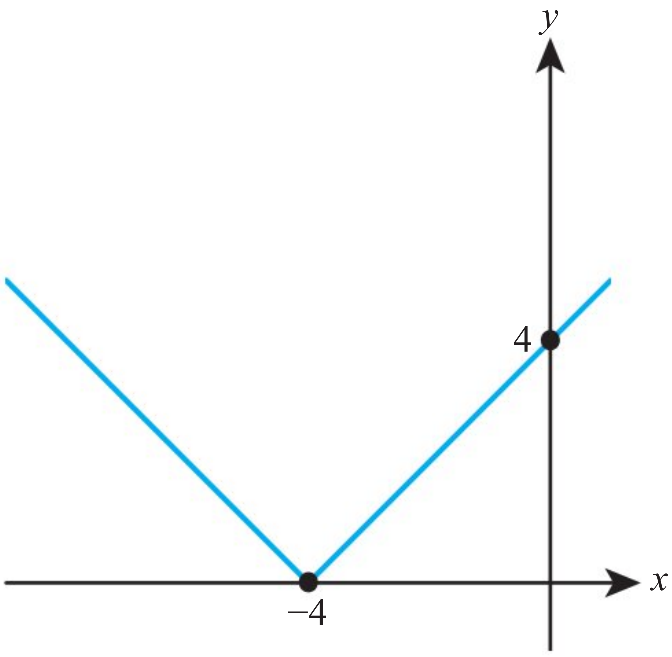


b $-a - 3b < x < -b$

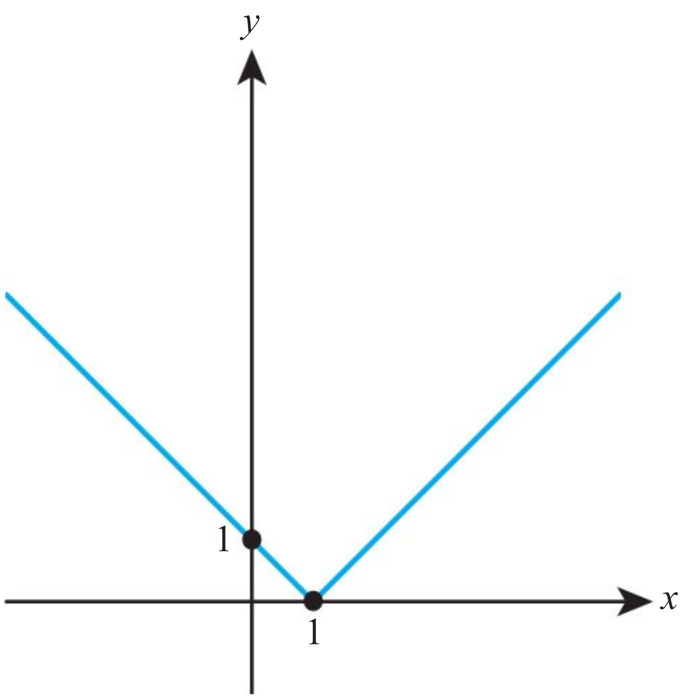
28 $p = 2,\ q = -0.5,\ r = 0.5$

Exercise 7C

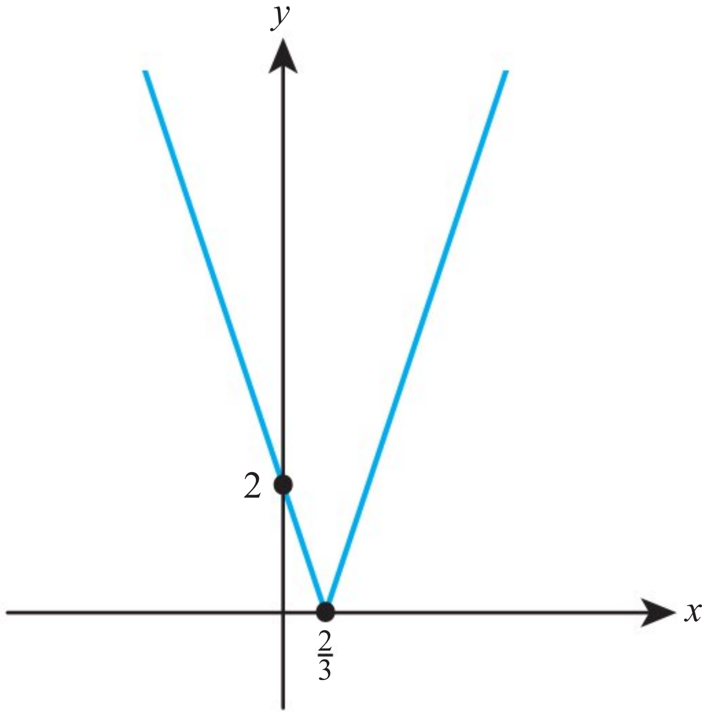
1 **a**



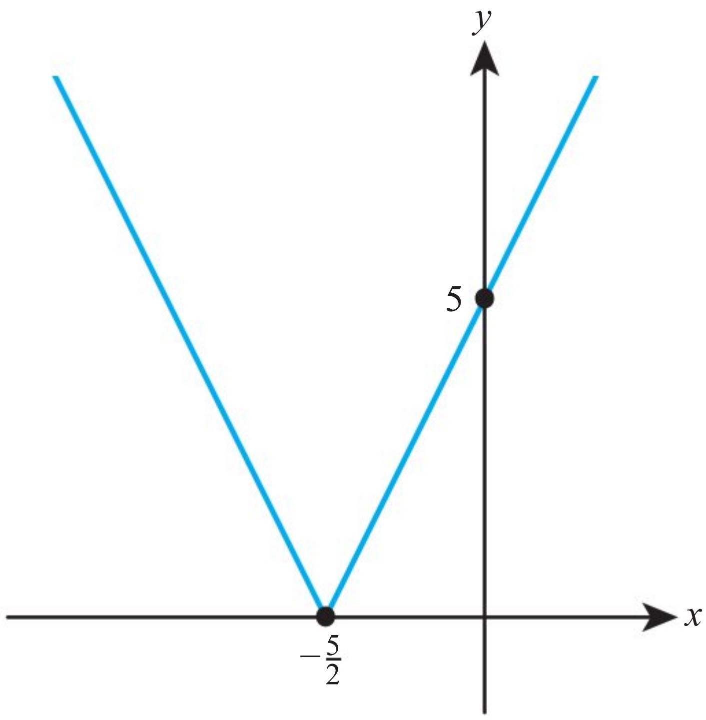
b



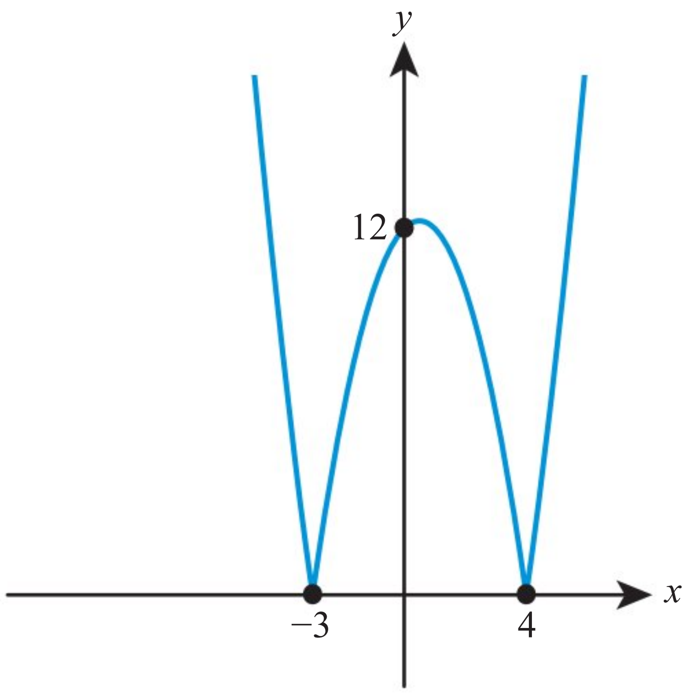
2 **a**



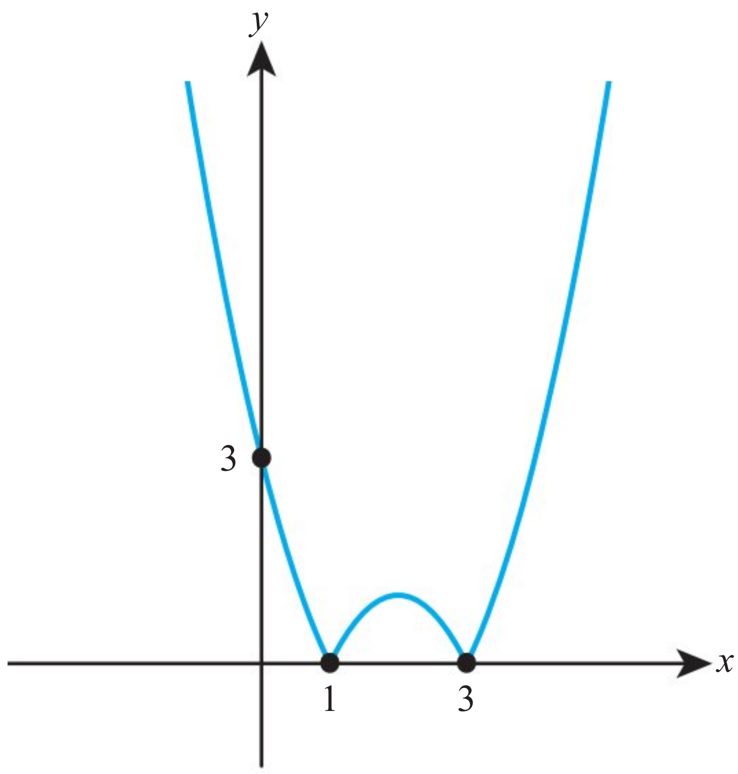
b



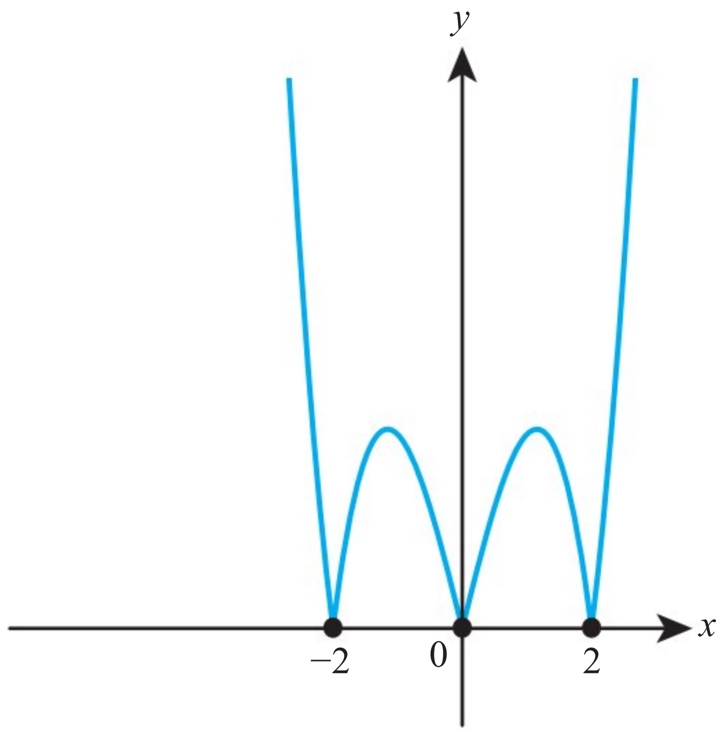
3 **a**



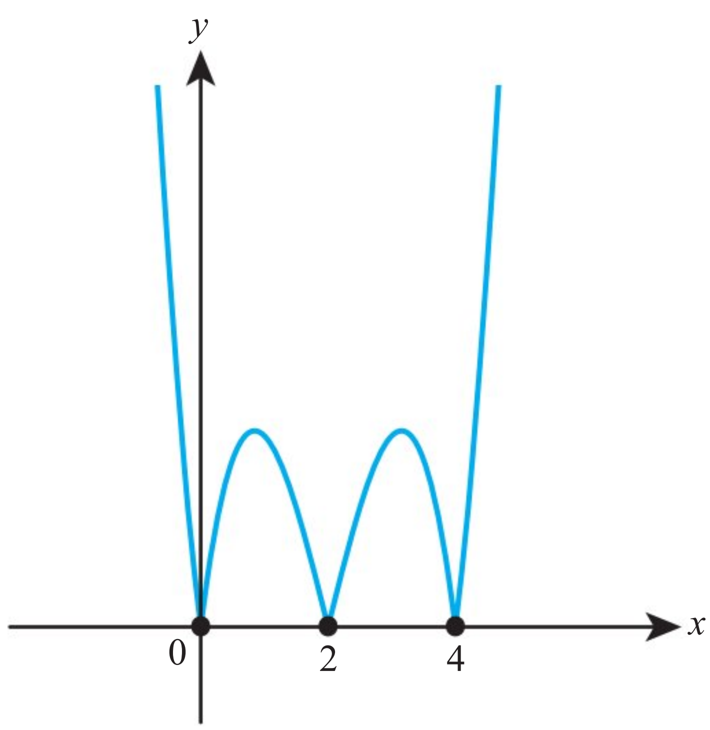
b



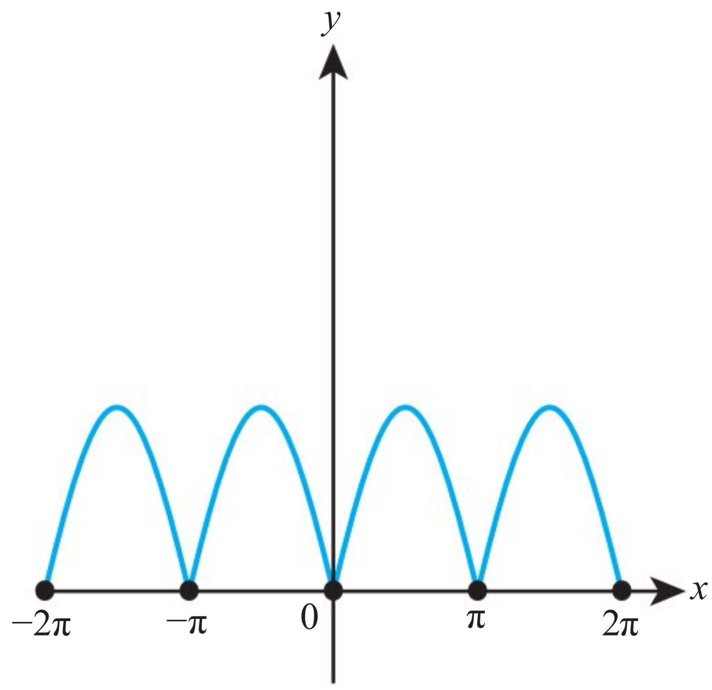
4 a



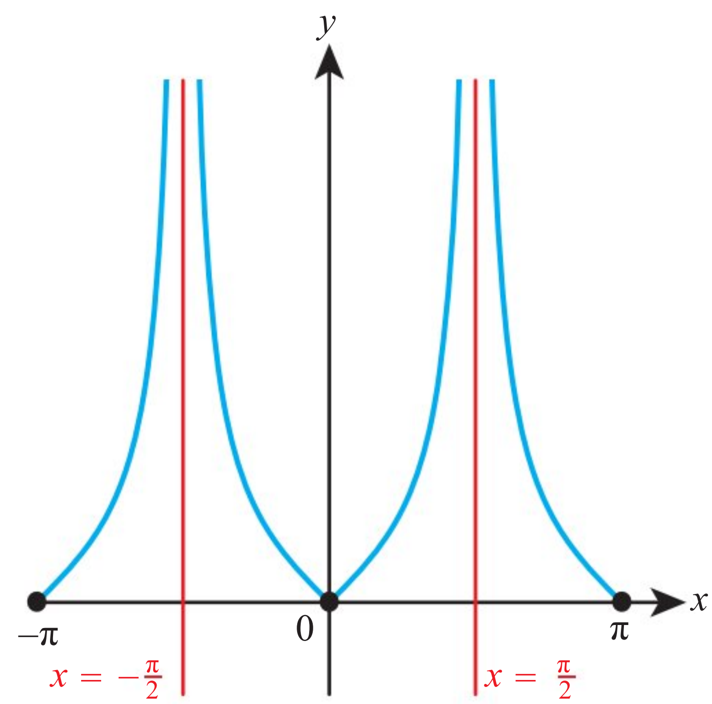
b



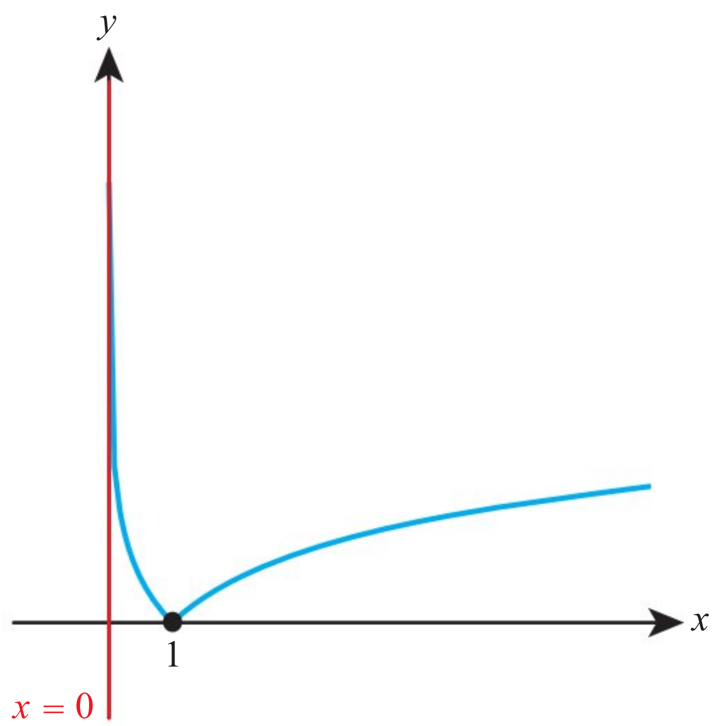
5 a



b



6 a



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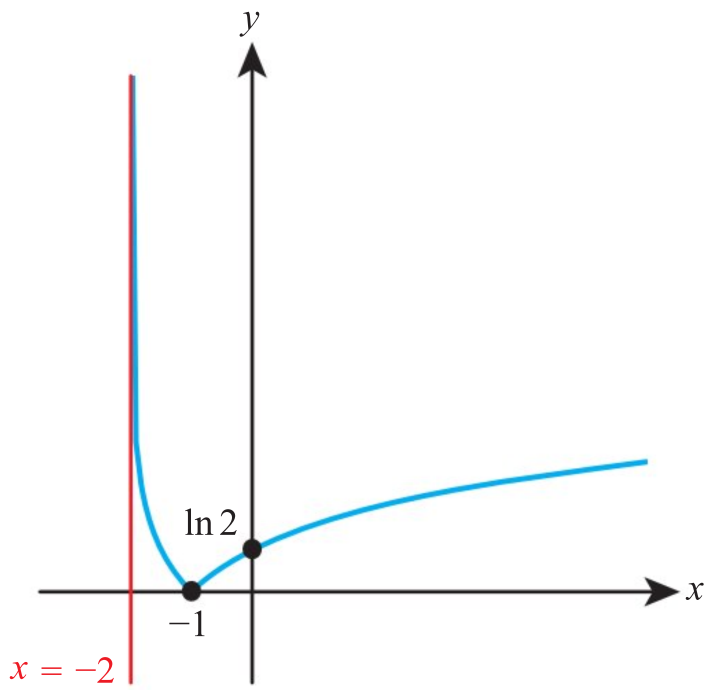
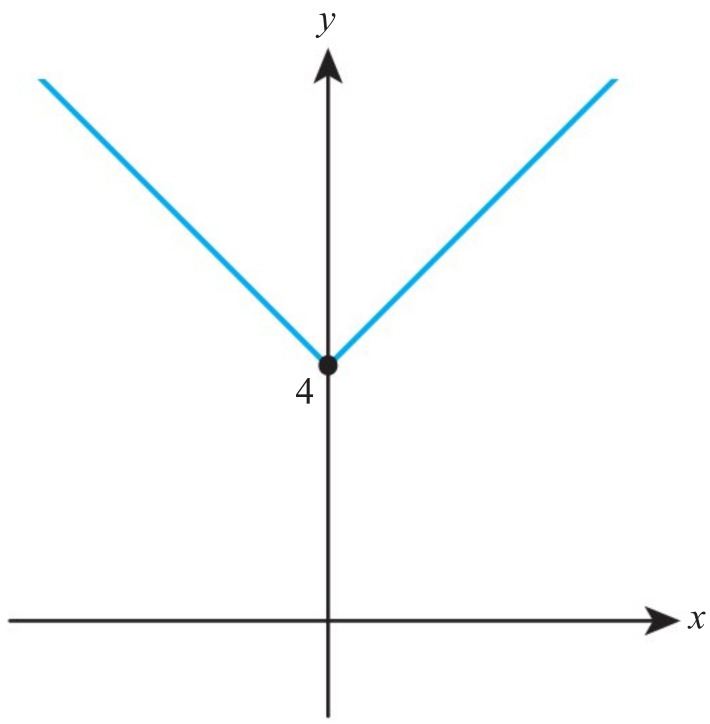
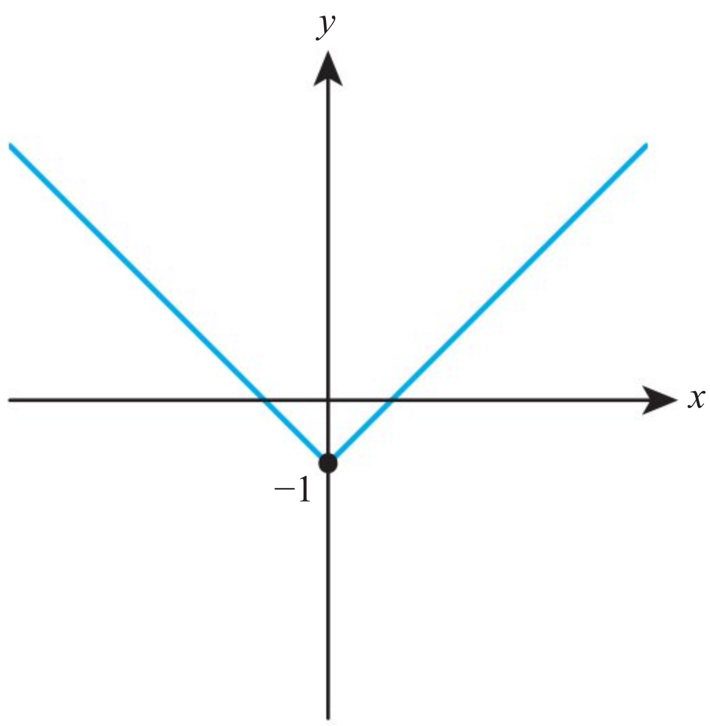


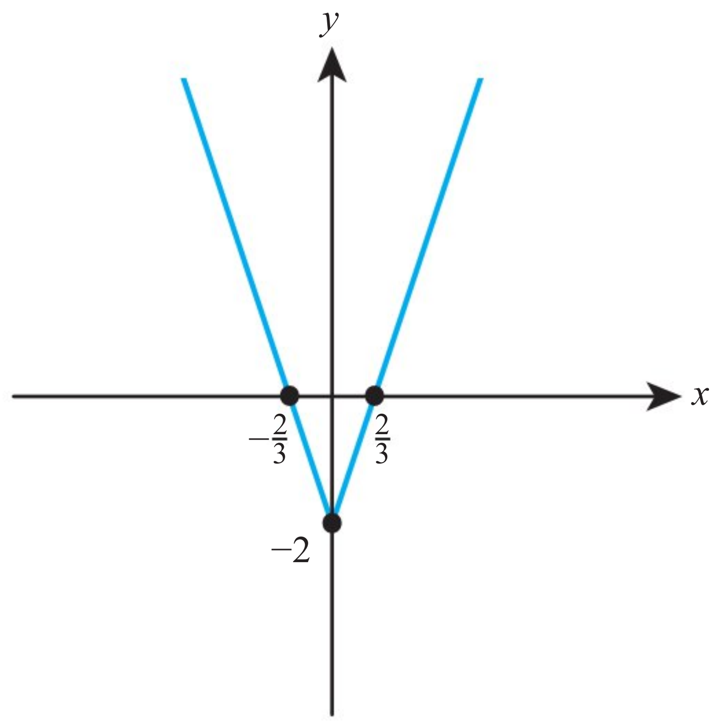
7 a



b



8 a



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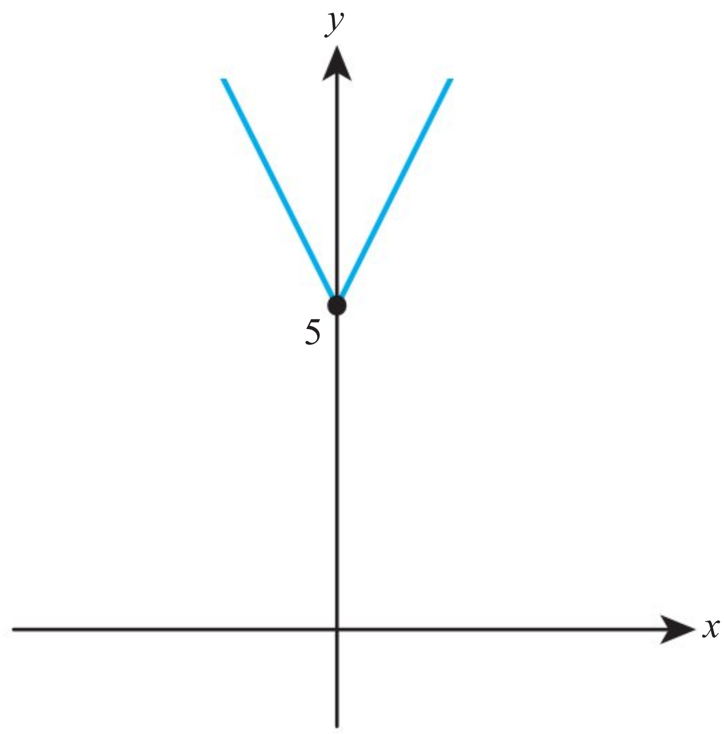
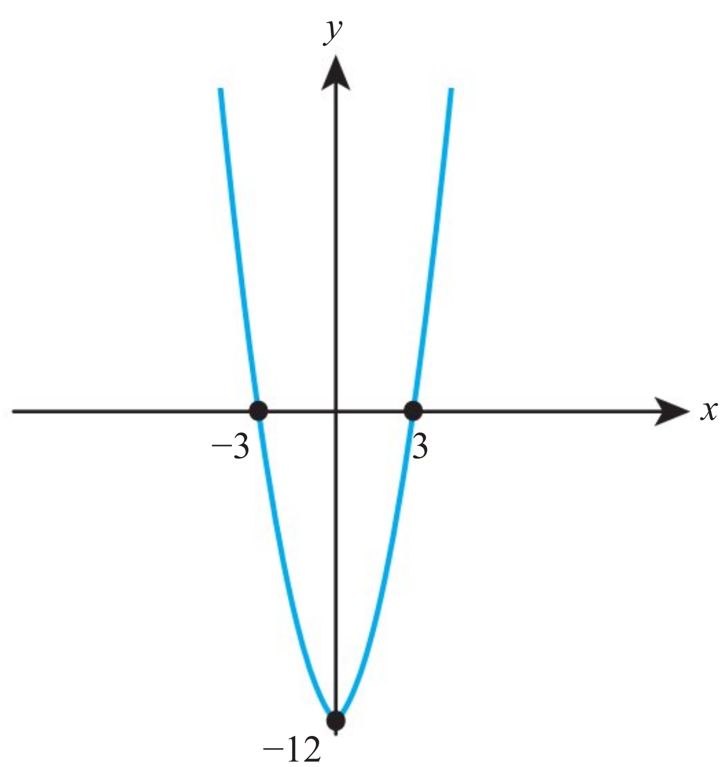


9 a



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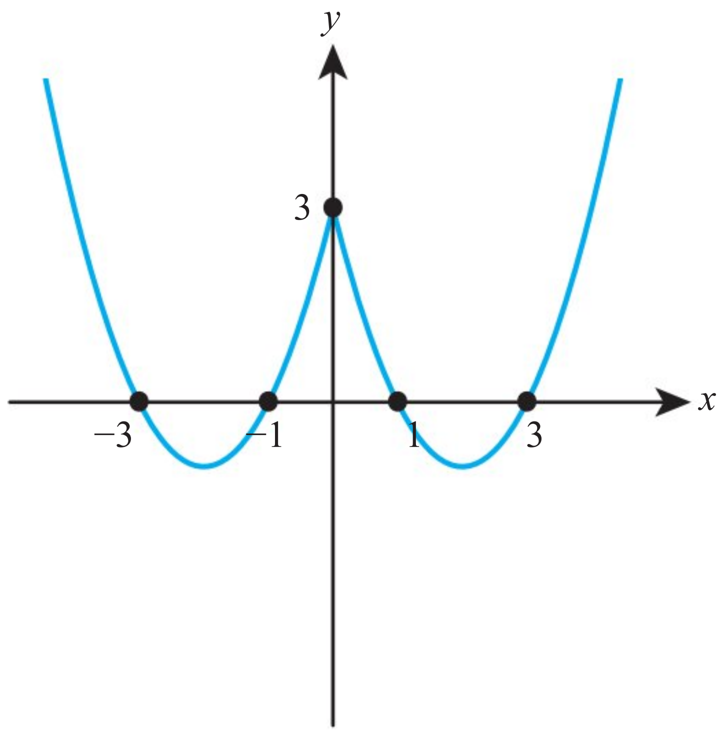
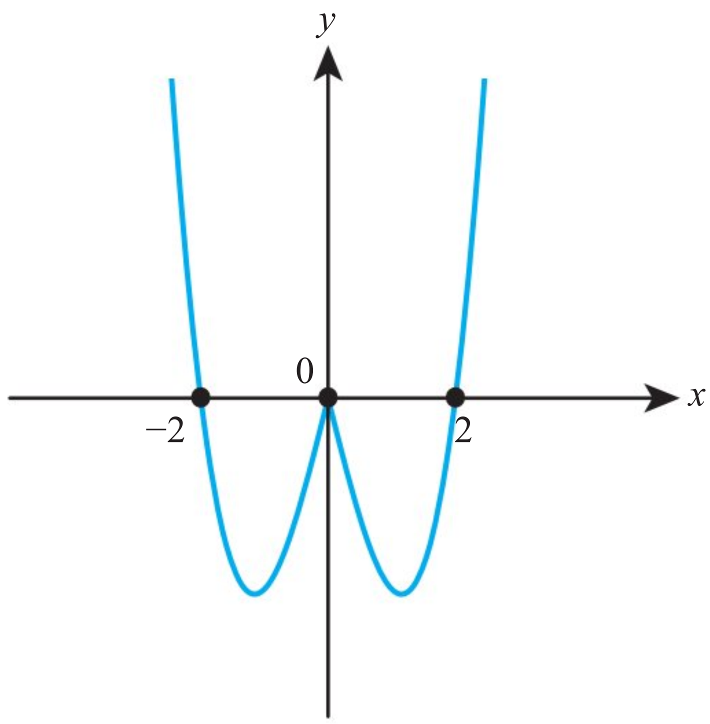


10 a



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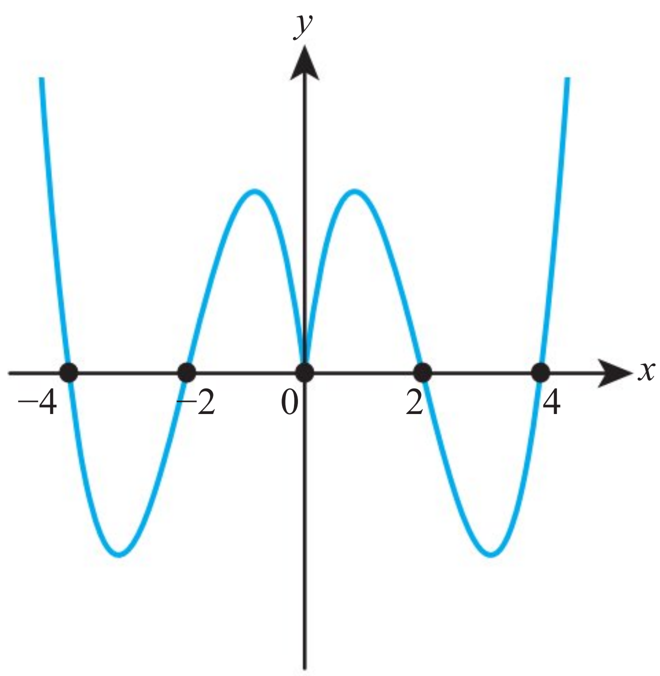
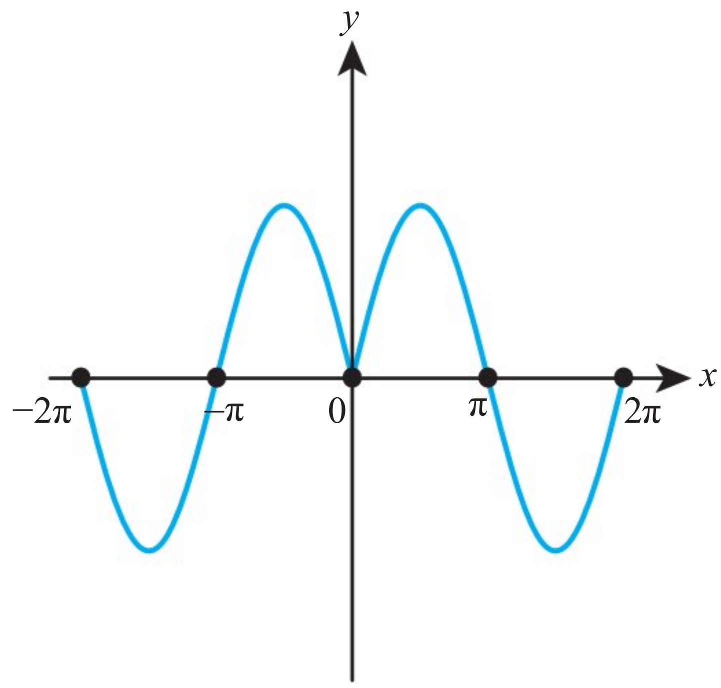
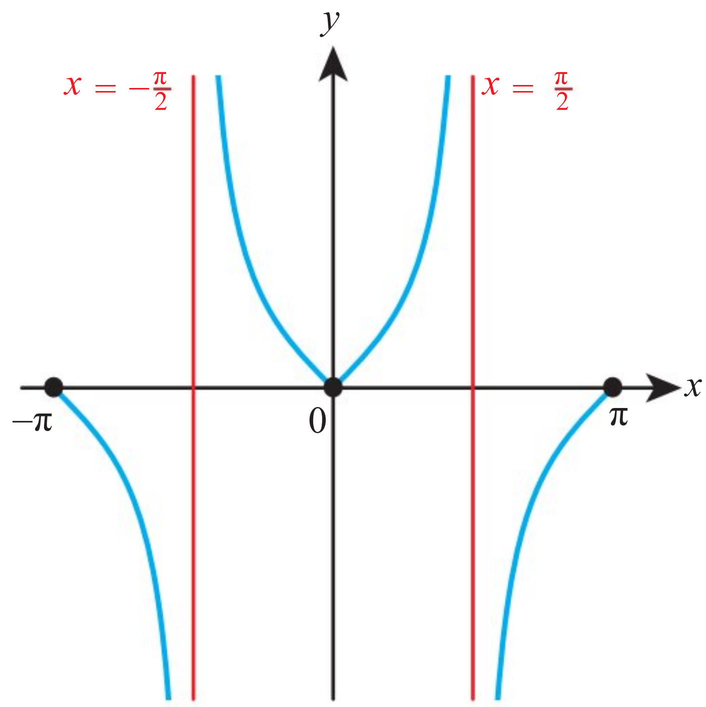


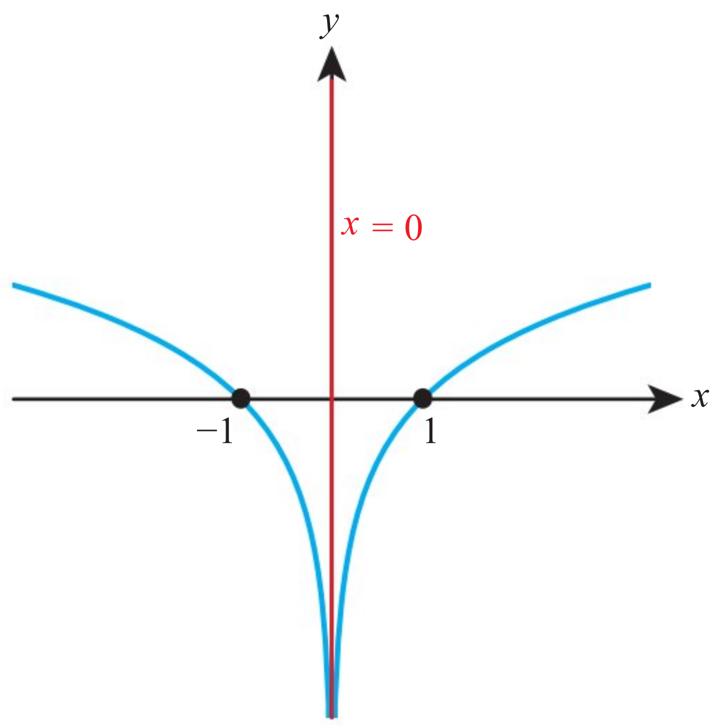
11 a



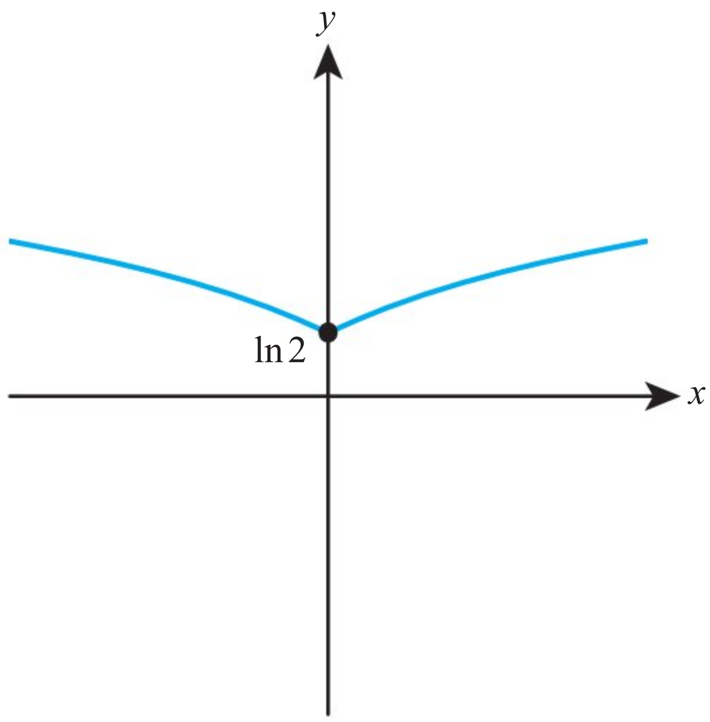
b



12 a



b



13 a $x = -2, 3$ **b** $x = -\frac{10}{3}, 2$

14 a $x = \frac{3}{4}, \frac{7}{2}$ **b** $x = -\frac{4}{3}, \frac{2}{5}$

15 a $x = -\frac{1}{3}$ **b** $x = -2, 3$

16 a $x = -4, -1, 0, 3$ **b** $x = 0, 2, 3, 5$

17 a $x = -2, 4, 6$ **b** $x = -3, -1$

18 a $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$ **b** $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

19 a $x < -2$ or $x > 3$ **b** $-\frac{10}{3} < x < 2$

20 a $\frac{3}{4} \leqslant x \leqslant \frac{7}{2}$ **b** $x \leqslant -\frac{4}{3}$ or $x \geqslant \frac{2}{5}$

21 a $x > -\frac{1}{3}$ **b** $-2 < x < 3$

22 a $x \leqslant -4$ or $-1 \leqslant x \leqslant 0$ or $x \geqslant 3$

b $0 \leqslant x \leqslant 2$ or $3 \leqslant x \leqslant 5$

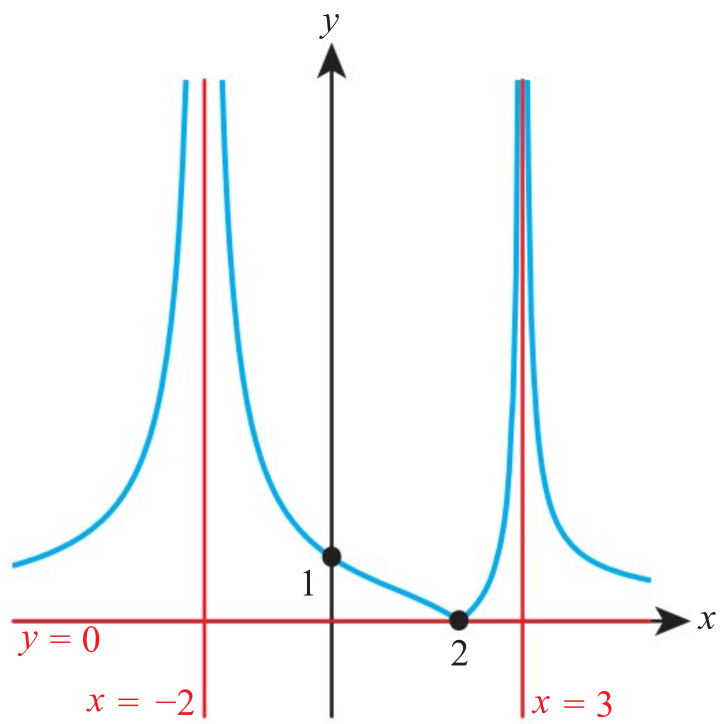
23 a $4 < x < 6$

b $x < -3$ or $x > -1$

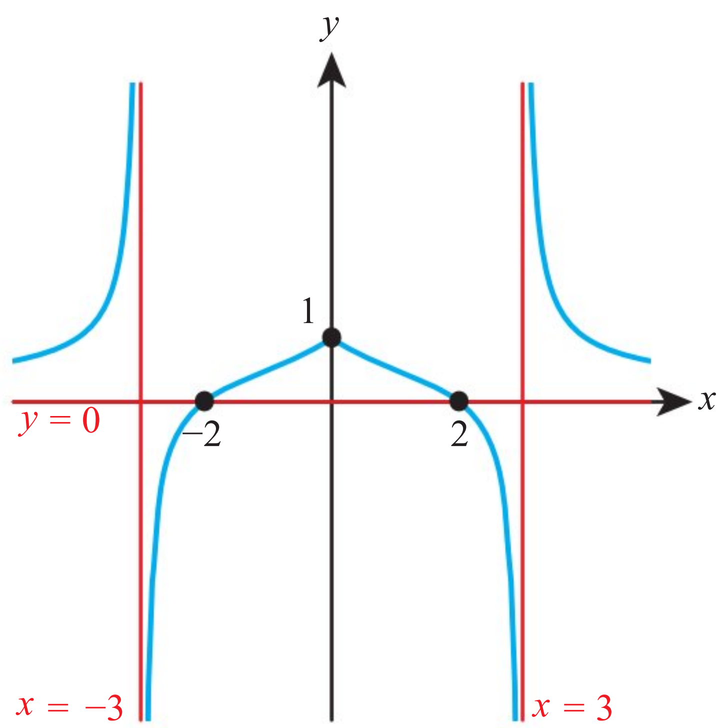
24 a $-\frac{2\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{2\pi}{3}$

b $-\frac{3\pi}{4} < x < -\frac{\pi}{4}$ or $\frac{\pi}{4} < x < \frac{3\pi}{4}$

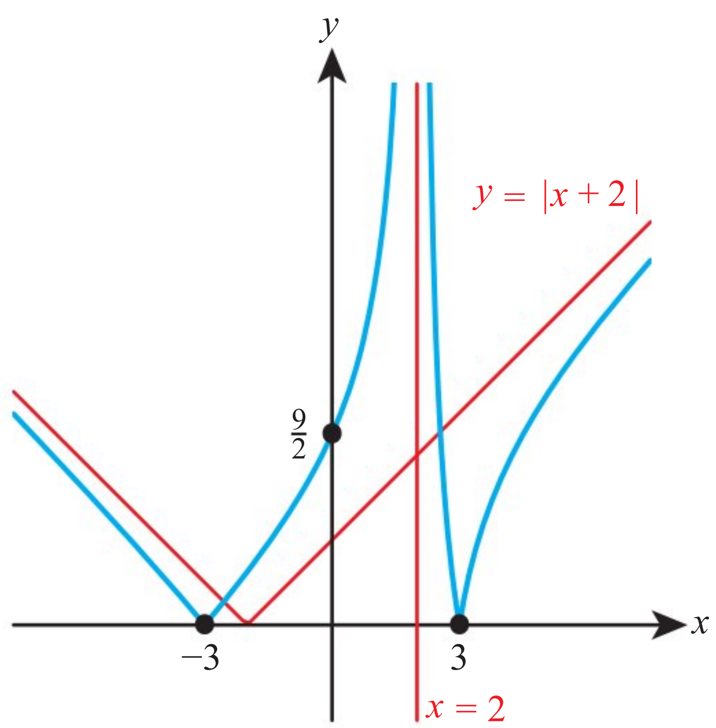
25 a



b



26 a



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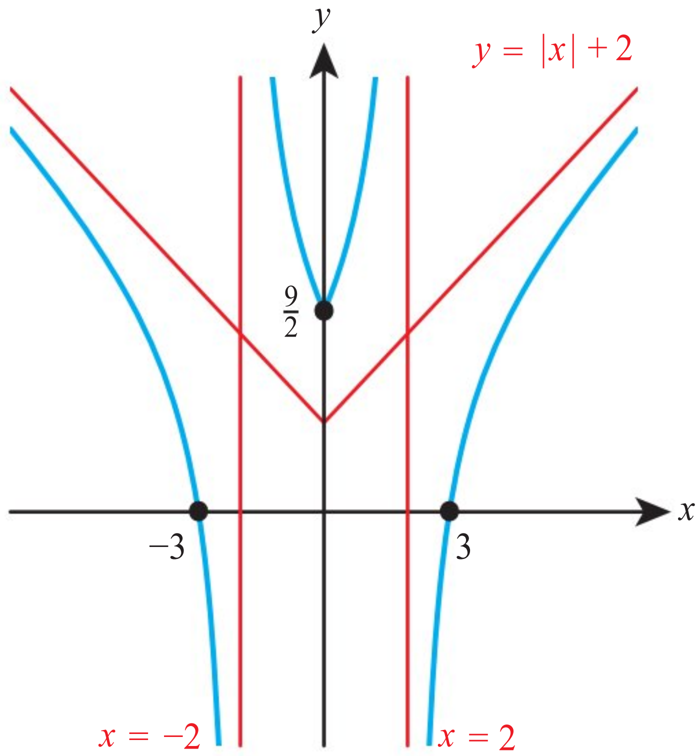
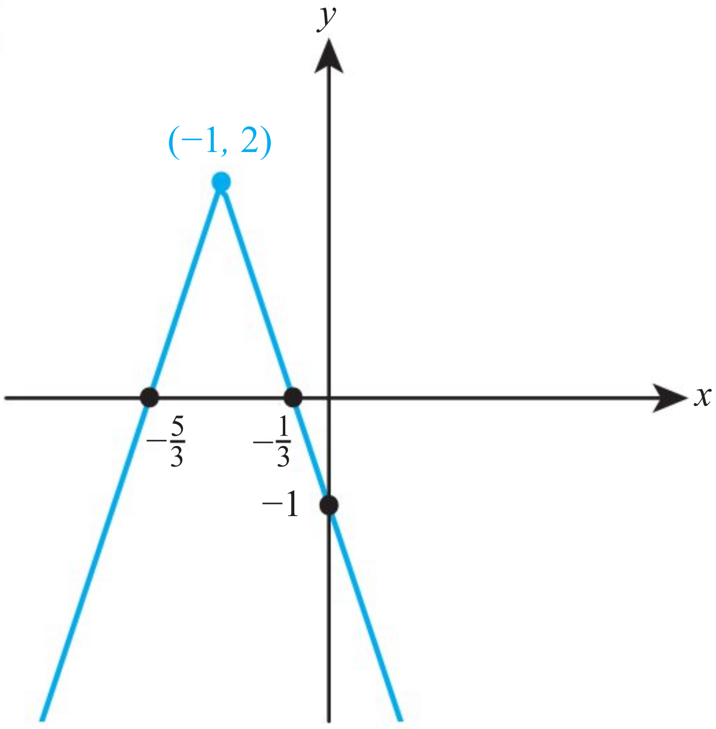
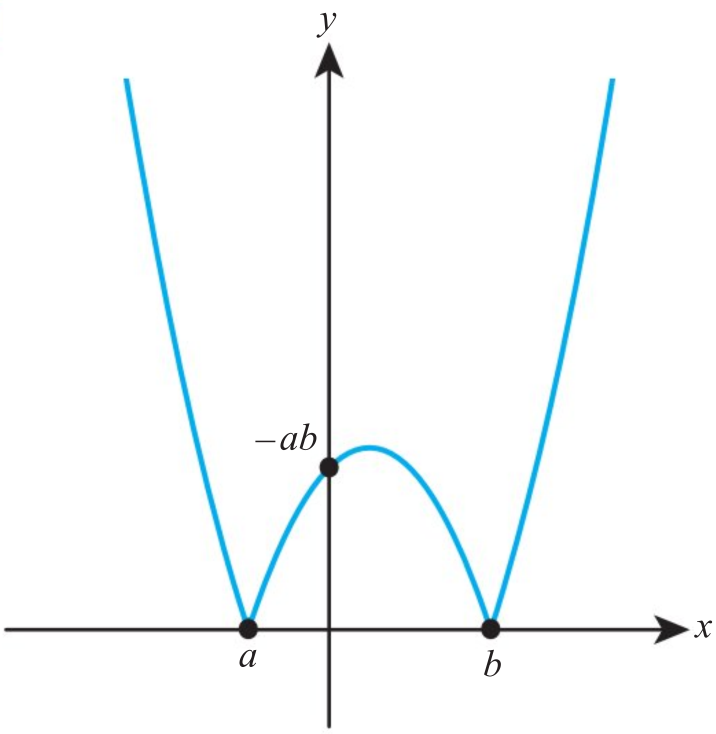


27



28



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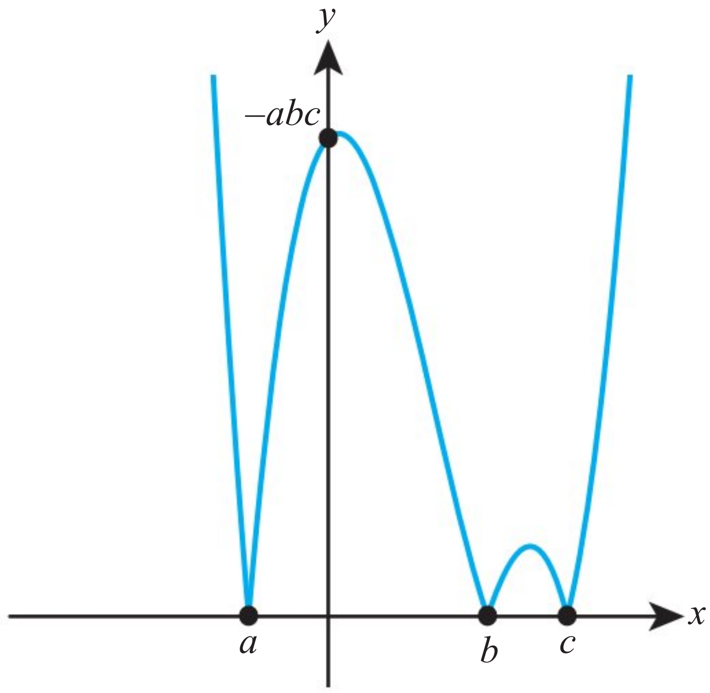
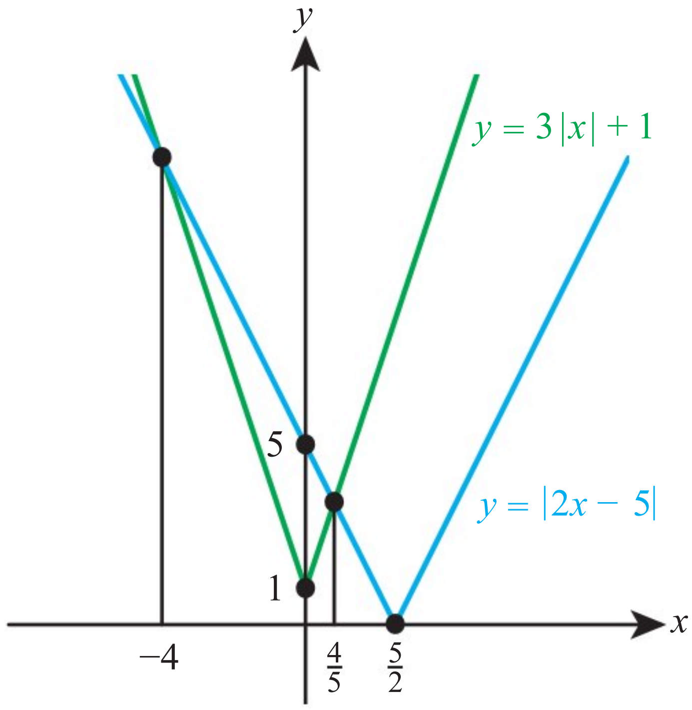


30 $-1.28 \leqslant x \leqslant 0.720$

31 $x \in (-\infty, -4.80) \cup (-3.32, 4.80)$

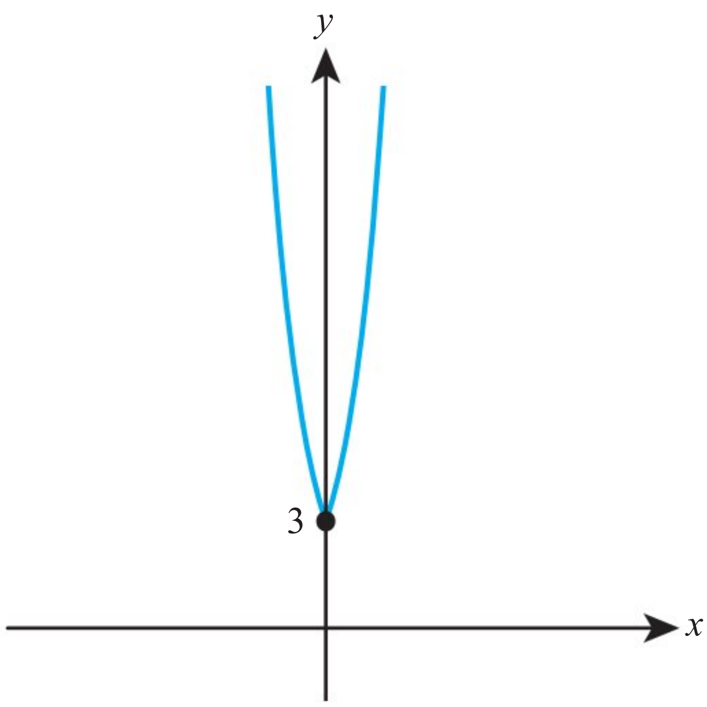
32 $x < -0.146$ or $0.180 < x < 0.967$

33 **a**



b $x < -4$ or $x > \frac{4}{5}$

34 **a**



b $x \in \left]-\infty, -\ln\frac{5}{3}\right] \cup \left[\ln\frac{5}{3}, \infty\right[$

35 $a = -2,\ b = 3,\ c = 5$

36 $x < -2$

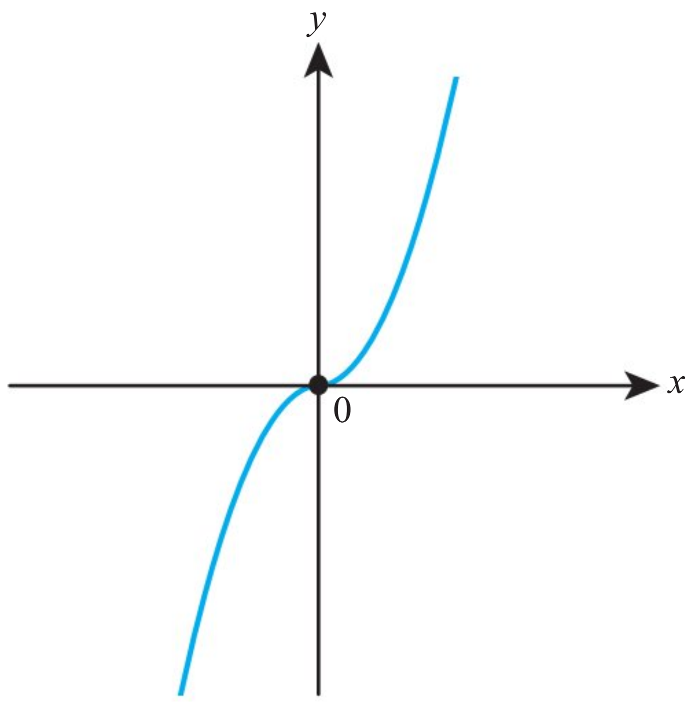
37 $x = -2,\ 2 - \sqrt{6}$

38 $x < \frac{5-\sqrt{17}}{2}$ or $2 < x < 3$ or $x > \frac{5+\sqrt{17}}{2}$

39 $x = 4,\ -\frac{4}{3}$

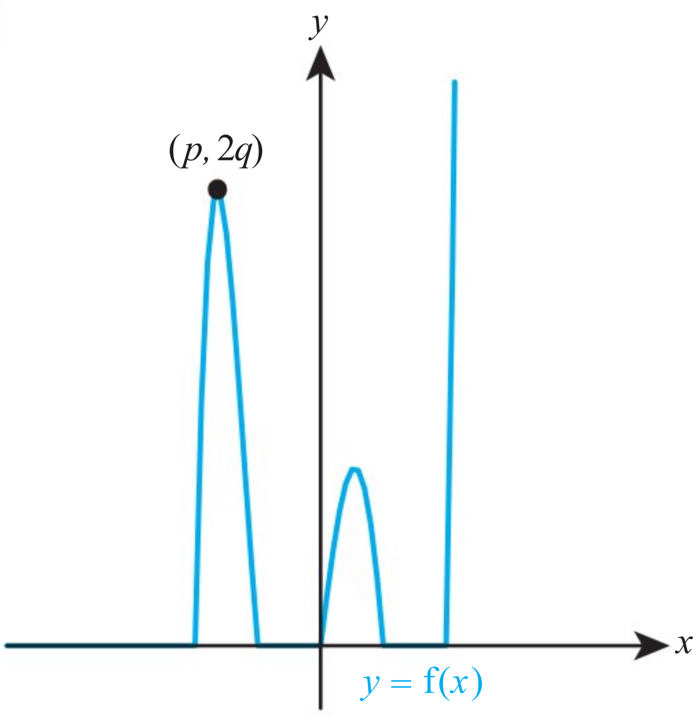
40 $x = 1,\ -\frac{1}{3}$

41 **a**



b $x = \pm k,\ 0$

42



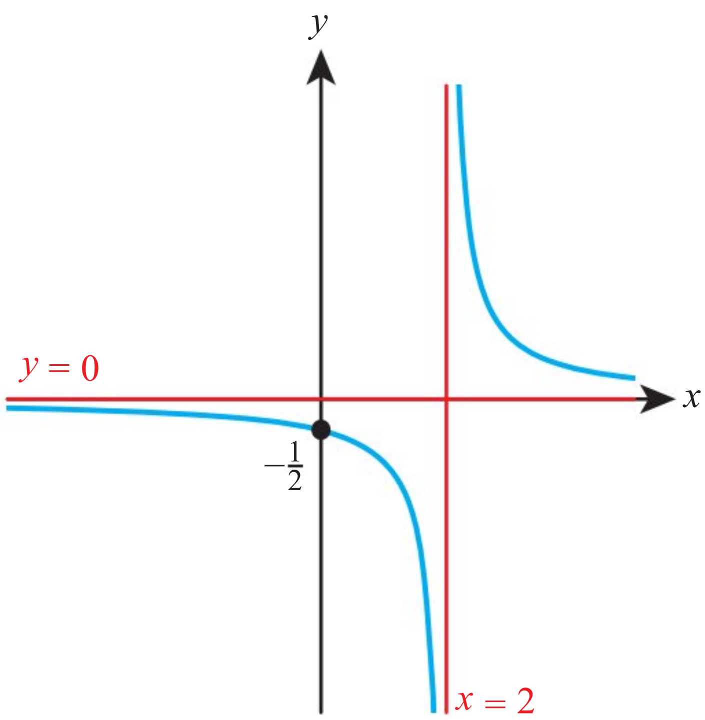
43 $\frac{a^2}{2}$

44 $0 < k < 11$

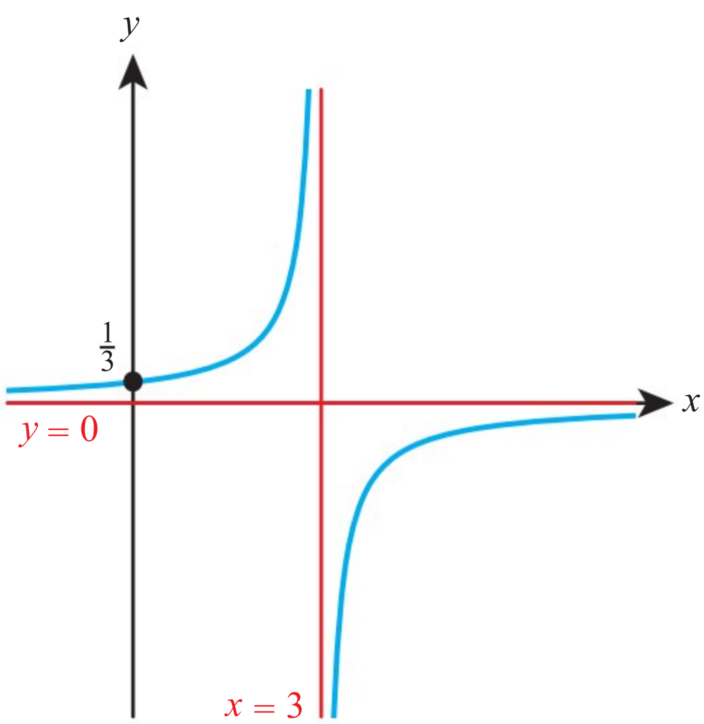
45 $12 < k < 20$

Exercise 7D

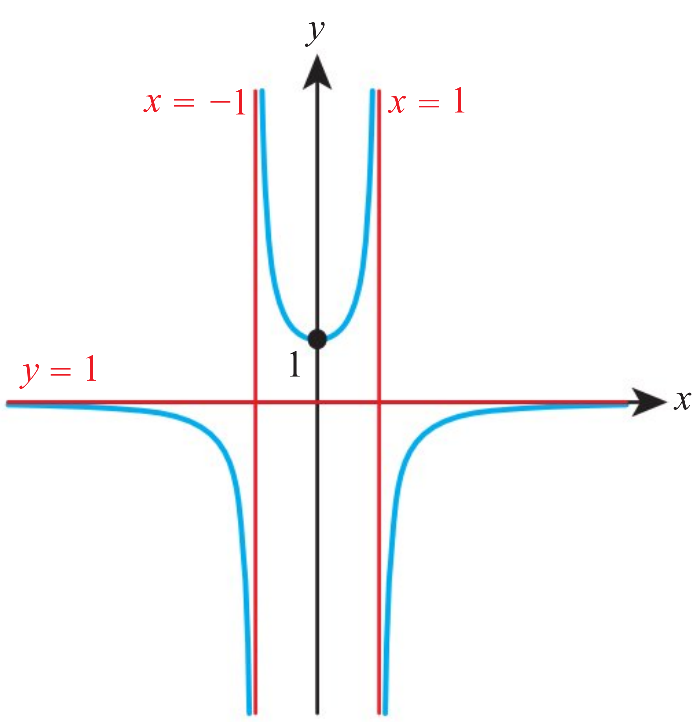
1 **a**



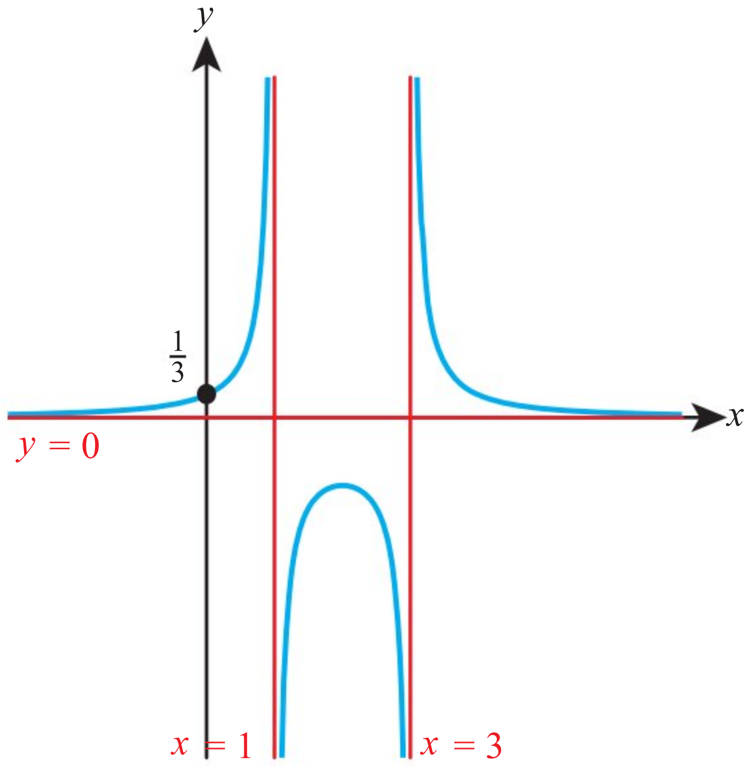
b



2 **a**

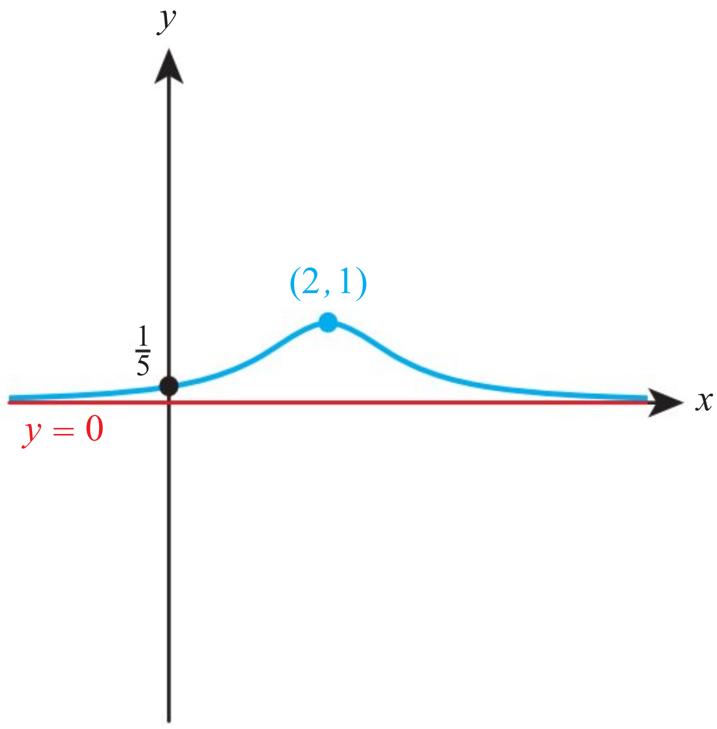


b



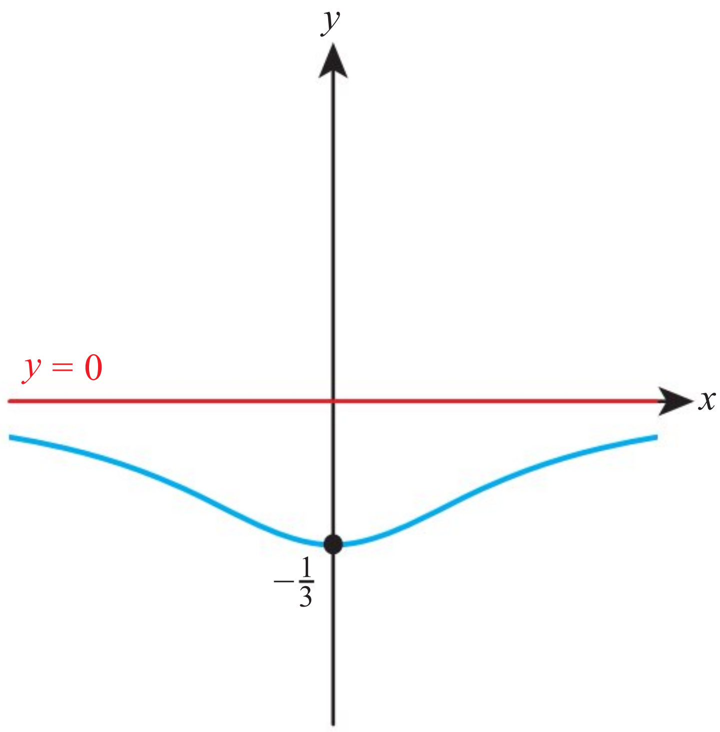
y
x
$\frac{1}{3}$
y = 0
x = 1
x = 3

3 a



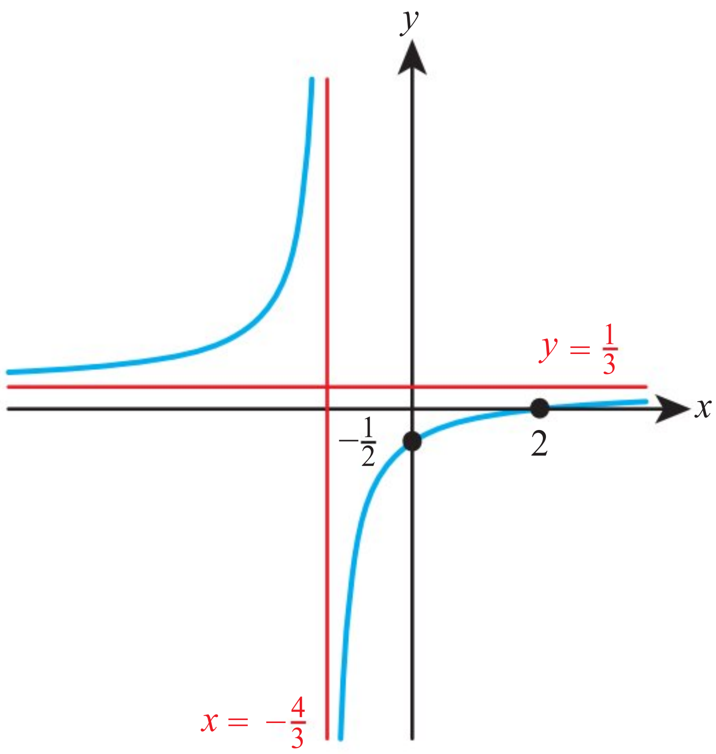
y
x
(2, 1)
$\frac{1}{5}$
y = 0

b



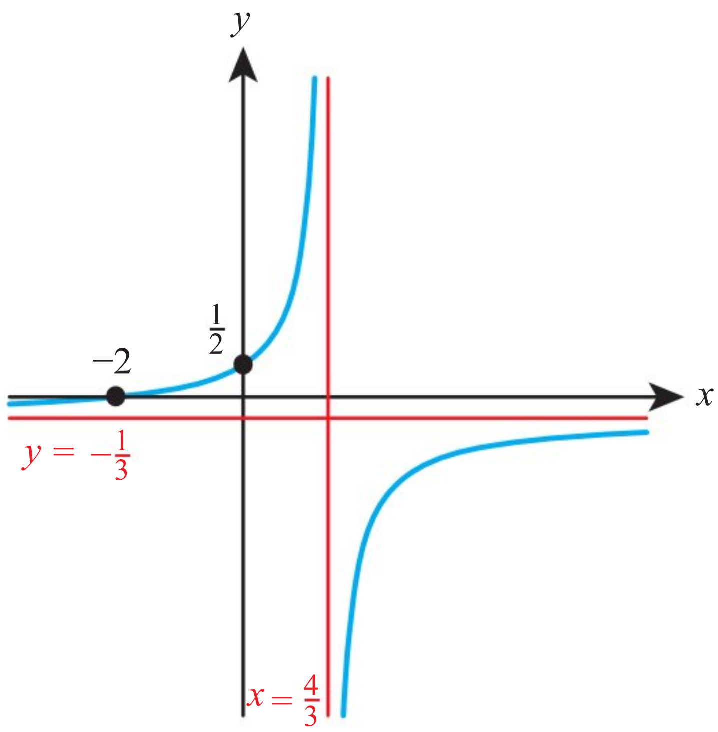
y
x
y = 0
$-\frac{1}{3}$

4 a



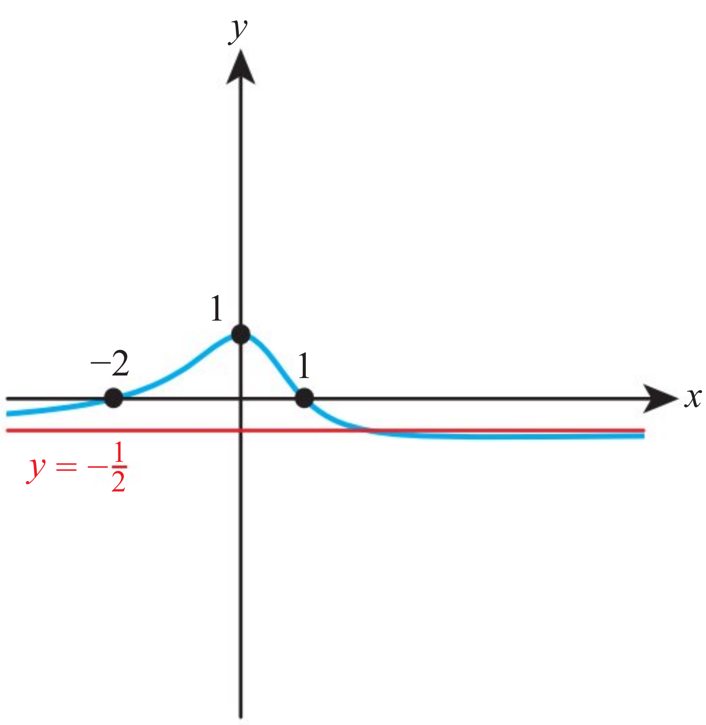
y
x
$y = \frac{1}{3}$
$-\frac{1}{2}$
2
$x = -\frac{4}{3}$

b



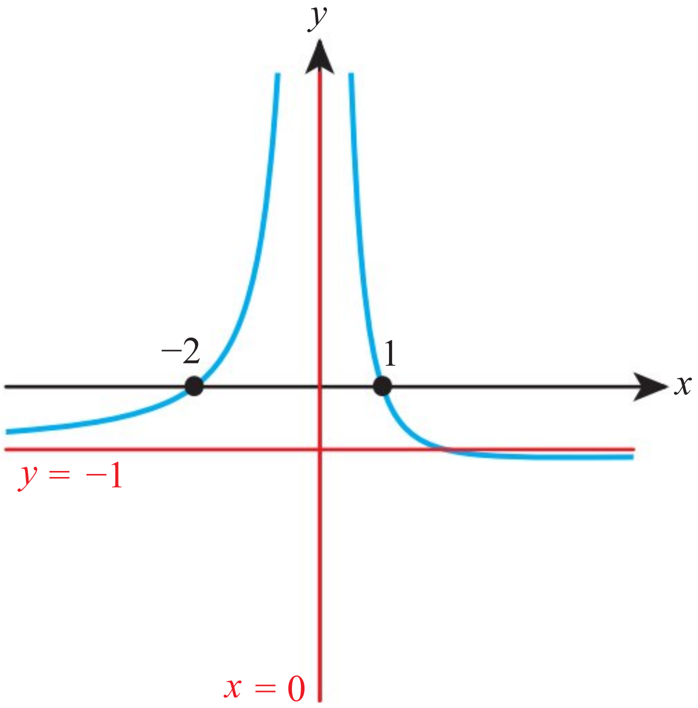
y
x
$\frac{1}{2}$
−2
$y = -\frac{1}{3}$
$x = \frac{4}{3}$

5 a

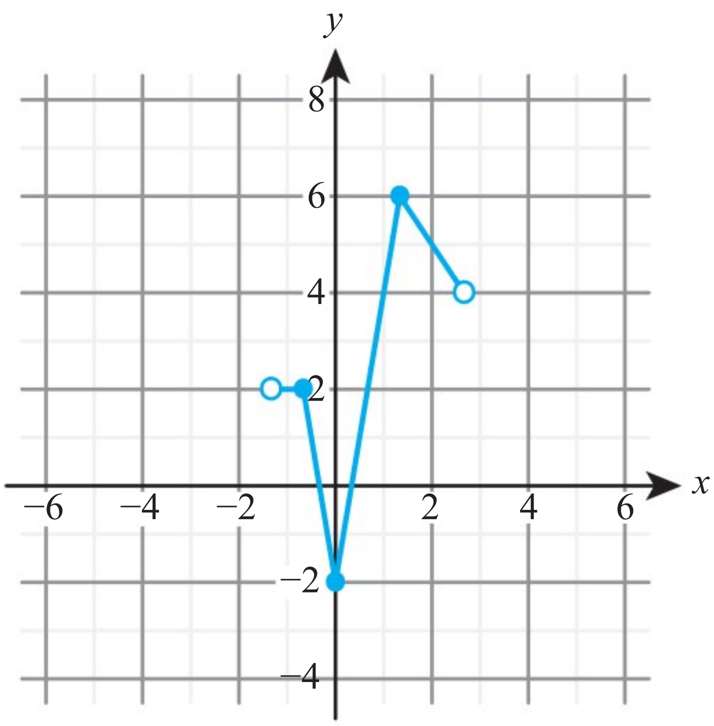


y
x
1
−2
1
$y = -\frac{1}{2}$

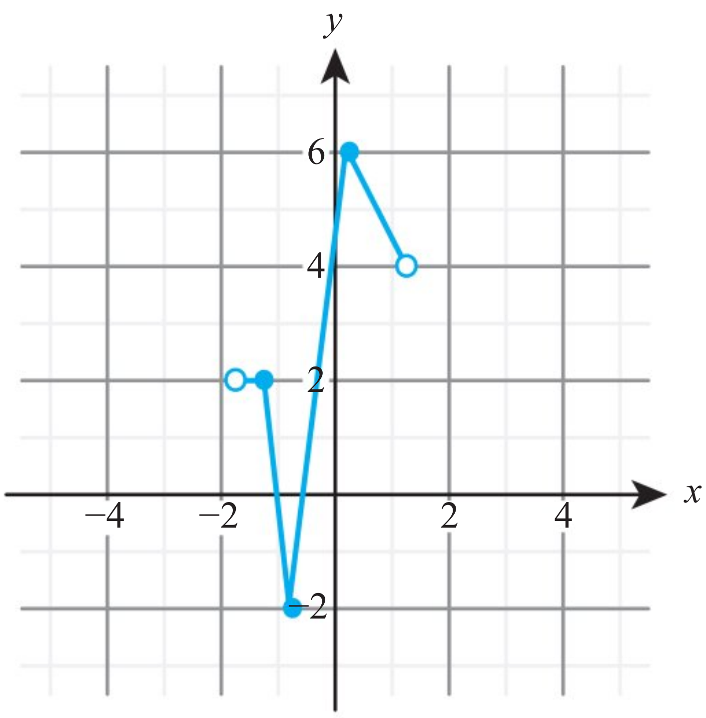
b


y
x
−2
1
y = −1
x = 0

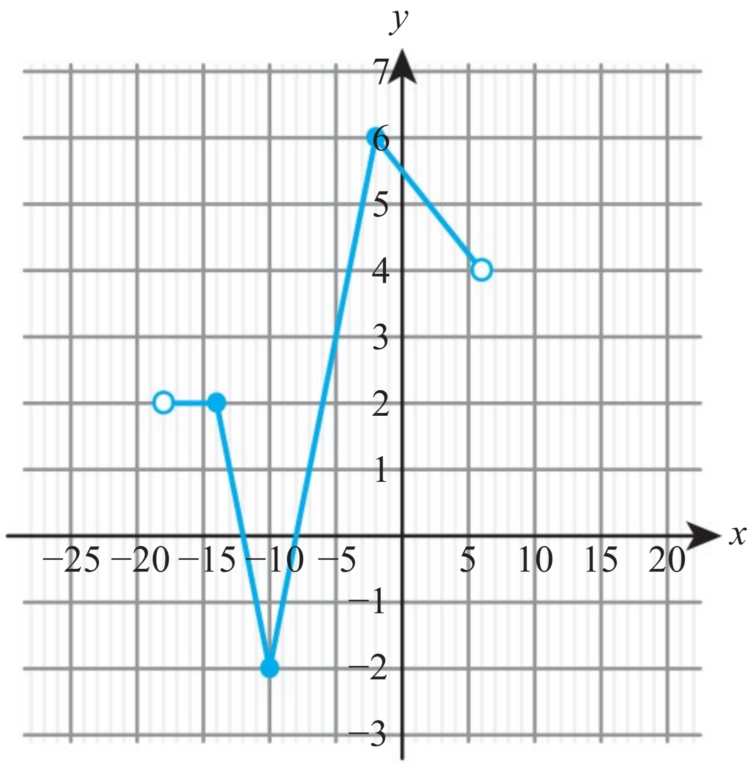
6 a


y
x
8
6
4
2
−2
−4
−6
−4
−2
2
4
6

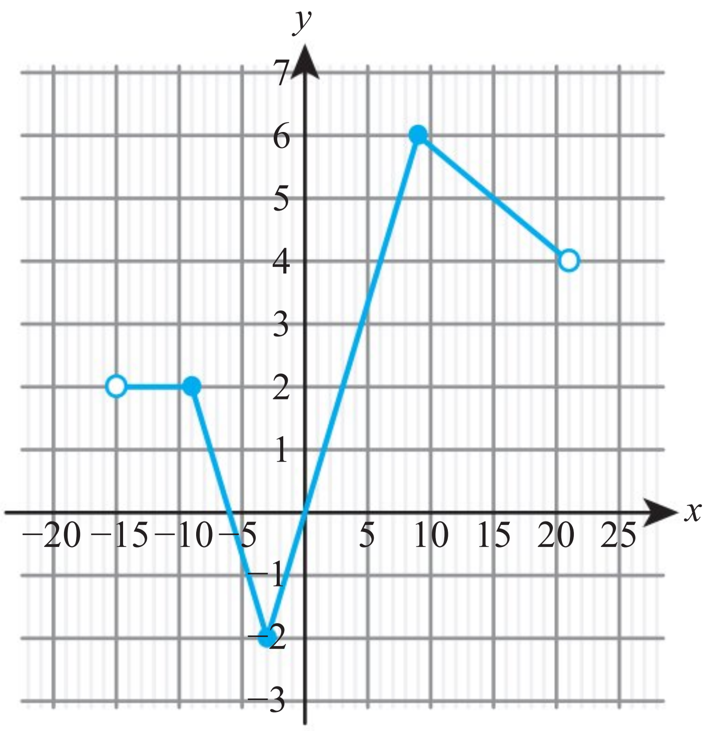
b


y
x
6
4
2
−2
−4
−2
2
4

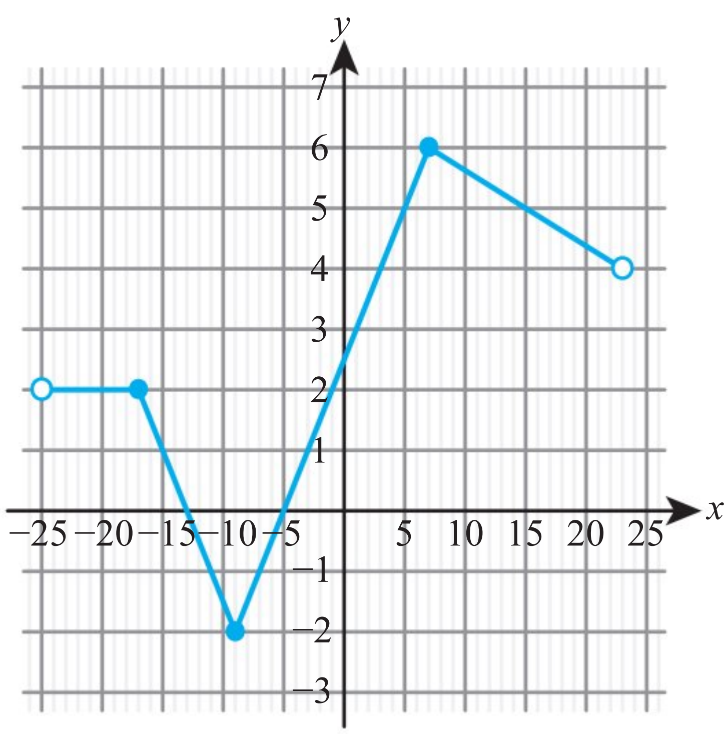
7 a


y
x
7
6
5
4
3
2
1
−1
−2
−3
−25 −20 −15 −10 −5
5 10 15 20

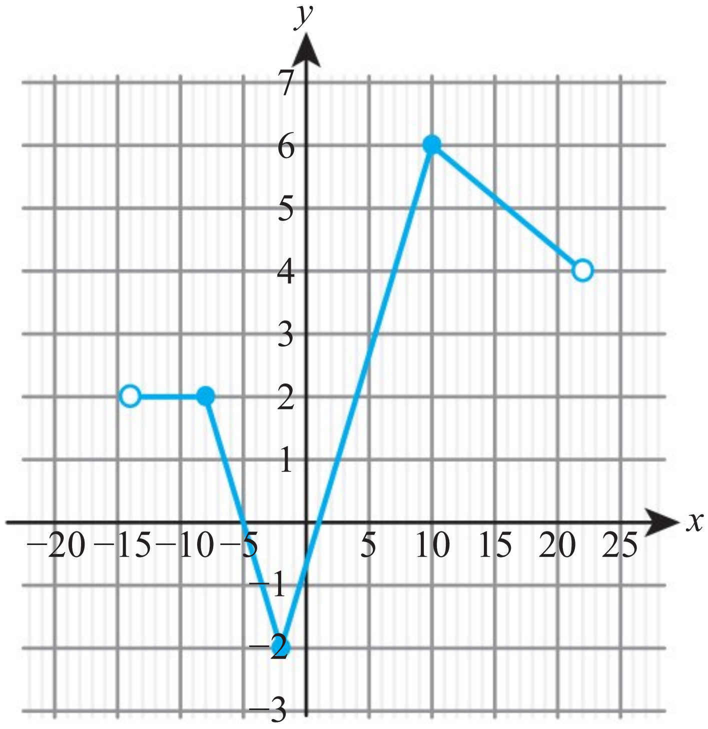
b


y
x
7
6
5
4
3
2
1
−1
−2
−3
−20 −15 −10 −5
5 10 15 20 25

8 a


y
x
7
6
5
4
3
2
1
−1
−2
−3
−25 −20 −15 −10 −5
5 10 15 20 25

b



9 a Translation right by 2 followed by horizontal stretch with scale factor 4

b Translation left by 1 followed by horizontal stretch with scale factor 3

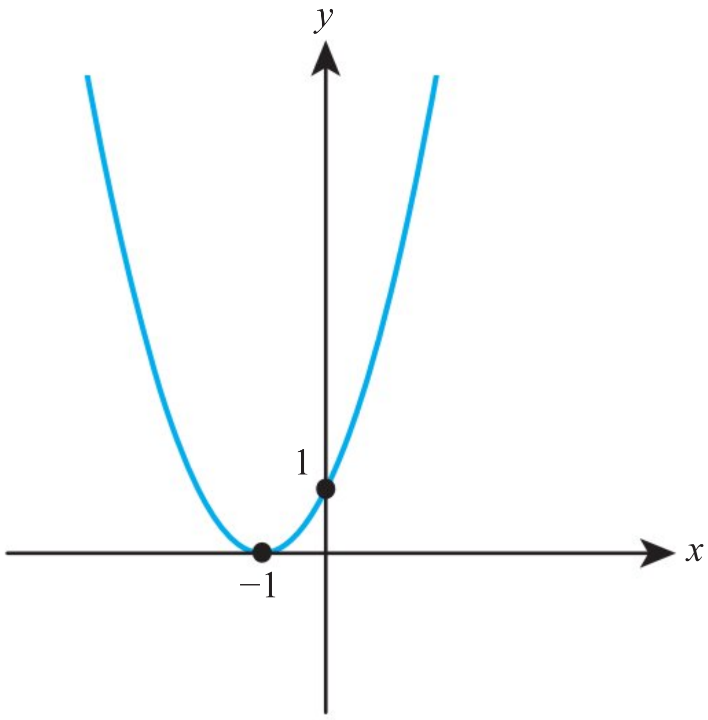
10 a Translation right by 1 followed by horizontal stretch with scale factor $\frac{1}{3}$

b Translation left by 3 followed by horizontal stretch with scale factor $\frac{1}{2}$

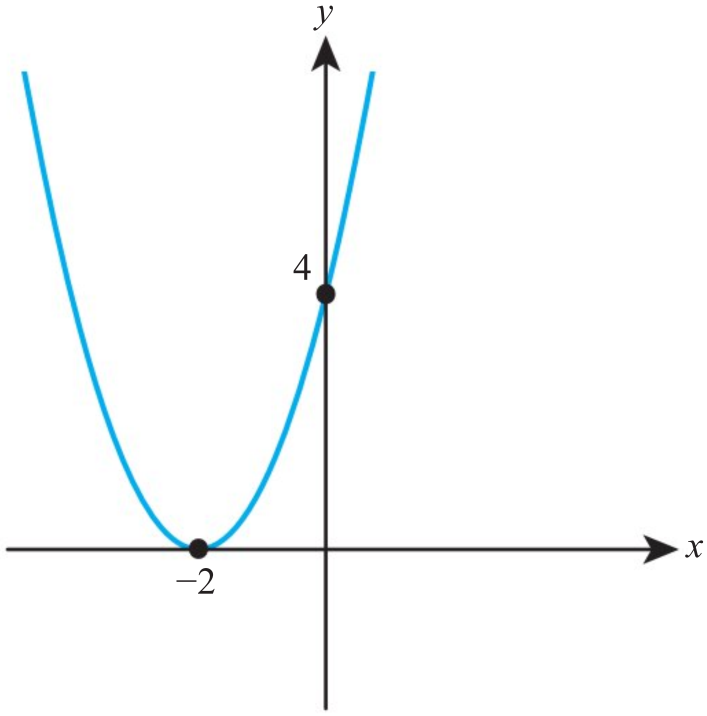
11 a Translation right by 3 followed by reflection in the y-axis or just translate right by 2

b Translation left by 2 followed by reflection in the y-axis or just translate right by 2

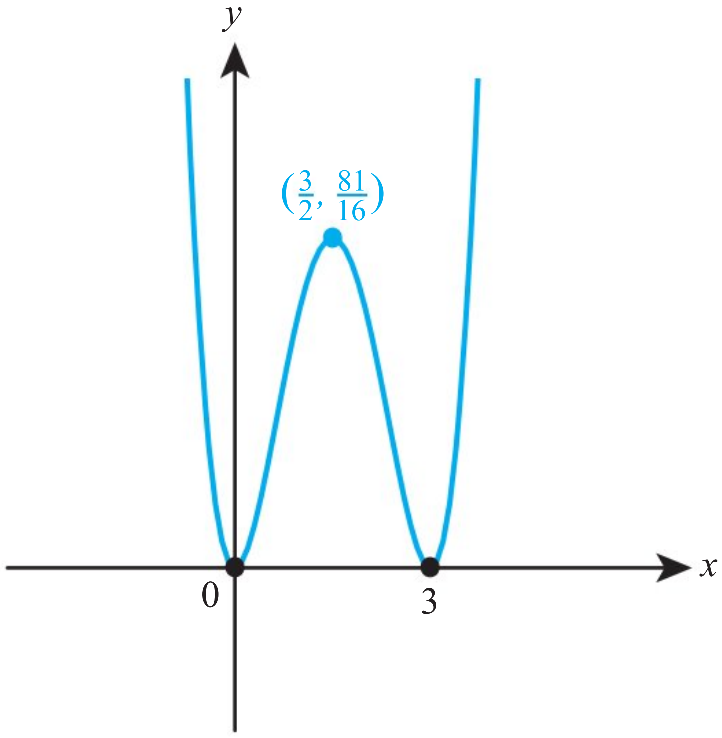
12 a



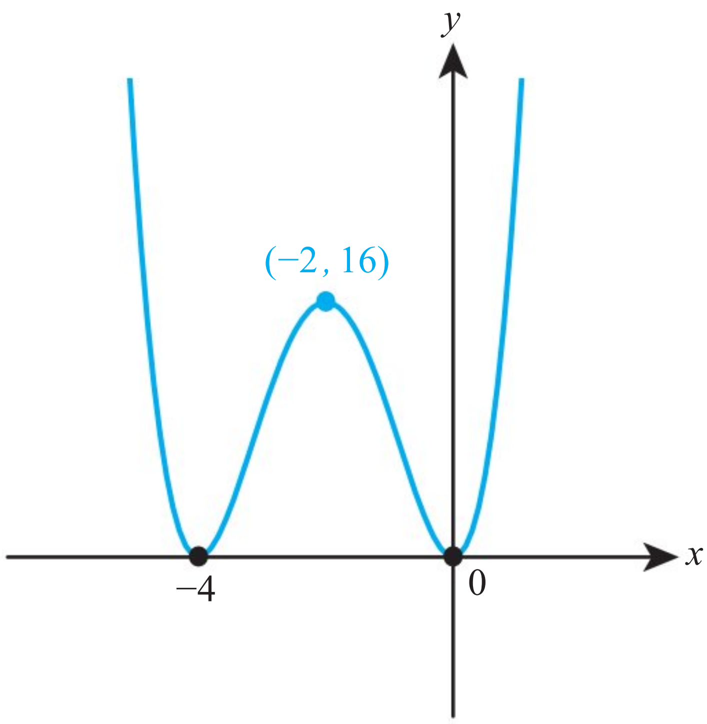
b



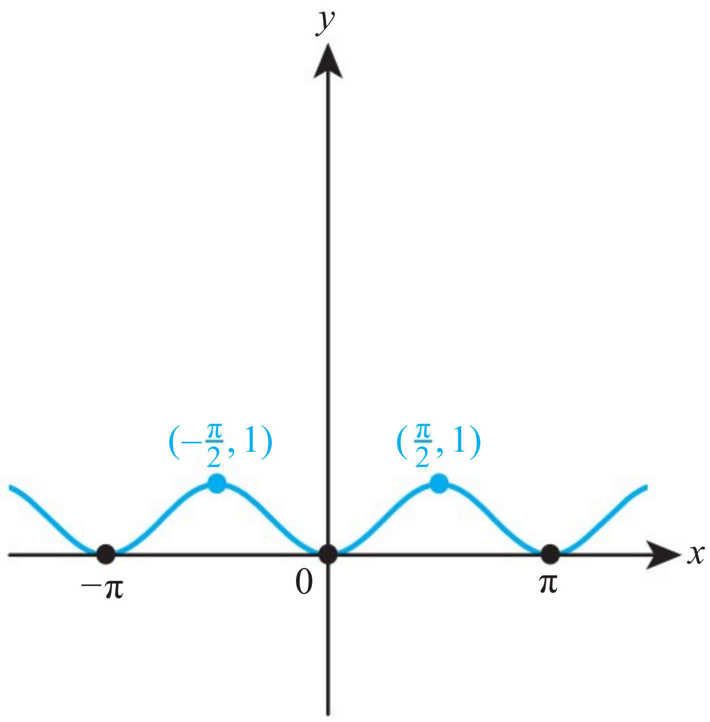
13 a



b



14 a



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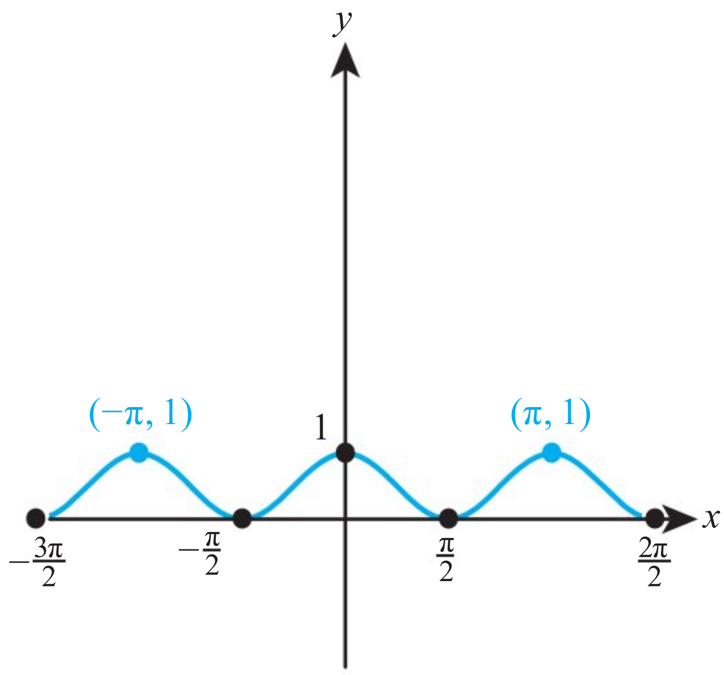
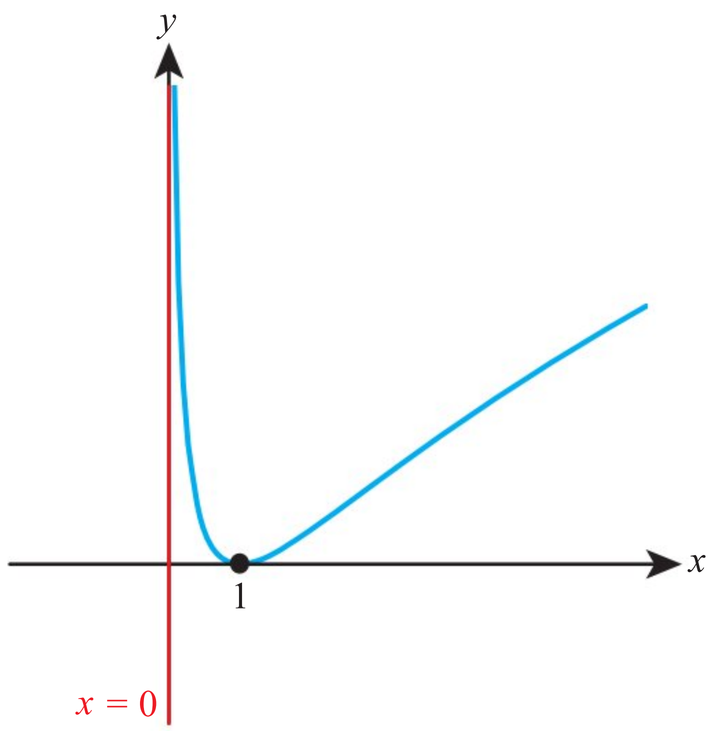
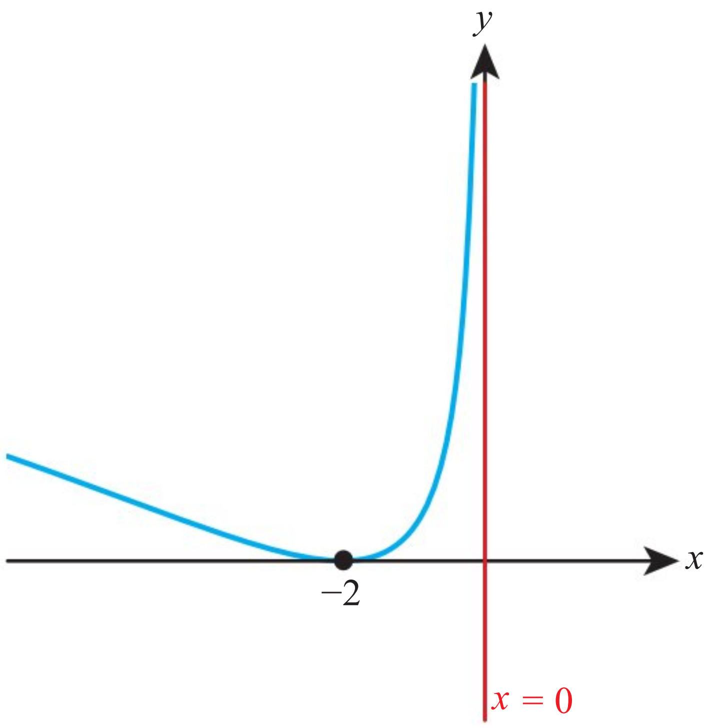


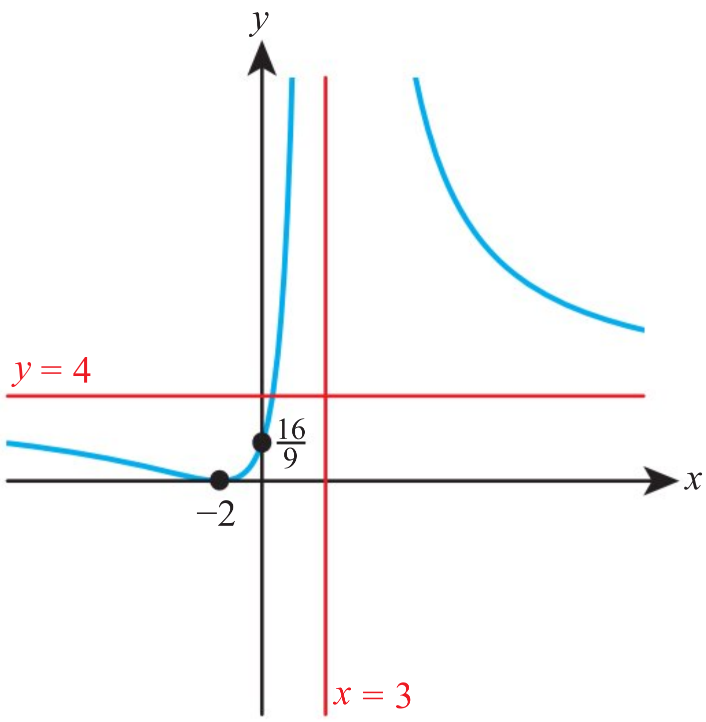
15 a



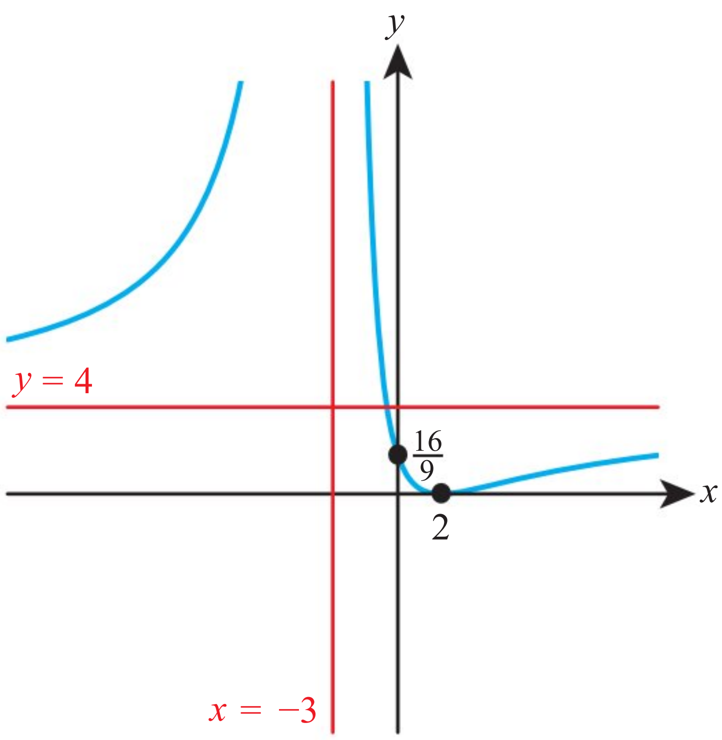
b



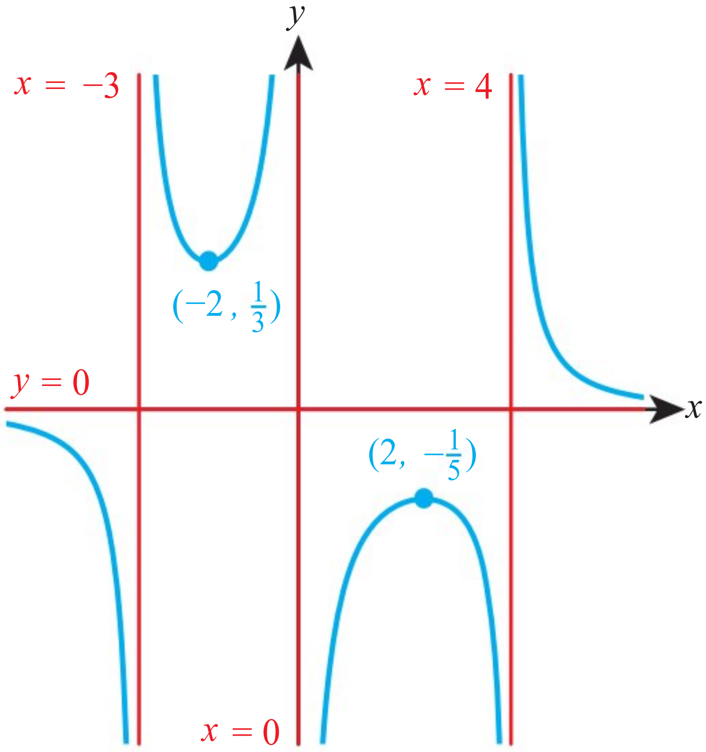
16 a



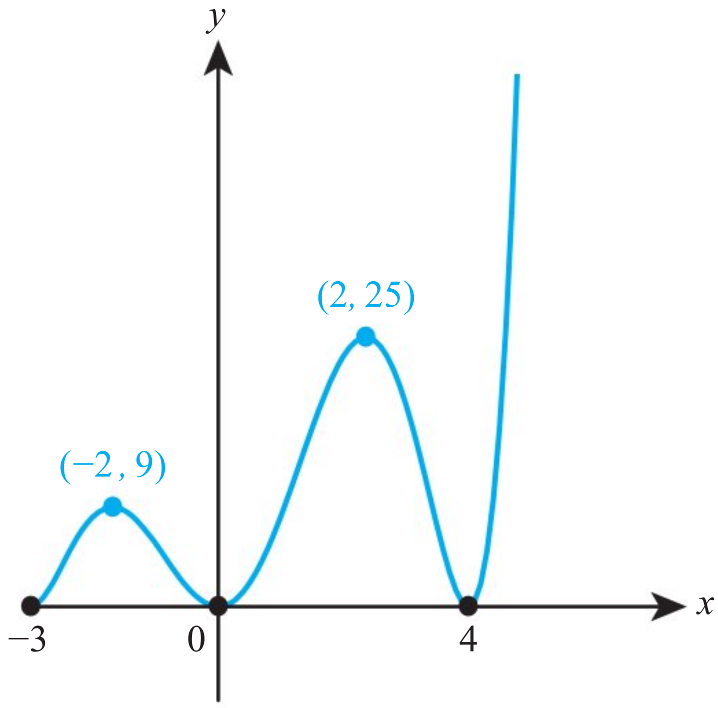
b



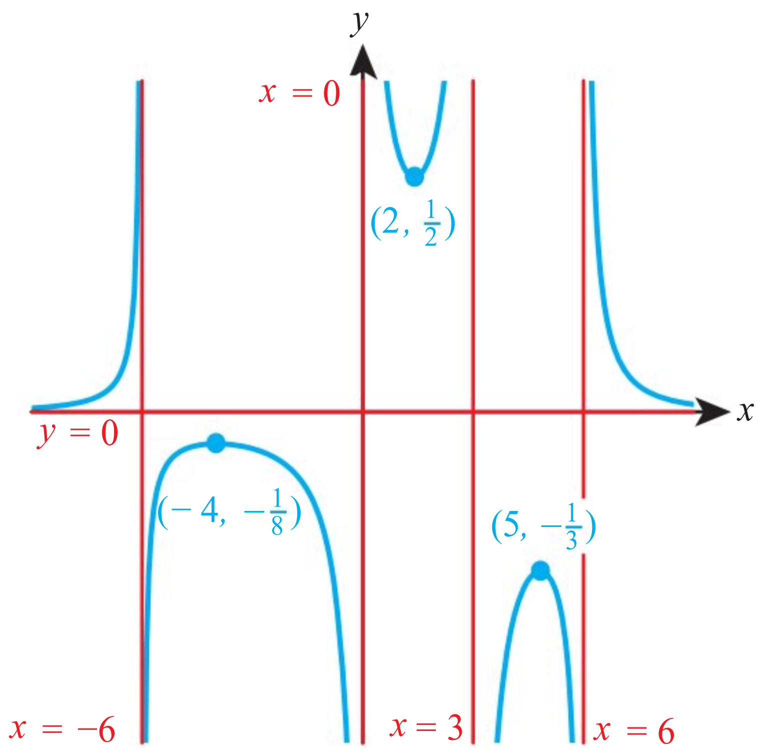
17 a



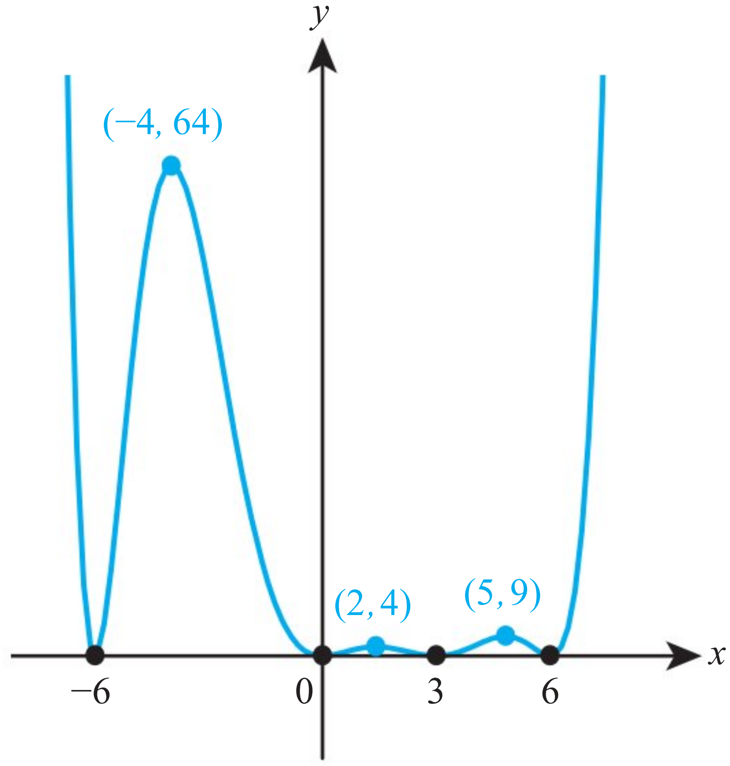
b



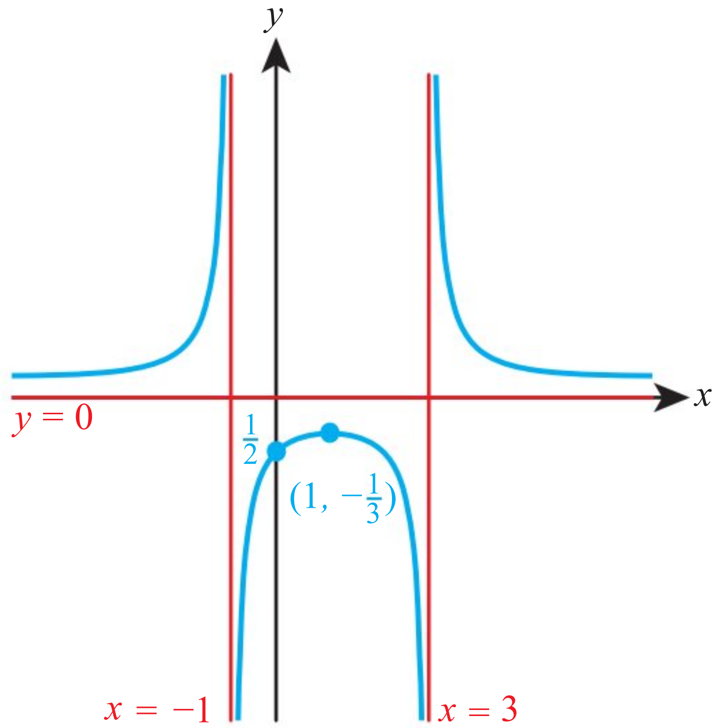
18 a



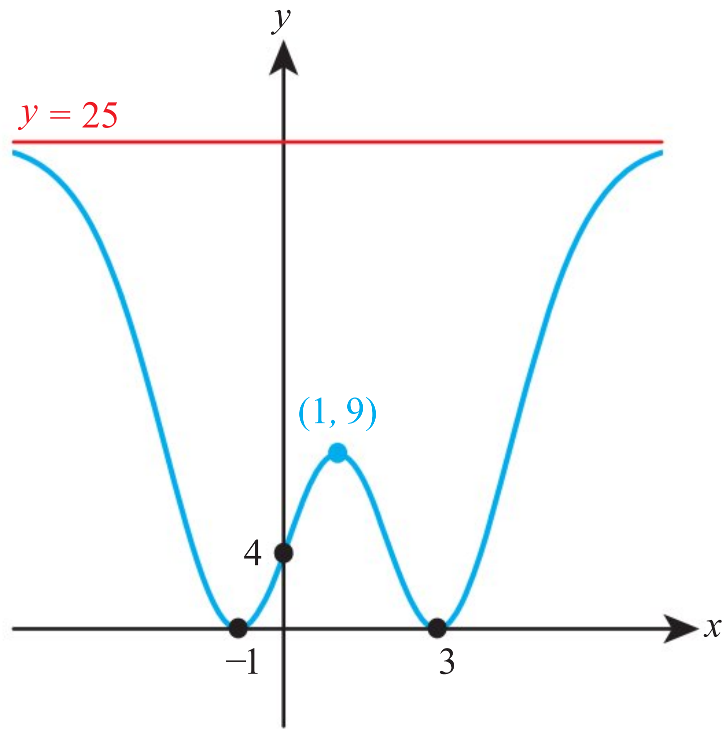
b



19 a



b



20 **a** $\left(3, -\frac{1}{4}\right)$ **b** $(3, 16)$ **c** $(2, -4)$

21 Translation right by $\frac{\pi}{4}$ followed by horizontal stretch with scale factor $\frac{1}{3}$

22 Translation left by 3 followed by horizontal stretch with scale factor $\frac{5}{2}$

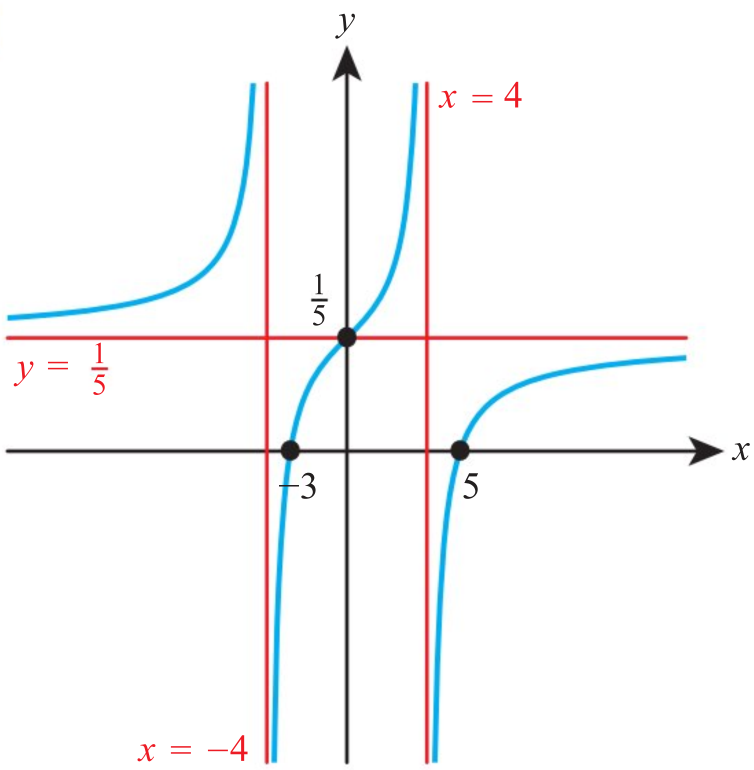
23 $y = 12x^2 - 16x + 4$

24 $y = 2\mathrm{f}(2x - 4) + 6$

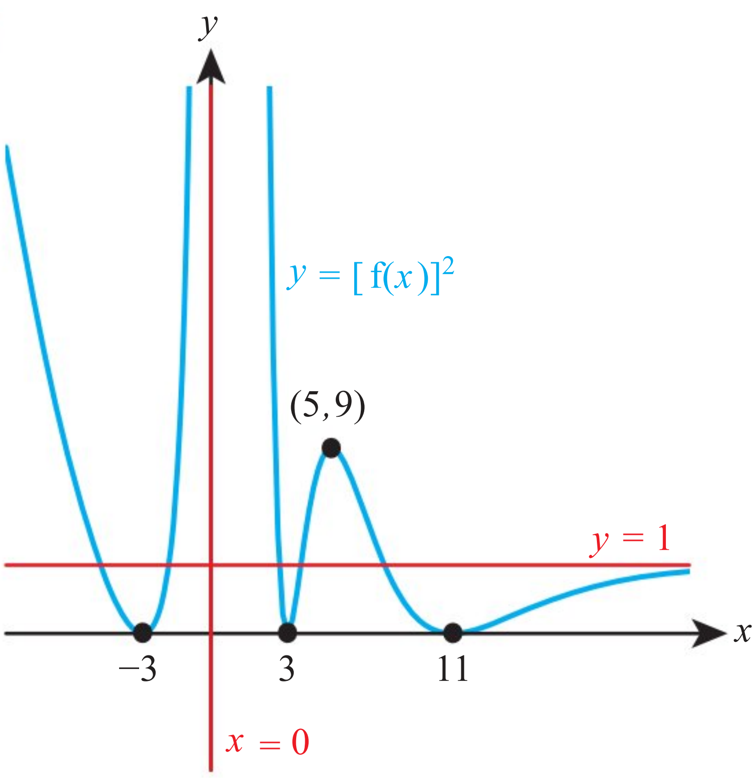
25 $y = \frac{1}{3}\mathrm{f}\left(\frac{x+1}{2}\right) - 4$

26 **a** $y = \mathrm{f}(-x + 5)$ **b** $y = \mathrm{f}(-x - 5)$

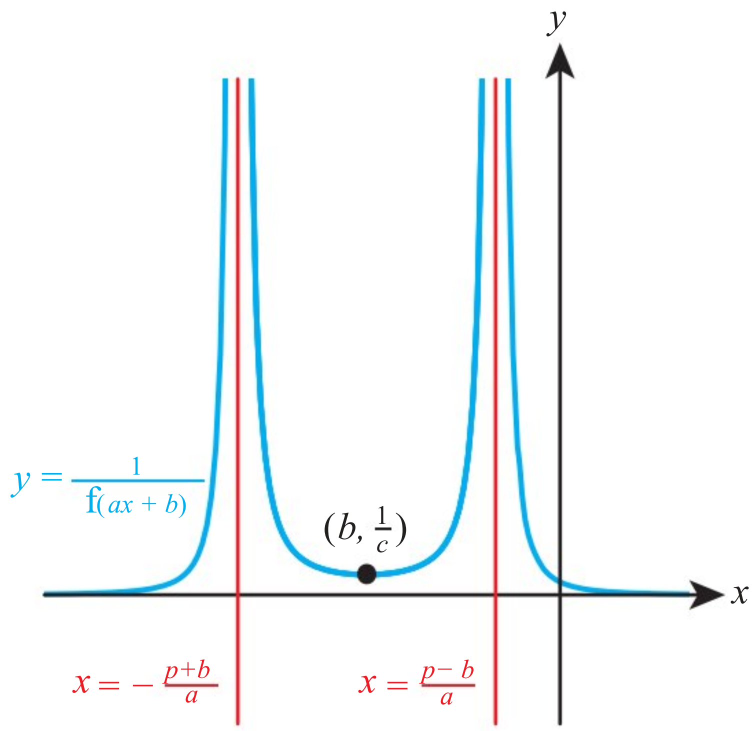
27



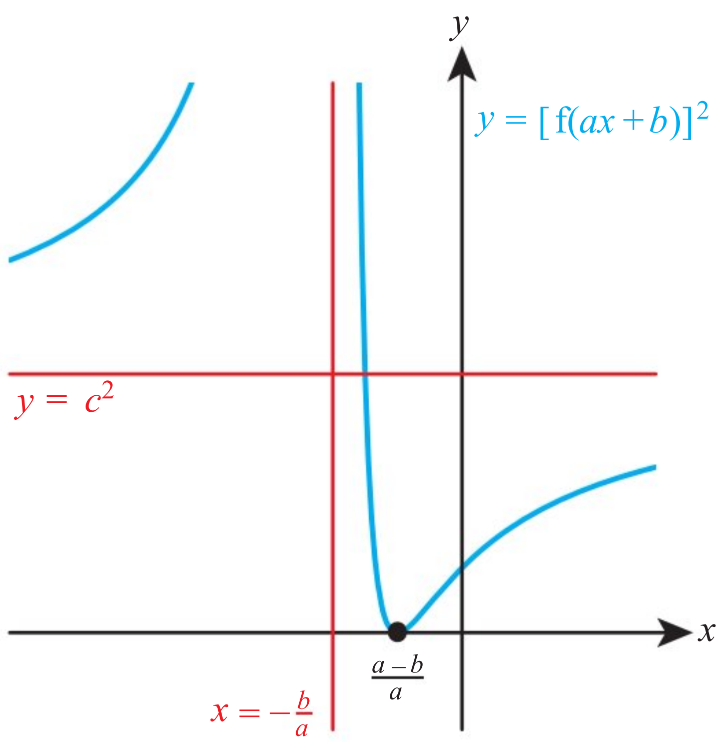
28



29



30



31 $a = 4,\ b = \frac{1}{2},\ c = -\frac{\pi}{3}$

32 $a = -1,\ b = 3,\ c = \frac{\pi}{2}$

33 Horizontal stretch with scale factor $\frac{1}{5}$ followed by translation right by $\frac{2}{5}$

34 $a = 9,\ b = 6,\ c = -10$

35 $a = \frac{1}{4},\ b = -2,\ c = 0$

36 Translation right by 1 followed by horizontal stretch with scale factor $\frac{2}{3}$

37 $y = \tan(-3x)$

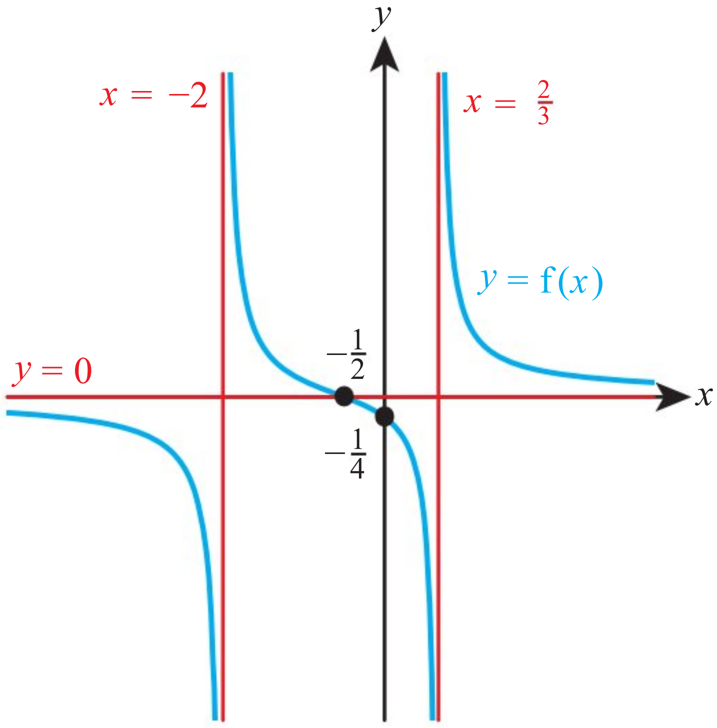
38 $c = -\log_2 5$

Exercise 7E

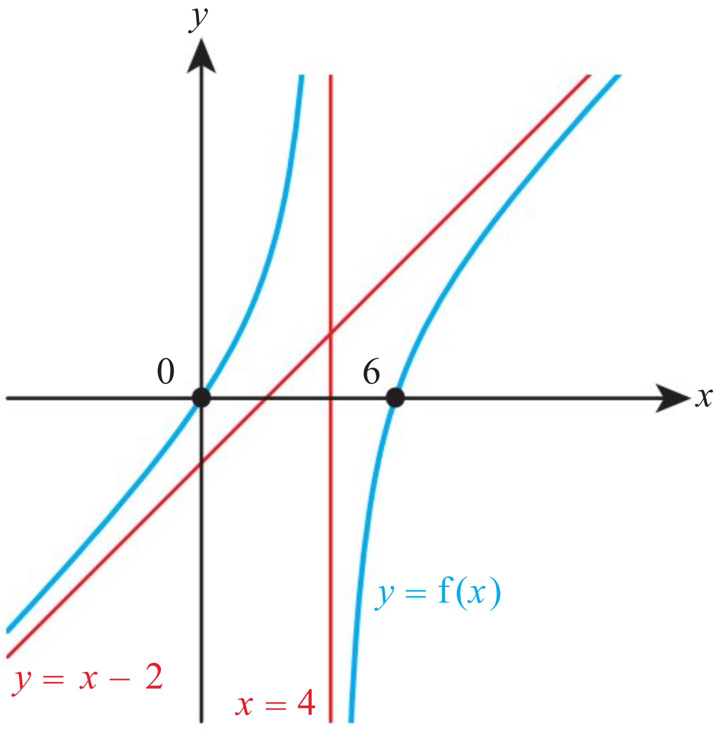
1 a Neither b Even
2 a Neither b Odd
3 a Neither b Even
4 a Even b Neither
5 a Odd b Even
6 a Neither b Even
7 a Odd b Neither
8 a Even b Neither
9 a $x \leqslant 2$ b $x \geqslant 2$
10 a $x \geqslant 4$ b $x \leqslant -2$
11 a $x \leqslant 1$ b $x \geqslant 3$
12 a Yes b Yes
13 a No b Yes
14 a No b Yes
15 Odd
16 Neither
17 Odd
18 a $k = -4$
b $f^{-1}(x) = -4 + \sqrt{x-3}, x \geqslant 3$
19 a $x \leqslant \frac{3}{2}$
b $f^{-1}(x) = \frac{3}{2} - \sqrt{x + \frac{5}{4}}, x \geqslant -\frac{5}{4}$
20 b $x \in \mathbb{R}$
21 b Neither
23 Even
24 a $-3 \leqslant x \leqslant -1$
b $-6 \leqslant x \leqslant -2$
25 a $k = 2$
b $x \geqslant -11$
26 a $x \leqslant \ln 4$
b $x \leqslant 4 - 4\ln 4$
27 b $x \neq \frac{2}{3}$
28 $a = -1$
30 $c = 2$

Chapter 7 Mixed Practice

1 a $x = \frac{2}{3}, x = -2$
b $\left(-\frac{1}{2}, 0\right)$ and $\left(0, -\frac{1}{4}\right)$
c

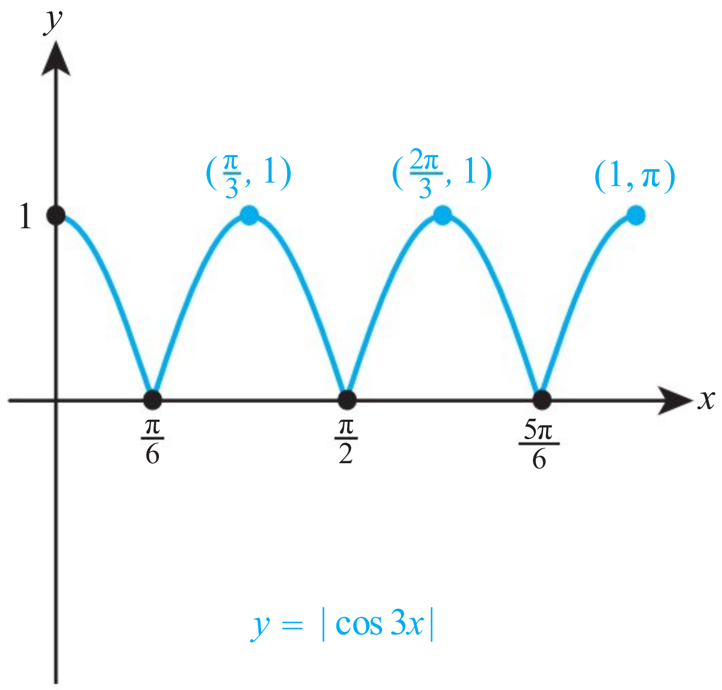


2 a i $x = 4$ ii $y = x - 2$
b (0, 0) and (6, 0)
c



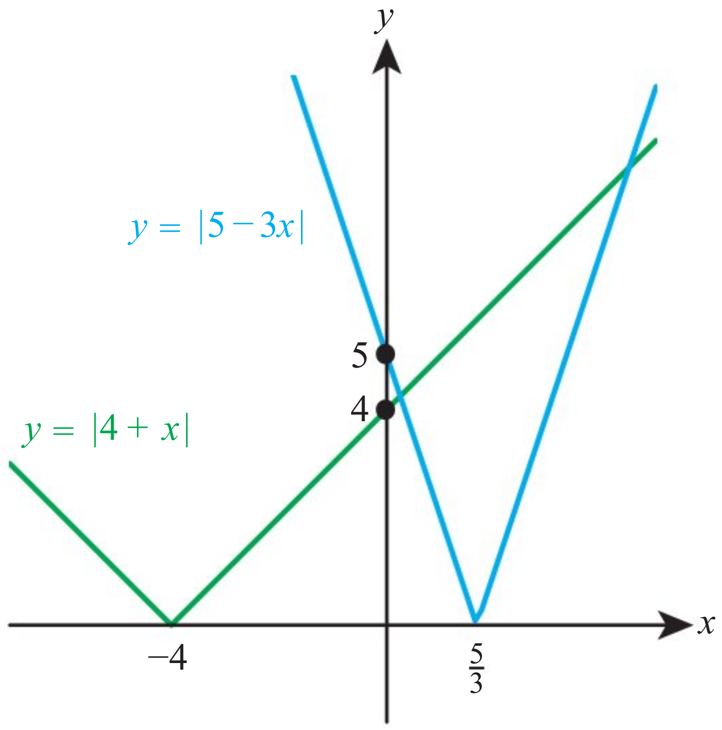
3 $\left]-\frac{3}{2}, 0\right[\cup \left]2, \infty\right[$
4 b $-2 \leqslant x \leqslant 1$ or $x \geqslant 4$
5 $-1.62 \leqslant x \leqslant -0.366$ or $0.618 \leqslant x \leqslant 1.37$
6 $0 < x \leqslant 3.04$ or $7.01 \leqslant x \leqslant 8.56$

7 a



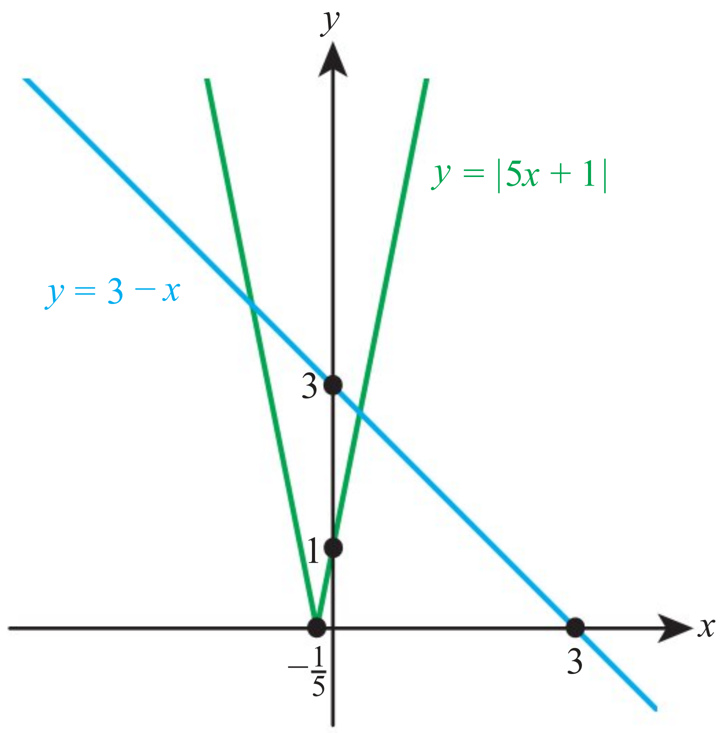
b $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

8 a



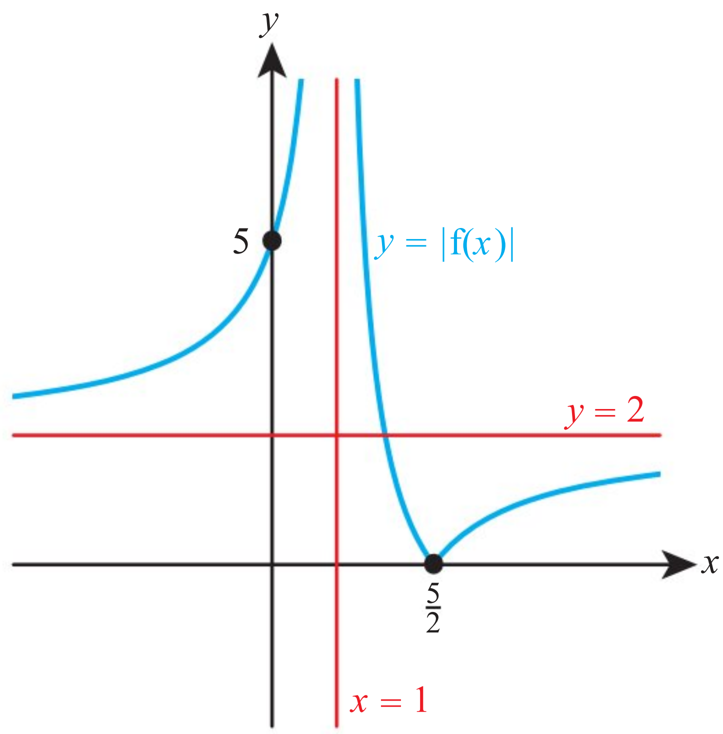
b $x \leqslant \frac{1}{4}$ or $x \geqslant \frac{9}{2}$

9 a

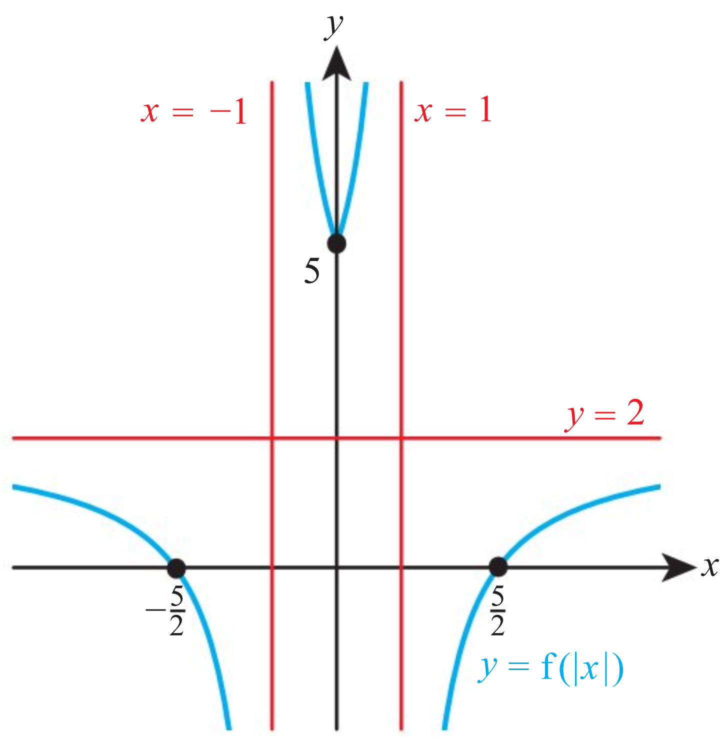


b $-1 < x < \frac{1}{3}$

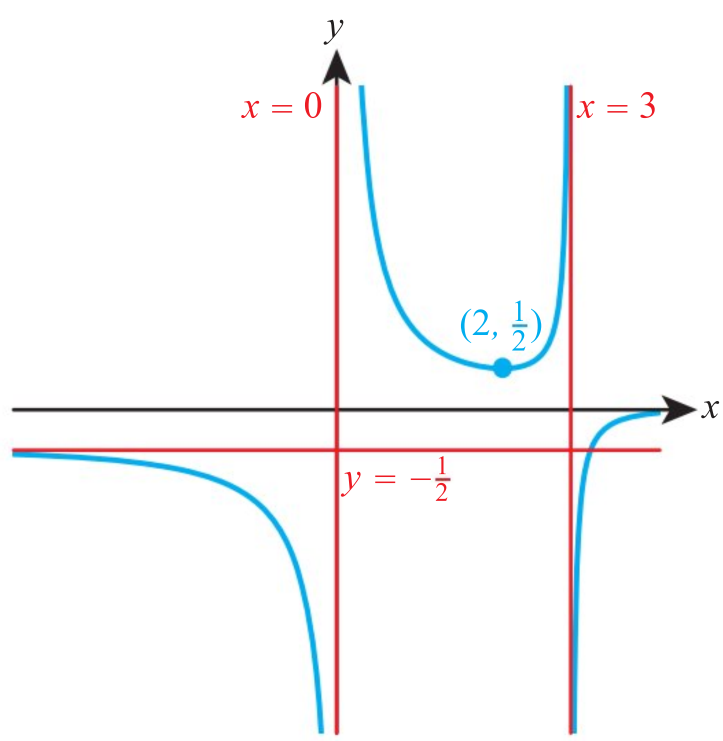
10 a



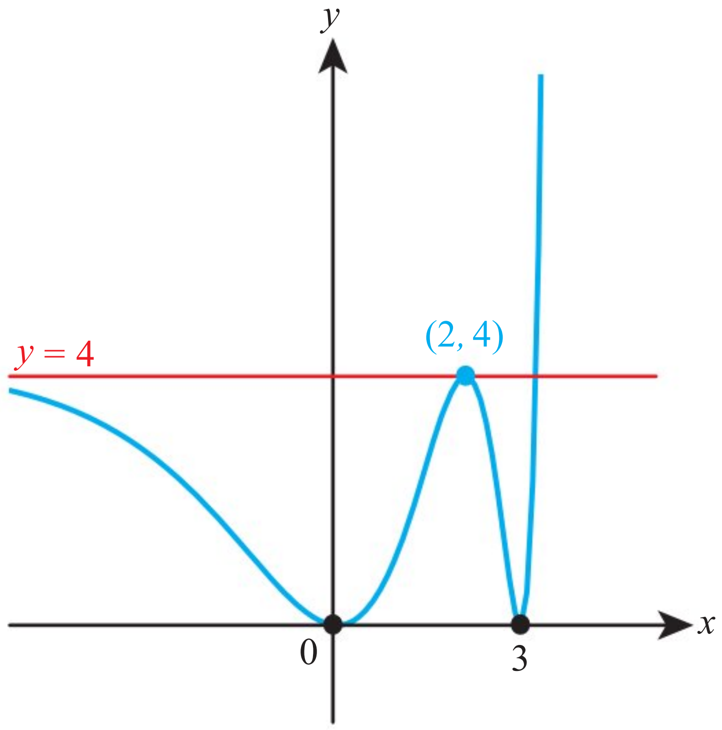
b



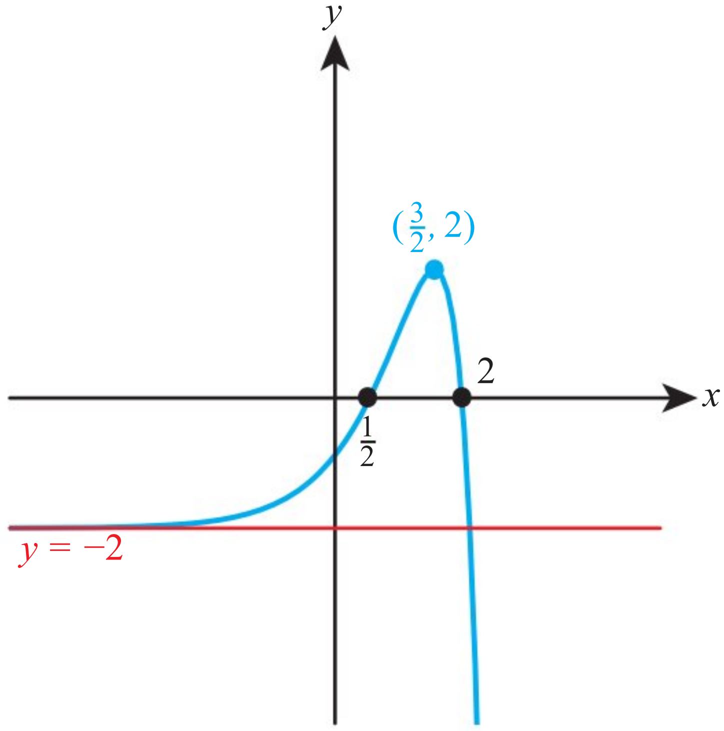
11 a



b



c



12 Even

13 a $k = 3$

b $f^{-1}(x) = 3 + \sqrt{5 - x},\ x \leqslant 5$

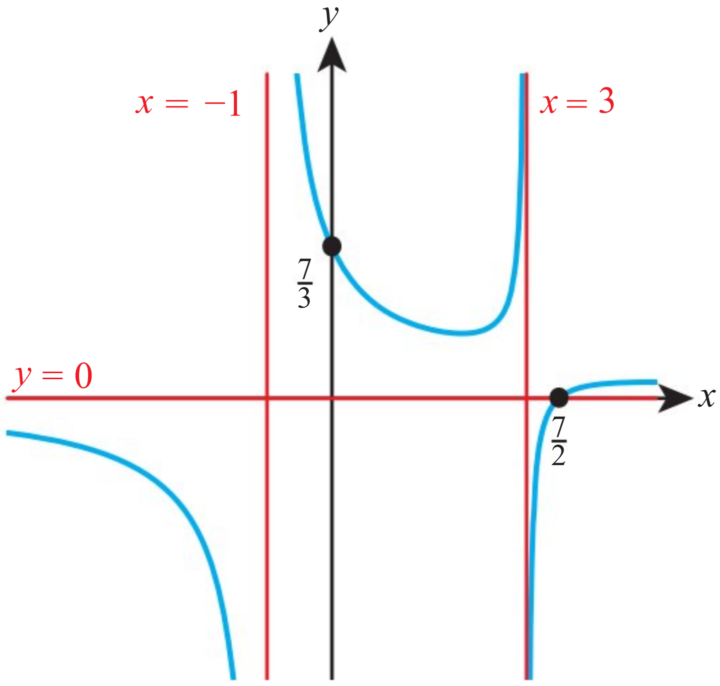
14 b $r = 0$

c $k(x) = 0$

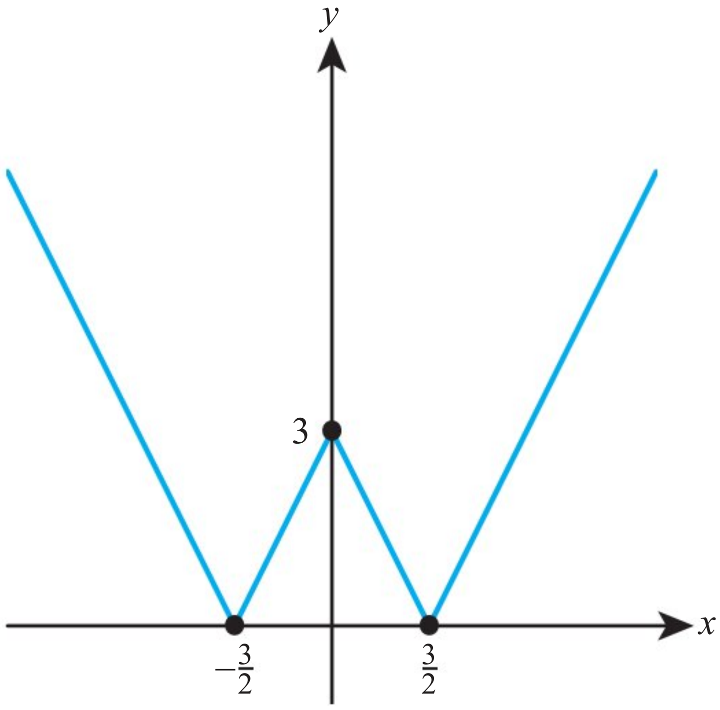
15 a i $k \leqslant \frac{1}{4}$ or $k \geqslant 1$

ii $f(x) \leqslant \frac{1}{4}$ or $f(x) \geqslant 1$

b



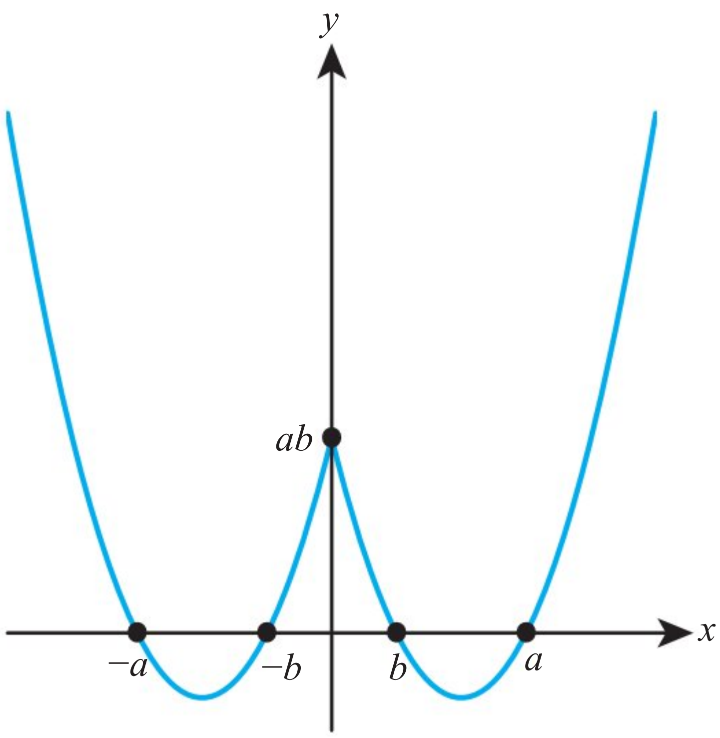
16 a



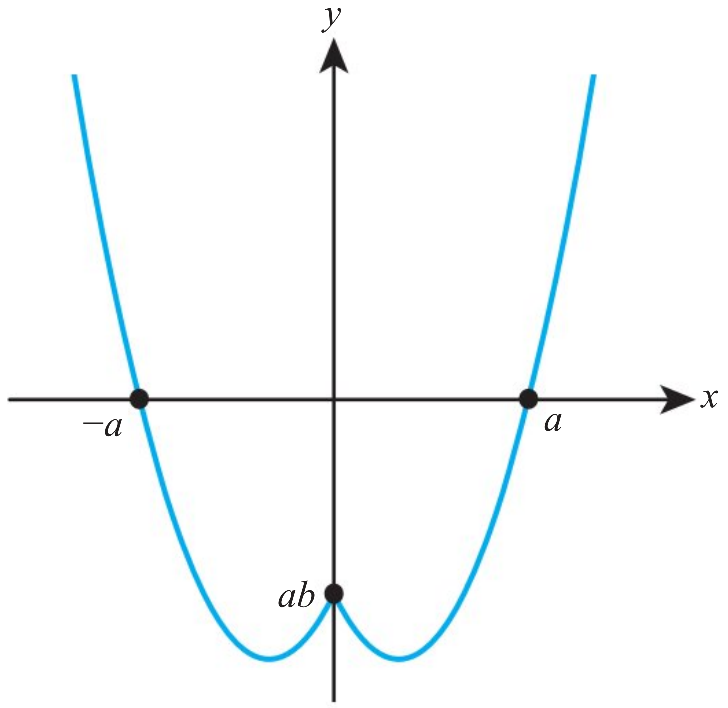
$\left(-\frac{3}{2}, 0\right), \left(\frac{3}{2}, 0\right), (0, 3)$

b $x = \pm\frac{1}{2}, \pm\frac{5}{2}$

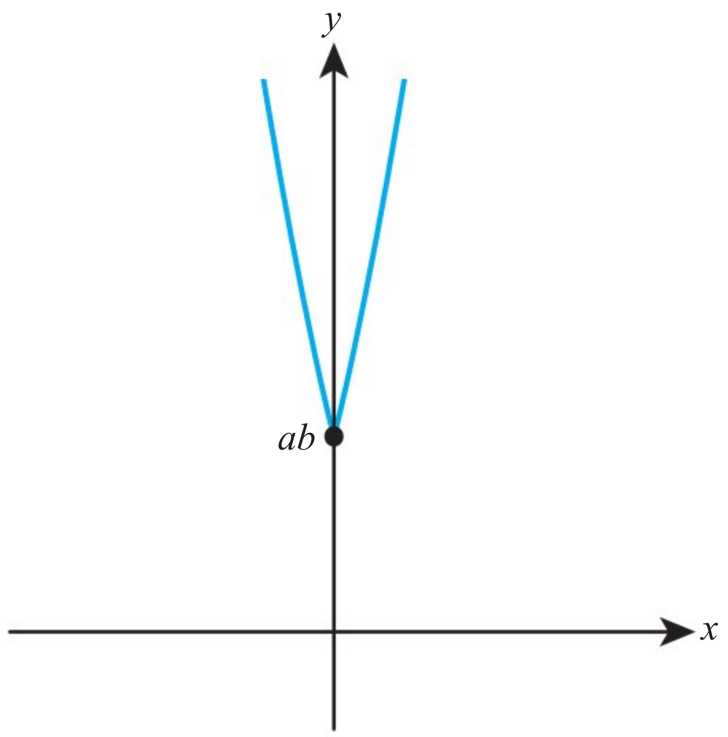
17 a



b



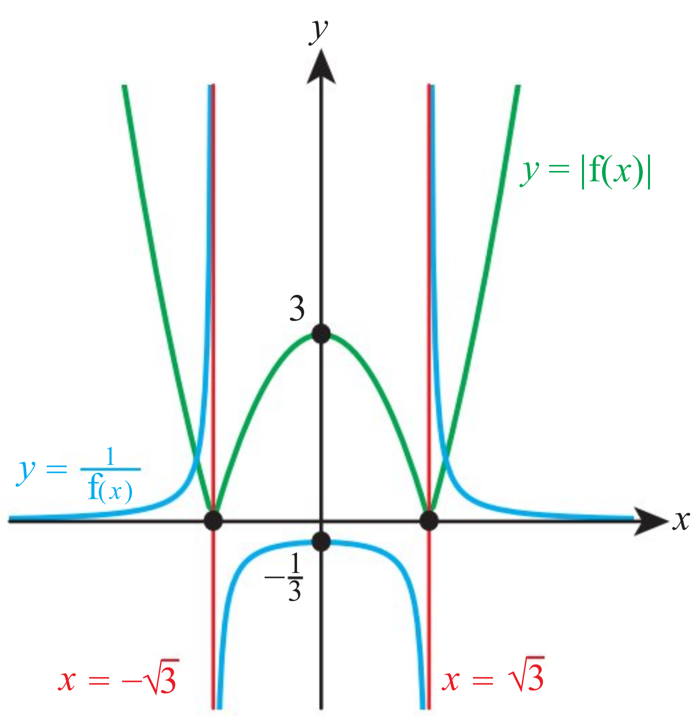
c



18 a Horizontal stretch with scale factor 3 followed by horizontal translation by +6

b Horizontal translation by +2 followed by horizontal stretch with scale factor 3

19 a



b $-2 \le x < -\sqrt{3}$ or $\sqrt{3} < x \le 2$

20 $a = -b$

21 a $k = \ln 4$

b $f^{-1}(x) = \ln(4 - \sqrt{x+9}),\ -9 \leqslant x < 7$

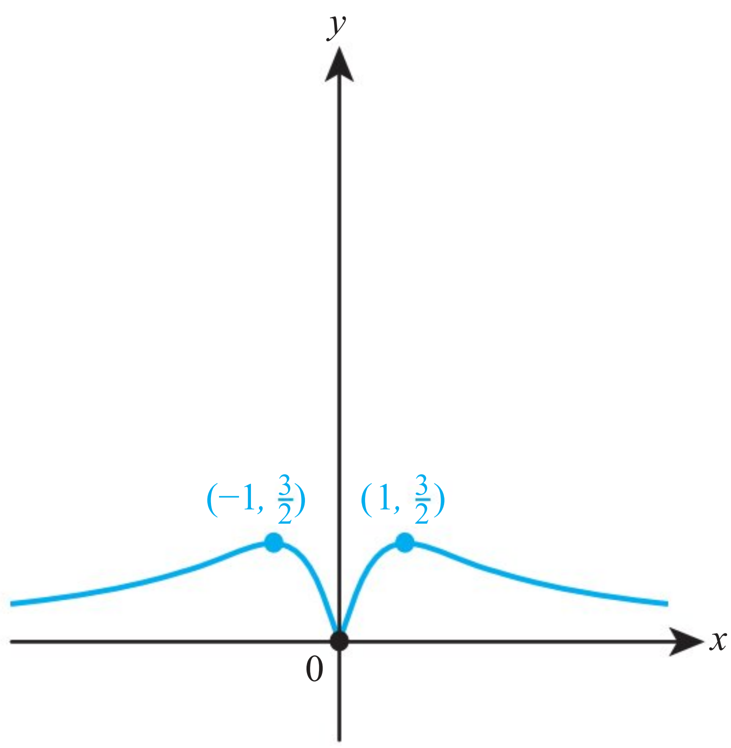
22 a $f'(x) = e^{\frac{x}{2}} + \frac{x}{2}e^{\frac{x}{2}},\ f''(x) = e^{\frac{x}{2}} + \frac{x}{4}e^{\frac{x}{2}}$

b $k = -2$ **c** $x \geqslant -2e^{-1}$

23 a ii Rotation 180° around the origin

b ii $\left(1, \frac{3}{2}\right), \left(-1, -\frac{3}{2}\right)$

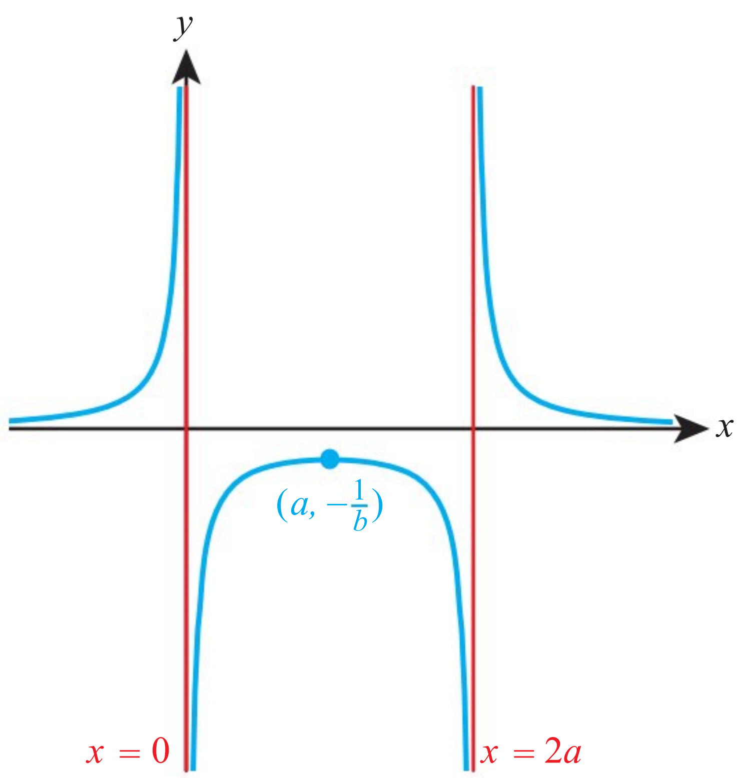
c



d $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

24 a $x \neq 0, 2a$

b

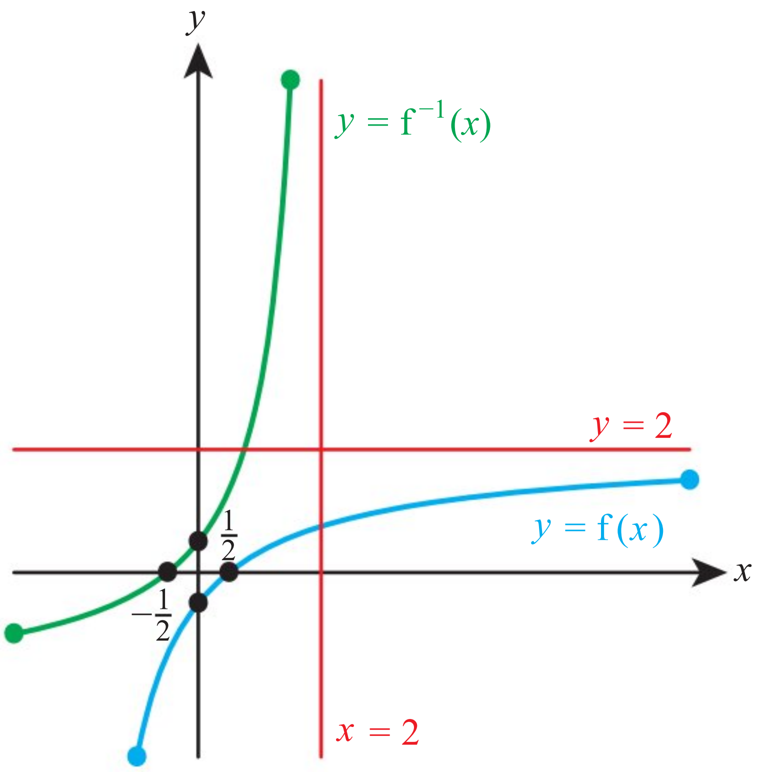


25 a $2 - \dfrac{5}{x+2}$

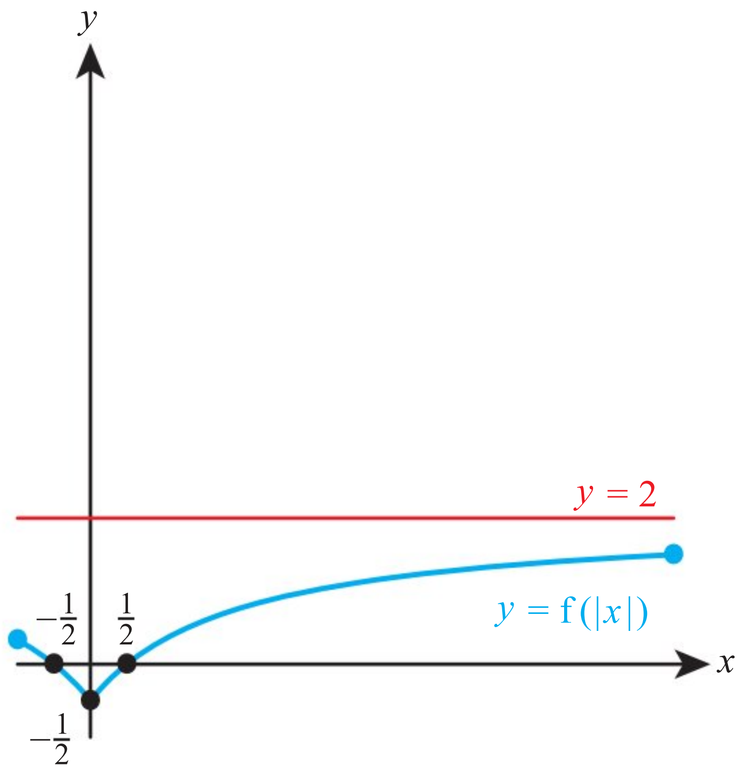
c $-3 \leqslant f(x) \leqslant 1.5$

d i $f^{-1}(x) = \dfrac{2x+1}{2-x}$

ii, iii



e i

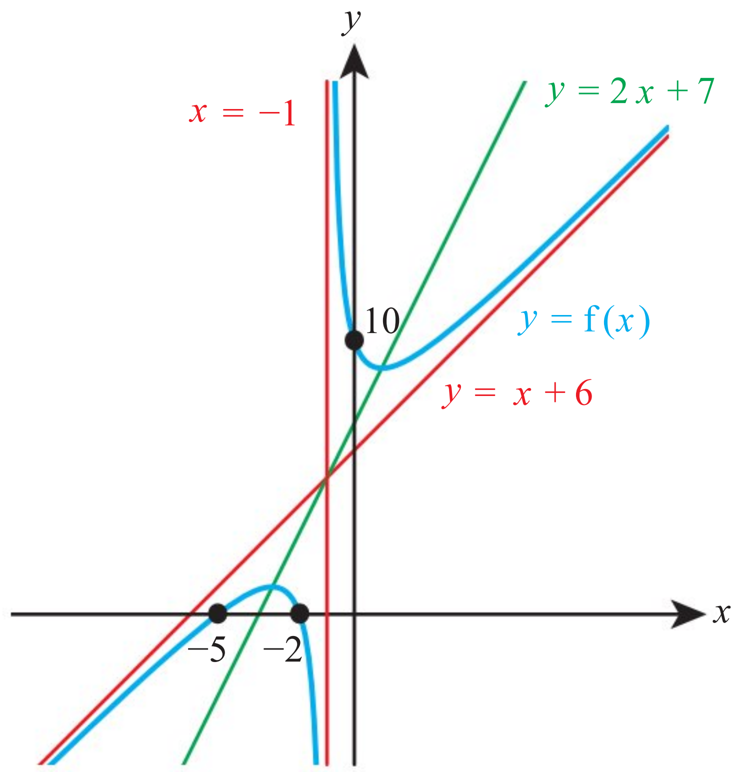


ii $x = \pm\dfrac{2}{9}$

26 a $y = x + 6$

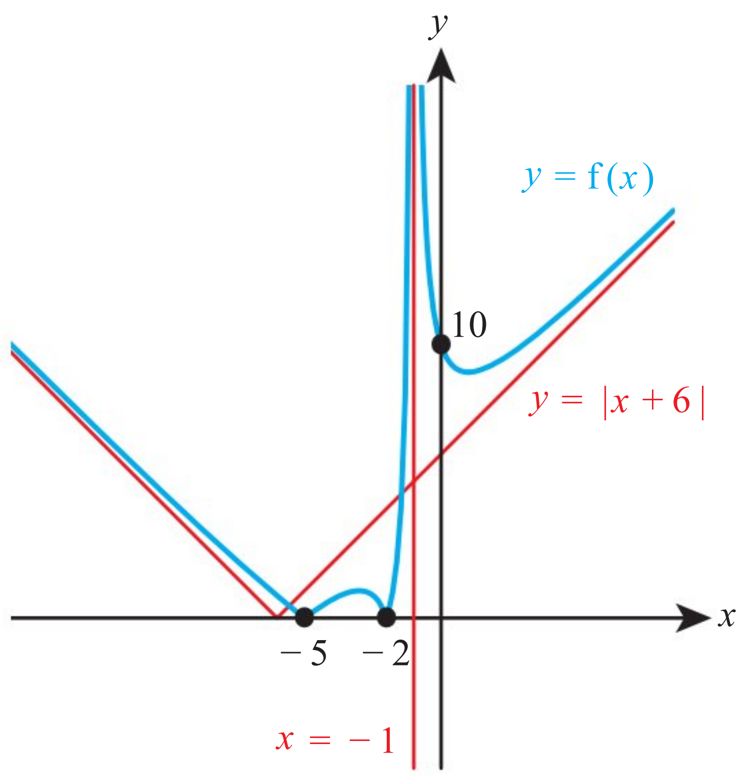
b (−3, 1) and (1, 9)

c



d $-3 < x < -1$ or $x > 1$

e



f $c = 0$ or $1 < c < 9$

27 $x = -4.5, -3.59, 1, 2.09$

28 $c = -3$

Chapter 8 Prior Knowledge

1 $2x + 5y = 23$

2 0

3 $x = \lambda,\ y = 13 - 7\lambda,\ z = 35 - 19\lambda$

Exercise 8A

1 $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2 $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

3 $\mathbf{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

4 $\mathbf{a} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

5 a $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ b $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$

6 a $-3\mathbf{i} + \mathbf{j}$ b $2\mathbf{i} - 2\mathbf{j}$

7 a $3\mathbf{i}$ b $-5\mathbf{j}$

8 a $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ b $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

9 a $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ b $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

10 a $\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$ b $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

11 a $\mathbf{a} + \mathbf{b}$ b $\mathbf{b} + \mathbf{c}$

12 a $-\mathbf{a} - \mathbf{b} - \mathbf{c}$ b $-\mathbf{b} - \mathbf{c}$

13 a $\begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}$ b $\begin{pmatrix} 3 \\ -7 \\ 3 \end{pmatrix}$

14 a $\begin{pmatrix} 11 \\ -4 \\ 21 \end{pmatrix}$ b $\begin{pmatrix} 22 \\ -21 \\ 36 \end{pmatrix}$

15 a $\begin{pmatrix} 11 \\ -17 \\ 15 \end{pmatrix}$ b $\begin{pmatrix} 2 \\ -9 \\ 0 \end{pmatrix}$

16 a $\mathbf{a} + \frac{4}{3}\mathbf{b}$ b $\mathbf{a} + \frac{1}{2}\mathbf{b}$

17 a $-\frac{3}{2}\mathbf{a} + \mathbf{b}$ b $-\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$

18 a $\frac{3}{2}\mathbf{a} - \mathbf{b}$ b $-\frac{4}{3}\mathbf{b} + \frac{1}{2}\mathbf{a}$

19 a $p = 3, q = 15$ b $p = 4, q = 16$

20 a $p = -6, q = 3$ b $p = 2, q = -8$

21 a $p = 4, q = 1$ b $p = 45, q = -1$

22 a $p = -3, q = -18$ b $p = 4, q = -2$

23 a $p = -2, q = -10$ b $p = -3, q = -3$

24 a $\mathbf{a} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ b $\mathbf{a} = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}$

25 a $\mathbf{a} = \begin{pmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$ b $\mathbf{a} = \begin{pmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$

26 a $\mathbf{a} = \frac{3}{\sqrt{11}}\mathbf{i} + \frac{1}{\sqrt{11}}\mathbf{j} - \frac{1}{\sqrt{11}}\mathbf{k}$

b $\mathbf{a} = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$

27 a $\mathbf{a} = \frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$ b $\mathbf{a} = \frac{2}{\sqrt{13}}\mathbf{j} - \frac{3}{\sqrt{13}}\mathbf{k}$

28 a $\mathbf{a} + \frac{1}{2}\mathbf{b}$ b $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$ c $2\mathbf{a} - \mathbf{b}$

29 a $\mathbf{a} - \mathbf{b}$ b $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ c $-\frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}$

30 a $\begin{pmatrix} 8 \\ 0 \\ -19 \end{pmatrix}$ b $\sqrt{146}$

31 $\pm 2\sqrt{11}$

32 $\dfrac{1 \pm \sqrt{41}}{10}$

33 $\begin{pmatrix} 2 \\ 0 \\ -3/4 \end{pmatrix}$

34 -2

35 a $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ b $\begin{pmatrix} 8 \\ -2 \\ 4\sqrt{2} \end{pmatrix}$

36 $-\frac{4}{3}$

37 -2

38 $\begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

39 $3, \frac{-5}{3}$

40 $\dfrac{\sqrt{65}}{3}$

41 $\sqrt{27 + 6\sqrt{2}}$

Exercise 8B

1 a $\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$ b $\begin{pmatrix} -6 \\ 3 \\ -1 \end{pmatrix}$

2 a $\begin{pmatrix} 8 \\ 0 \\ 10 \end{pmatrix}$ b $\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$

3 a $\begin{pmatrix} -2 \\ -3 \\ -9 \end{pmatrix}$ b $\begin{pmatrix} -8 \\ 0 \\ -10 \end{pmatrix}$

4 a $\sqrt{53}$ b $\sqrt{26}$

5 a $\sqrt{2}$ b $\sqrt{2}$

6 a $\sqrt{94}$ b $\sqrt{53}$

7 a Yes, no b Yes, no

8 a No b No

9 a Yes, yes b Yes, yes

10 a $p = -15, q = 22$ b $p = 1, q = -3$

11 a $p = 10, q = 11$ b $p = 0, q = -3$

12 a $p = 3, q = -1$ b $p = -\frac{1}{3}, q = -6$

13 a $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ b $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$

14 a $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ b $\frac{1}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} + 2\mathbf{k}$

15 a $\frac{1}{2}\mathbf{j} + \frac{7}{2}\mathbf{k}$ b $-\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{7}{2}\mathbf{k}$

16 $\frac{3}{2}$

17 b $1:3$

18 a $\sqrt{30}$ b $\frac{3}{2}\mathbf{i} - \mathbf{j} - \frac{3}{2}\mathbf{k}$

19 a $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ b $\sqrt{227}$

20 $3\mathbf{i} - 4\mathbf{j}$

21 $\begin{pmatrix} 2 \\ 13 \\ -1 \end{pmatrix}$

22 a $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ b No

23 a $(13, 4, -6)$ b $(6, 3, 1)$

24 a $\overrightarrow{BC} = \mathbf{c} - \mathbf{b},\ \overrightarrow{MN} = \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{b}$

b Parallel, $MN = \frac{1}{2}BC$

26 a $p = 6, q = 5$ b $3:2$

27 $\frac{11}{5}\mathbf{i} - \mathbf{j} - \frac{12}{5}\mathbf{k}$

28 $-2, -\frac{23}{15}$

29 a $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 2\mathbf{k}$ b $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

30 a $\frac{1}{2}\mathbf{a}+\frac{1}{3}\mathbf{b}+\frac{1}{6}\mathbf{c}$ c $2:1$

31 a $\mathbf{d}=\frac{3}{5}\mathbf{b}+\frac{2}{5}\mathbf{c},\ \mathbf{e}=\frac{3}{2}\mathbf{a}-\frac{1}{2}\mathbf{c},\ \mathbf{f}=\frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$

Exercise 8C

1 a 35 b 20

2 a 67.3 b 9.64

3 a –54.0 b –36.9

4 a 16 b –56

5 a –16 b 10

6 a 9 b 9

7 a 64° b 67°

8 a 108° b 101°

9 a 61° b 65°

10 a 96° b 111°

11 a 25.5 b 58.5

12 a –4.5 b –3

13 a 156.5 b 615.5

14 a 51.5 b 420.5

15 a $\frac{2}{7}$ b $-\frac{1}{2}$

16 a 3 b 2

17 a 9 b $\frac{16}{7}$

18 a $\frac{4}{5}$ b –12

19 a 19 b 7 c 32

20 141°

21 40.0°

22 $\frac{2}{3}$

23 48.2°

24 98.0°

26 3

27 61.0°, 74.5°, 44.5°

28 94.3°, 54.2°, 31.5°

29 b 41.8°, 48.2°
c 161

30 6

31 $-\frac{3}{4}$

32 $0, \frac{3}{2}$

33 2

34 a 1.6 b 68.7°, 21.3°, 90°
c 61.8

36 a $\mathbf{a}+\mathbf{b},\ \mathbf{b}-\mathbf{a}$ b $|\mathbf{b}|^2-|\mathbf{a}|^2$

37 b 2 c $4\sqrt{5}$

Exercise 8D

1 a $\mathbf{r}=\begin{pmatrix}2\\1\\5\end{pmatrix}+t\begin{pmatrix}-1\\2\\2\end{pmatrix}$, yes

b $\mathbf{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\1\\5\end{pmatrix}$, yes

2 a $\mathbf{r}=\begin{pmatrix}4\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\0\\3\end{pmatrix}$, no

b $\mathbf{r}=\begin{pmatrix}-1\\5\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\7\end{pmatrix}$, no

3 a $\mathbf{r}=\begin{pmatrix}4\\1\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\end{pmatrix}$, no

b $\mathbf{r}=\begin{pmatrix}2\\7\end{pmatrix}+\lambda\begin{pmatrix}2\\-9\end{pmatrix}$, yes

4 a $\mathbf{r}=\begin{pmatrix}1\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\-3\end{pmatrix}$

b $\mathbf{r}=\begin{pmatrix}-1\\1\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\2\end{pmatrix}$

5 a $\mathbf{r}=\begin{pmatrix}0\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

6 a $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

7 a $x = 3 + 8\lambda,\ y = 5 + 2\lambda,\ z = 2 + 4\lambda$

b $x = 4 - 2\lambda,\ y = -1 + 3\lambda,\ z = 2 + 5\lambda$

8 a $x = 2 + \lambda,\ y = 3,\ z = -4\lambda$

b $x = 2\lambda,\ y = 3 + \lambda,\ z = -1$

9 a $x = -1,\ y = 3 + 5\lambda$ b $x = 4 - 3\lambda,\ y = 2\lambda$

10 a $x - 3y = 17$ b $3x + 4y = -2$

11 a $2x + 3y = 6$ b $5x - 3y = -12$

12 a $y = 4$ b $x = 2$

13 a $\frac{x-1}{-1} = \frac{y-7}{1} = \frac{z-2}{2}$

b $\frac{x-3}{2} = \frac{y+1}{-4} = \frac{z}{5}$

14 a $\frac{2x-1}{4} = \frac{y+2}{3} = \frac{4-3z}{6}$

b $\frac{2-3x}{3} = \frac{y-2}{1} = \frac{2z+1}{6}$

15 a $\frac{2x-2}{1} = \frac{3y}{1} = \frac{2-z}{3}$

b $\frac{x-3}{4} = \frac{2-2y}{1} = \frac{3z+21}{2}$

16 a $x = -1, \frac{y-5}{-2} = \frac{z}{2}$

b $x = 1, \frac{y}{3} = \frac{z-3}{-1}$

17 a $\frac{x+1}{5} = \frac{3-z}{2},\ y = 2$

b $\frac{x-2}{-3} = \frac{y+1}{1},\ z = -3$

18 a $y = 1,\ z = 5$

b $x = 4,\ z = -3$

19 a $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$

20 a $\mathbf{r} = \begin{pmatrix} 3/2 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} -1/3 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4/3 \\ 1 \\ -2 \end{pmatrix}$

21 a $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

22 a $\mathbf{r} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

23 a 31.4° b 31.8°

24 a 38.1° b 80.0°

25 a 83.7° b 7.13°

26 a $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix}$ b No

27 $\frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-2}{-3}$

28 81.8°

30 No

31 a $2.55\,\text{m s}^{-1}$

b $\mathbf{r} = (12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}) + t(0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$

c No

32 16.8 m

33 a $p = 12,\ q = 5$ b 76.4°

34 b (0, 3, 0)

35 a $\mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$ b (–5, –5 –11)

36 a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ b 7

c $(-8, 16, -26), (12, -14, 34)$

37 a $\frac{x-1}{3} = \frac{4-y}{2} = \frac{z+1}{3}$ b $\frac{1}{\sqrt{22}}\begin{pmatrix}3\\2\\3\end{pmatrix}$

38 a $\mathbf{r} = \begin{pmatrix}3\\-1\\-1/3\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\4/3\end{pmatrix}$ b $\frac{1}{3}$

39 a $\mathbf{r} = \begin{pmatrix}1/2\\7\\2\end{pmatrix} + \lambda\begin{pmatrix}3/2\\0\\-4\end{pmatrix}$ b 69.4°

40 13.2°

41 $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$

42 a $\begin{pmatrix}3\\-1\\4\end{pmatrix}$ b $3.32\,\text{m s}^{-1}$ c 14.9 m

43 a $\mathbf{r}_1 = 3\mathbf{i} + t(-2\mathbf{i} + 5\mathbf{j}),\ \mathbf{r}_2 = 5\mathbf{j} + t(4\mathbf{i} + \mathbf{j})$

b $\sqrt{52t^2 - 76t + 34}$

c 2.50 m

44 $\mathbf{r} = \begin{pmatrix}2\\0\\0\end{pmatrix} + t\begin{pmatrix}596\\-596\\298\end{pmatrix}$

45 a $(9, -5, 8)$ c $(3, 4, -3)$

46 b 48.5° d $\frac{11\sqrt{11}}{6}$ $(= 6.08)$

e 4.55

47 3

48 $\sqrt{\frac{6}{11}}$

Exercise 8E

1 a $(10, -7, -2)$ b $(-1, 1, 6)$

2 a $(0.5, 0, 1)$ b $(4.5, 0, 0)$

3 a $(3, 3, 1)$ b $(7, 2, 4)$

6 a Parallel b Parallel

7 a Not parallel b Not parallel

8 a Same line b Same line

9 $(4, 3, 3)$

10 Skew

11 a $x = 2z - 3,\ y = 2z - 9; x = \frac{11-3z}{5},\ y = \frac{z-27}{5}$

b $(1, -5, 2)$

12 $(3, -2, 1)$

14 a $(0, -22, 0)$

15 a $(8, 7, 1)$

16 3; $(3, 7, 2)$

17 b No

18 a $\sqrt{54}$, 3 b No

19 b 1.58 m

20 a 0.1 b 47.2 km/h

21 a $7\lambda + 5\mu = 11$ b 3

Exercise 8F

1 a 60.6 b 34.6

2 a 251 b 11.5

3 a 64.3 b 25.8

4 a $\begin{pmatrix}-2\\6\\-1\end{pmatrix}$ b $\begin{pmatrix}-1\\-10\\7\end{pmatrix}$

5 a $\begin{pmatrix}-9\\-19\\2\end{pmatrix}$ b $\begin{pmatrix}-23\\1\\8\end{pmatrix}$

6 a $-5\mathbf{i} - 11\mathbf{j} - 2\mathbf{k}$ b $12\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$

7 a 50 b 30

8 a 120 b 80

9 a 10 b 30

10 a 70 b 90

11 a $\frac{\sqrt{153}}{2}$ b $\sqrt{33}$

12 a $\frac{15\sqrt{3}}{2}$ b $\frac{9}{2}$

13 a $\frac{\sqrt{446}}{2}$ b $\frac{3\sqrt{66}}{2}$

14 17.5

15 0.775

16 0.630

17 $5\sqrt{6}$

18 $-8\mathbf{i}-5\mathbf{j}+\mathbf{k}$

19 a $\begin{pmatrix}0\\-1\\1\end{pmatrix}$ b $\begin{pmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

20 $\begin{pmatrix}3/14\\1/14\\-2/14\end{pmatrix}$

21 b $13\mathbf{a}\times\mathbf{b}$

22 b 0

23 a $\begin{pmatrix}18\\-12\\72\end{pmatrix},\begin{pmatrix}-18\\12\\-72\end{pmatrix}$ b $\mathbf{p}=-\mathbf{q}$

25 b 42.6

26 $\frac{\sqrt{19}}{2}$

27 a $(11, -2, 0)$ b 21.9

30 a $C(5,4,0),\ F(5,0,2),\ G(5,4,2),\ H(0,4,2)$
b 11.9

Exercise 8G

1 a $\mathbf{r}=\begin{pmatrix}3\\4\\-2\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\2\end{pmatrix}$, yes
b $\mathbf{r}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\1\end{pmatrix}+\mu\begin{pmatrix}3\\-1\\0\end{pmatrix}$, yes

2 a $\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\5\\2\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\3\end{pmatrix}$, no
b $\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\4\\-1\end{pmatrix}+\mu\begin{pmatrix}5\\3\\0\end{pmatrix}$, no

3 a $\mathbf{r}=(\mathbf{j}+\mathbf{k})+\lambda(3\mathbf{i}+\mathbf{j}-3\mathbf{k})+\mu(\mathbf{i}-3\mathbf{j})$, no
b $\mathbf{r}=(\mathbf{i}-6\mathbf{j}+2\mathbf{k})+\lambda(5\mathbf{i}-6\mathbf{j})+\mu(-\mathbf{i}+3\mathbf{j}-\mathbf{k})$, yes

4 a i $\mathbf{r}\cdot\begin{pmatrix}3\\-5\\2\end{pmatrix}=-4$ ii $3x-5y+2z=-4$
b i $\mathbf{r}\cdot\begin{pmatrix}6\\-1\\2\end{pmatrix}=19$ ii $6x-y+2z=19$

5 a i $\mathbf{r}\cdot(3\mathbf{i}-2\mathbf{j}+5\mathbf{k})=5$
ii $3x-2y+5z=5$
b i $\mathbf{r}\cdot(4\mathbf{i}+\mathbf{j}-2\mathbf{k})=-2$
ii $4x+y-2z=-2$

6 a i $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\0\end{pmatrix}=-9$ ii $3x-y=-9$
b i $\mathbf{r}\cdot\begin{pmatrix}4\\0\\-5\end{pmatrix}=-10$ ii $4x-5z=-10$

7 a $10x+13y-12z=38$
b $3x+y+z=1$

8 a $x+5y=22$
b $x+20y+7z=152$

9 a $x+y+z=10$
b $40x+5y+8z=580$

10 a $\mathbf{r}=\begin{pmatrix}12\\4\\10\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-5\end{pmatrix}+\mu\begin{pmatrix}3\\-8\\-10\end{pmatrix}$
b $\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}$

11 a $\mathbf{r}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$
b $\mathbf{r}=\begin{pmatrix}-1\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}-6\\2\\-4\end{pmatrix}$

12 a $\mathbf{r}=\begin{pmatrix}9\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-11\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-8\\-1\\2\end{pmatrix}$
b $\mathbf{r}=\begin{pmatrix}11\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}-10\\21\\-1\end{pmatrix}+\mu\begin{pmatrix}-16\\17\\-3\end{pmatrix}$

13 Yes

14 a $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

b No

15 a $4x - y + 7z = 39$ b No

16 a $5\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ b 14

17 a $\frac{1}{3\sqrt{10}}(\mathbf{i} + 5\mathbf{j} - 8\mathbf{k})$ b $p = 30,\ q = 1$

18 a $(1, -3, 14)$ b $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

c $2x - 3y + z = 25$

19 a $(10, 11, -6)$ b $\begin{pmatrix} 7 \\ -9 \\ -5 \end{pmatrix}$

c $7x - 9y - 5z = 1$

20 $\mathbf{r} = \begin{pmatrix} 11 \\ 12 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 15 \\ 6 \end{pmatrix}$

21 b $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ c $x + 2z = 8$

Exercise 8H

1 a 46.4° b 10.8°

2 a 17.5° b 51.0°

3 a 75.8° b 47.6°

4 a 17.7° b 51.9°

5 a 48.2° b 51.9°

6 a 60° b 60°

7 a (7, 3, 11) b (5, 3, 6)

8 a (1, 3, 5) b (10, 4, –3)

9 a (–7, 0, 3) b (–7, 1, 5)

10 a $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$

11 a $\mathbf{r} = \begin{pmatrix} 0 \\ 8 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -5 \\ -11 \end{pmatrix}$

12 a $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

13 a (–11, 64, 88) b (–2, 1, 0)

14 a Prism b Prism

15 a Π_1 and Π_2 are parallel

b Π_1 and Π_3 are parallel

16 a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

17 a $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

18 a $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

19 a $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ b 8°

20 a $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

c (3, 3, 1) d 9

21 a $8\mathbf{i} - 16\mathbf{j} + 24\mathbf{k}$ b $x - 2y + 3z = 9$

c 69.1°

22 a (9, 6, 7) b 7.42° c 12.1

23 a 32.5° b (1, 1, 5) c 3.39

24 a $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ b $(3, 4, -3)$

c $(2, 7, -8)$

25 a $\begin{pmatrix}0\\5\\5\end{pmatrix}$ b $2x - y + z = 0$

d $\mathbf{r} = \begin{pmatrix}3\\4\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ e (3, 4, 0)

f 47.1°

26 a 63

b Intersect along a line

27 a Prism

b $a = -2$, $\mathbf{r} = \begin{pmatrix}0.4\\0\\-0.8\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$

28 a $\mathbf{r} = \begin{pmatrix}3\\0\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\4\end{pmatrix}$ c Prism

29 a $c = 1$ b $\begin{pmatrix}20\\-15\\5\end{pmatrix}$

c $\frac{x+3}{4} = \frac{y-4}{-3} = \frac{z-1}{1}$

30 a $\begin{pmatrix}-2\\-12\\62\end{pmatrix}$ b 31.6

c $x + 6y - 31z = -86$ e $(-4.75, 3.49, 3.30)$

f $\frac{248}{3}$

31 b $\frac{8}{15}$ c $\mathbf{r} = \begin{pmatrix}2\\0\\8/15\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix}$

d 132

32 b $4\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$ d $\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-5}{-4}$

33 a $k \neq 1$ b $k = 1, c = -4$

c $k = 1, c \neq -4$

Chapter 8 Mixed Practice

1 a $\frac{1}{2}\mathbf{b} - \mathbf{a}$ b $\frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$

2 a -14 b $\frac{19}{16}$

3 $\frac{5\pi}{12}$

4 a $6\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ b $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

c $\sqrt{65}$

5 a $128\,\mathrm{m\,s^{-1}}$ b 83.3 seconds

6 a $\begin{pmatrix}2\\0\\k-7\end{pmatrix}$ c (3, 6, 1)

d $\frac{1}{\sqrt{130}}$

7 a $\mathbf{r} = \begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\5\\-1\end{pmatrix}$ b 88.5°

c 24 d 23.6

8 a $\begin{pmatrix}-2\\7\\-3\end{pmatrix}$ b (3, 3, 8)

c $\begin{pmatrix}-2\\7\\-3\end{pmatrix}$ d $2x - 7y + 3z = 9$

9 a $3x + y - z = 6$ b $\frac{5}{4}$

10 a i $\frac{1}{2}(\mathbf{c} - \mathbf{a})$ b i $\frac{1}{3}(\mathbf{a} - \mathbf{b})$

11 $\sqrt{6}\mathbf{i} - \frac{\sqrt{6}}{2}\mathbf{j} + \frac{\sqrt{6}}{2}\mathbf{k}$

12 $\frac{19}{3}$

13 b 107°, 73.2° c $\frac{5}{4}$

15 $\sqrt{133}$

16 $p = \frac{3}{8}, q = \frac{1}{8}$

17 a $\sqrt{2(1-\cos\alpha)}, \sqrt{2(1+\cos\alpha)}$

b $72.9°$

18 a Perpendicular b No

19 a $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ b ii $(1, -3, 14)$

c $2x - 3y + z = 25$

20 a $\mathbf{r} = \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$ b $\left(-33.28, \frac{12}{7}, \frac{45}{18}\right)$

c 2.63

21 a $\boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$ b $\left(\frac{22}{17}, -\frac{8}{17}, -\frac{9}{17}\right)$

c 1.62

22 a $\boldsymbol{r} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ b $\frac{1}{2}$

c 6 d 3

23 b $\begin{pmatrix} 3 \\ 10 \\ 15 \end{pmatrix}$ c $3x + 10y + 15z = 21$

24 a $r = \begin{pmatrix} 1/2 \\ -2 \\ 4/3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

b Yes $\left(\text{at}\left(\frac{11}{6}, 0, 0\right)\right)$

c $61.0°$

25 a $\frac{x-3}{3} = \frac{y-1}{-1} = \frac{z+4}{-1}$ b $(0, 2, -3)$

c $(-3, 3, -2)$ e $3\sqrt{2}$

26 a $15\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ b $3.74\,\text{m s}^{-1}$ c No

27 a $\begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$ b $30\,\text{km}$

28 355

29 c $\frac{x-2}{3} = \frac{y-2}{7} = z - 3$

30 b $\boldsymbol{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} = -5$ c $\sqrt{30}$ d 5

31 5

32 a $51.7°$ b $(1, 1, -1)$ d 91.2

33 a $\frac{\sin 3\alpha - 1}{2}$ b $\frac{\pi}{6}$ c 0; Parallel

34 b 1

c $k = -1, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

35 a $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ b $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c i $\frac{\sqrt{3}}{2},$ ii $-x + y + z = 3$

d $\left(\frac{1}{3}, \frac{5}{3}, \frac{5}{3}\right)$

36 a $(4, 1, -2)$ c $(1, 1, 2)$ d $\frac{5\sqrt{26}}{2}$

37 $t = \frac{1}{3}, d = \sqrt{\frac{14}{3}}$

38 8

39 a $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ b $3x + y = 11$

40 b $(5, -7, 6), (-1, 5, -6)$

c $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

[second possible solution]:

$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

41 a $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$ b $\frac{\sqrt{29}}{3}$

c $\left(\frac{65}{9}, -\frac{10}{9}, \frac{17}{9}\right)$

42 a $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ b $|\mathbf{c}|\cos\theta$

d $\frac{1}{3}$ e $\frac{2}{\sqrt{99}}$

f B $\left(h = \frac{2}{\sqrt{203}}\right)$

43 c $(4, -1, 3)$ d $3x - 2y + z = 10$

e $\mathbf{r} = \begin{pmatrix}1\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\1\\-1\end{pmatrix}$

44 b $\frac{7\sqrt{6}}{18}$ c $9\sqrt{5}$ d $\frac{7\sqrt{30}}{2}$

45 b 1

c $-\frac{8}{5}$

46 a $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$ d $\frac{44}{\sqrt{26}}$

47 11:52

Chapter 9 Prior Knowledge

1 a $\frac{19}{39}$ b $\frac{1}{3}$

2 a 0.4 b 2.5

3 a 2 b $e^{\frac{3}{2}}$

Exercise 9A

1 a 0.243 b 0.913

2 a 0.462 b 0.226

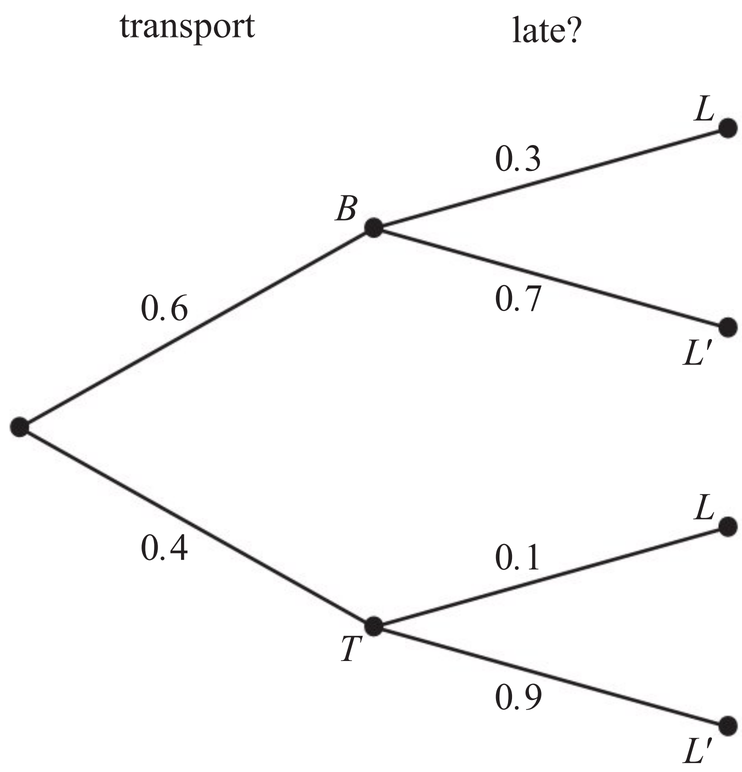
3 a 0.667 b 0.875

4 a 0.136 b 0.238

5 a 0.182 b 0.205

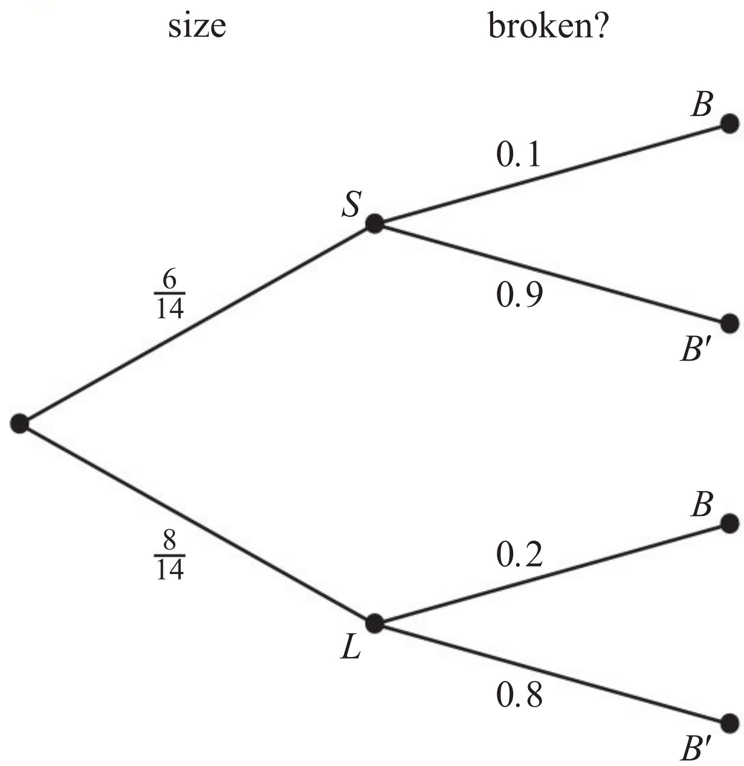
6 a 0.0714 b 0.286

7 a



b 0.78 c 0.538

8 a



b 0.157 c 0.273

9 a 0.16 b 0.5

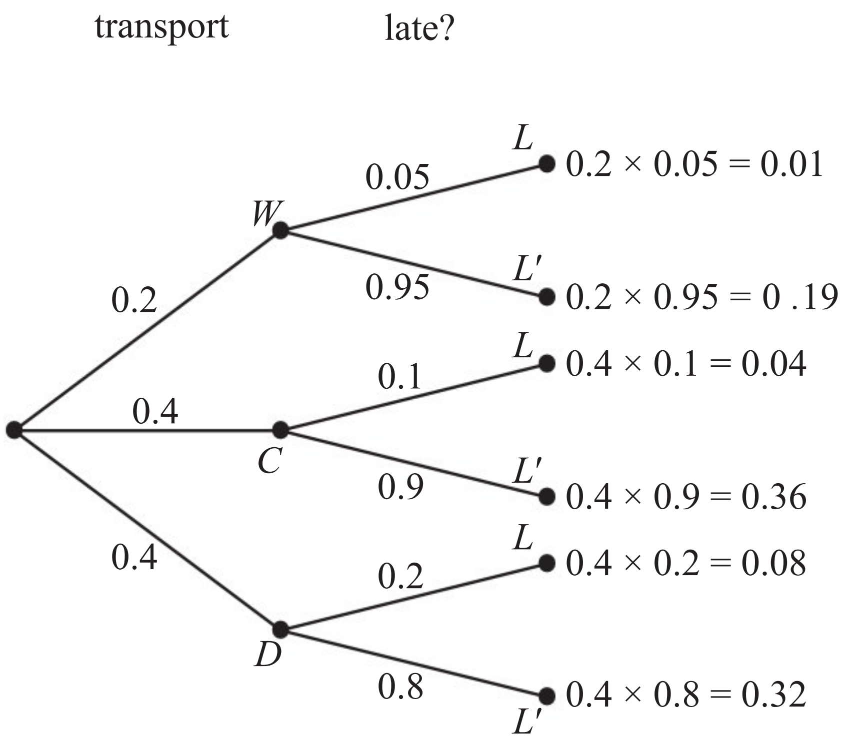
10 a 0.523 b $\frac{8}{17}$ c 0.5

11 a 0.3 b 0.3

12 a 0.87 b 0.724

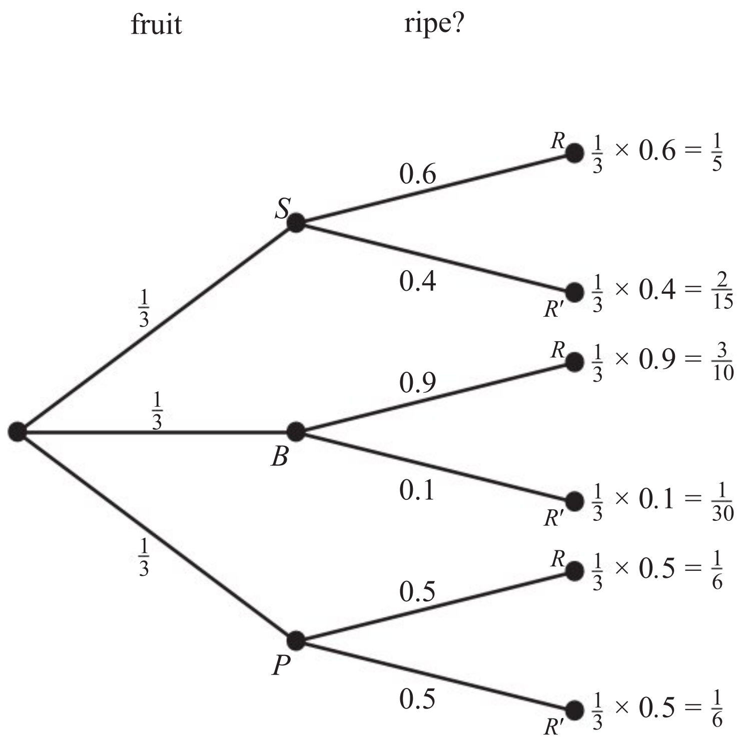
13 a 0.914 b 0.482

14 a



b 0.13 c 0.0769

15 a



b 0.667 c 0.3

16 0.471

17 0.451

18 $\frac{2}{7}$

19 $\frac{3}{7}$

20 2.86%

21 27.7%

22 0.460

23 a $\frac{m(m-1)+n(n-1)}{(m+n)\ (m+n-1)}$ b 0.435

Exercise 9B

1 a 1 b 1

2 a 2.5 b 14.5

3 a 1 b 1

4 a 0.388 b 0.595

5 a $E(Y)=11,\ Var(Y)=28$
b $E(Y)=47,\ Var(Y)=80$

6 a $E(Y)=1,\ Var(Y)=28$
b $E(Y)=41,\ Var(Y)=80$

7 a $E(Y)=4,\ Var(Y)=\frac{2}{9}$
b $E(Y)=4,\ Var(Y)=\frac{3}{5}$

8 a $E(Y)=-30.5,\ Var(Y)=33.3$
b $E(Y)=-55,\ Var(Y)=180$

9 a $E(Y)=4.4,\ Var(Y)=0.2$
b $E(Y)=2.8,\ Var(Y)=0.3$

10 a $E(Y)=24,\ Var(Y)=18$
b $E(Y)=30,\ Var(Y)=12$

11 a $E(Y)=2,\ Var(Y)=\frac{1}{6}$
b $E(Y)=3,\ Var(Y)=\frac{2}{5}$

12 a 0.3 b 1.9 c 2.69

13 a 0.4 b 1.17

14 a 4.2, 4.96 b 13.6, 44.64

15 a $\frac{8}{15}$ c 13.8

16 a $\frac{25}{8}$ b 34.25, 85.9

17 a 4.1, 3.09
b

y	7	10	16	25
$P(Y=y)$	0.2	0.3	0.4	0.1

18 a 6, 4
b

w	−3	−1	1	3
$P(W=w)$	0.1	0.2	0.3	0.4

19 $E(B)=3.1,\ Var(B)=0.3$

20 $E(V)=-6,\ Var(V)=2.56$

21 **a**

y	0	1	2	3	4
$P(Y = y)$	0.1	0.2	0.2	0.3	0.2

b $E(Y) = 2.3,\ Var(Y) = 1.61$

c $E(X) = 233,\ Var(Y) = 16100$

22 **a**

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b $\frac{3}{2}$

23 Mean 3.67, s.d. 0.972

24 **b** $a + b = 0.3$ **c** 1

25 0.85

26 **a** $5, \frac{25}{6}$ **b** $50 - c, \frac{2500}{6}$ **c** 50 cents

27 14

28 1.5

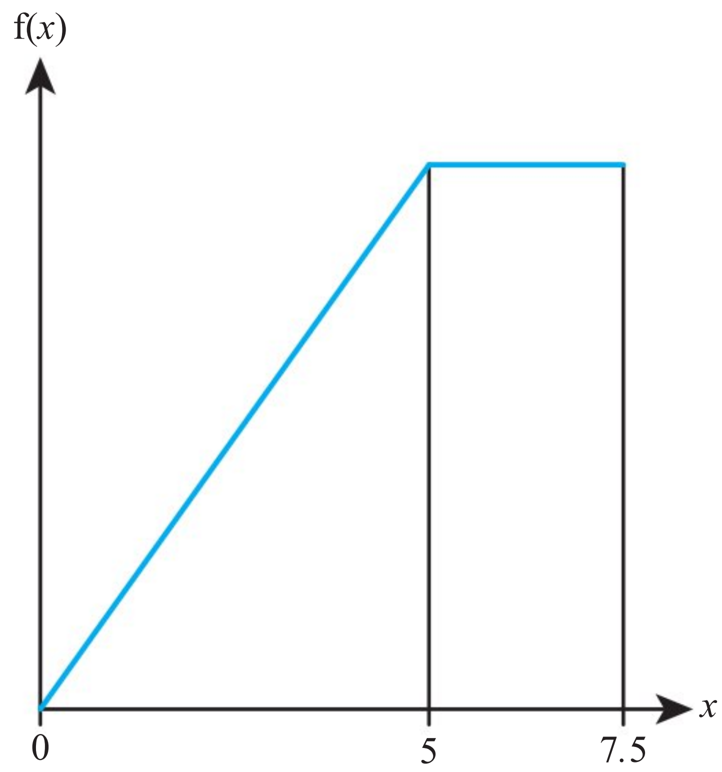
Exercise 9C

1 **a** $\frac{4}{65}$, 0.355 **b** $\frac{3}{14}$, 0.599

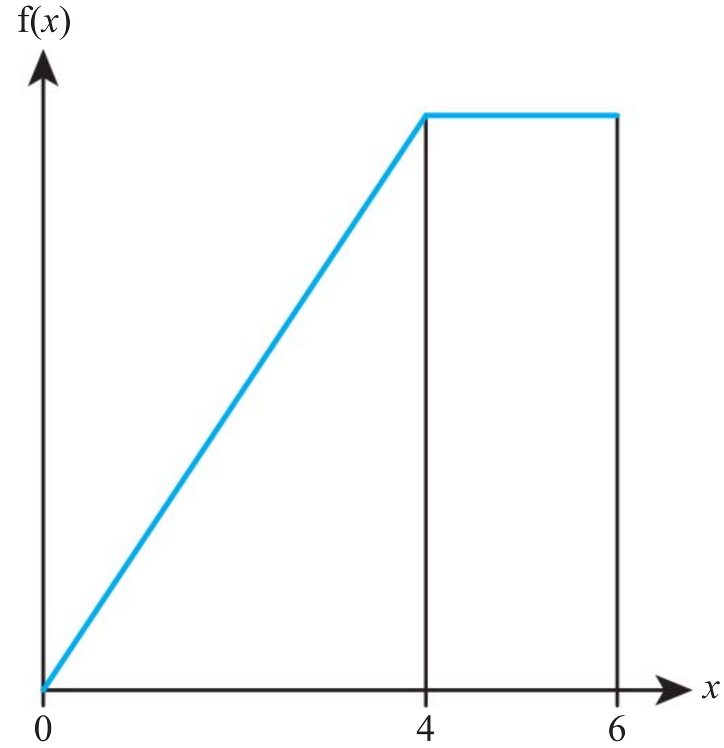
2 **a** $\frac{11}{60}, \frac{1}{3}$ **b** $\frac{1}{12}, \frac{1}{6}$

3 **a** $\sqrt{2}, \frac{1}{16}$ **b** $\sqrt[3]{\frac{3}{7}}, \frac{19}{56}$

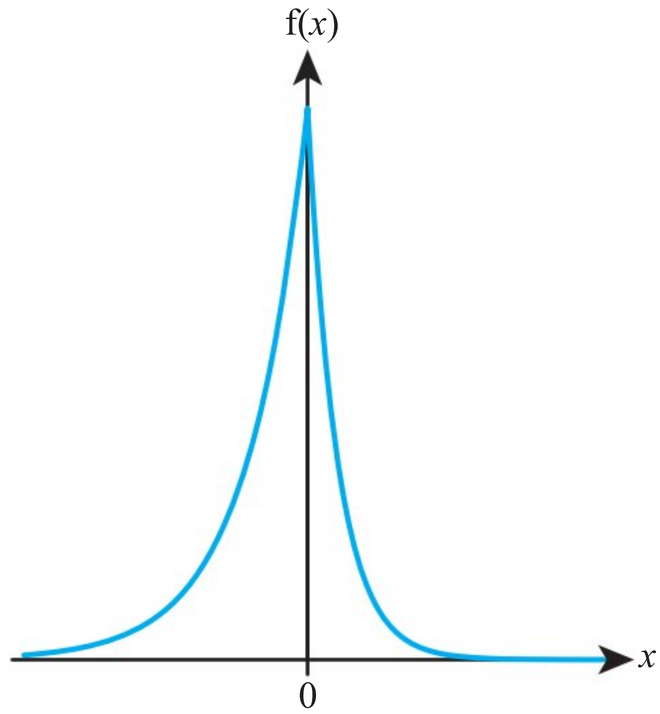
4 **a** 0.62



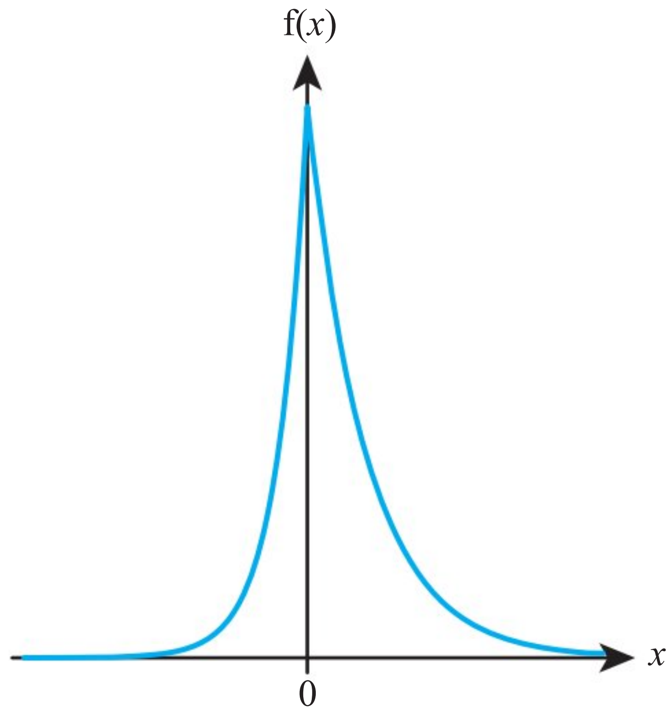
b 0.469



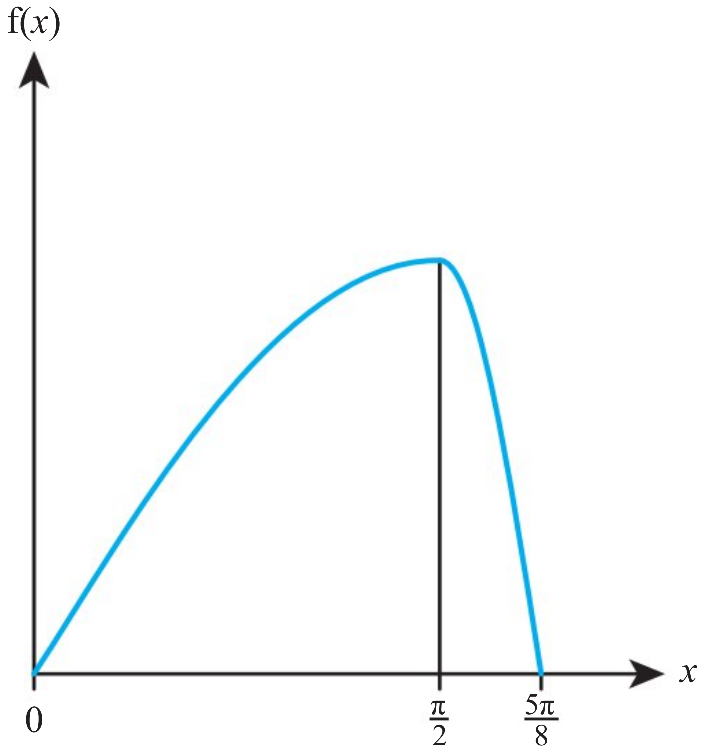
5 **a** 0.583



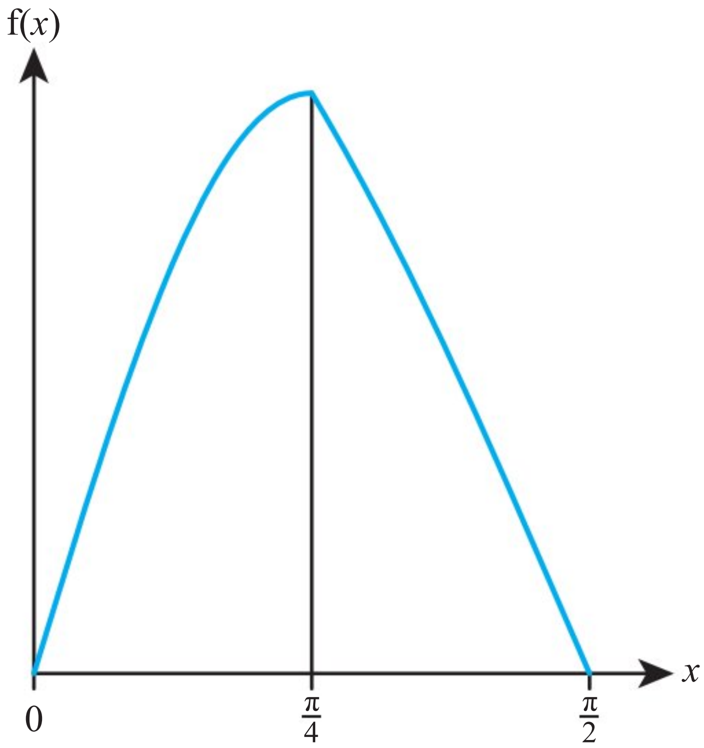
b 0.865



6 a 0.449



b 0.729



7 a 1 **b** 8

8 a 2.67 **b** 0.167

9 a $0, \pi$ **b** $\frac{\pi}{2}, \frac{3\pi}{2}$

10 a 5.66 **b** 4.47

11 a $\frac{1}{3}\ln 2$ **b** $\frac{1}{2}\ln 2$

12 a 0.572 **b** $\sqrt{e}$

13 a 1.19 **b** 0.742

14 a 2.73 **b** 3.82

15 a 4 **b** 5

16 a $\frac{16}{3}, \frac{32}{9}$ **b** $\frac{13}{3}, \frac{11}{9}$

17 a 0.571, 0.142 **b** 0.386, 0.0391

18 a $\frac{12}{5}, \frac{16}{25}$ **b** $\frac{3}{20}, \frac{1}{400}$

19 a 4.79, 3.08 **b** 3.83, 1.97

20 a $5, \frac{25}{6}$ **b** $4, \frac{8}{3}$

21 a 1.14, 0.197 **b** 0.750, 0.111

22 a 0.172 **b** 0.467

23 a $\frac{11}{32}$ **b** $\frac{20}{27}$

24 a 20.2 **b** 0.129 **c** 3.97

25 a $\frac{1}{9}$ **b** $\frac{9}{4}$

26 0.293

27 a $\frac{3}{4}$ **b** $\sqrt{2}$

28 a 3.87 **b** 2.25 **c** 3.63 **d** 2.47

29 a 0 **b** 0.467

30 a 0.821 **b** 1 **c** 0.709

30 a $\ln 3$ **b** $\sqrt{3}$

32 a $\frac{1}{\sqrt{3}}$ **b** $\frac{8}{15}$

33 a 22 minutes **b** 8

34 a i 0.237 **ii** 0.881

b 4.63 minutes

35 b $\frac{k}{\sqrt{2}}$ **c** $0.9k$

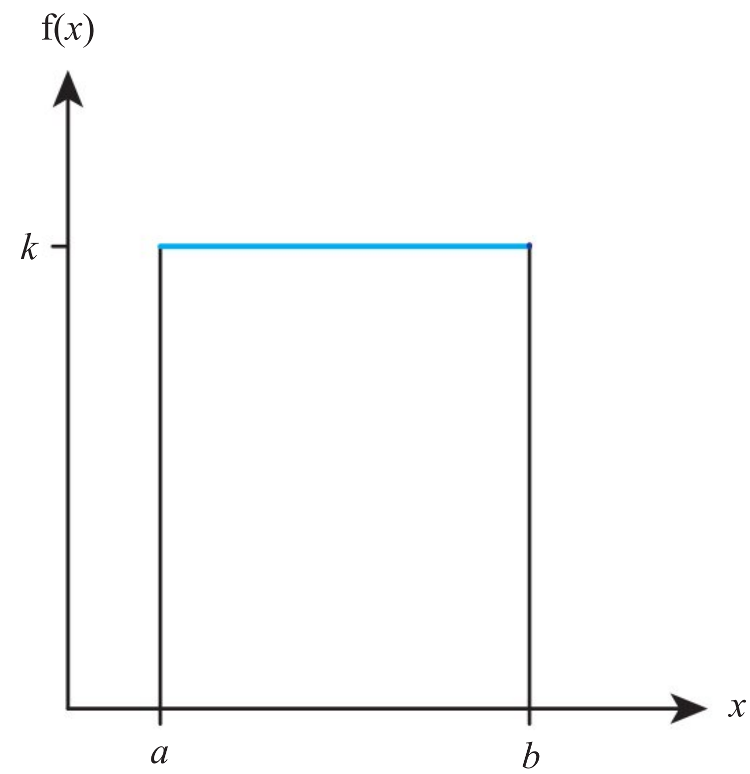
36 a −0.418 **b** 0.0820

37 a 2.95 **b** 0.836

38 0.833

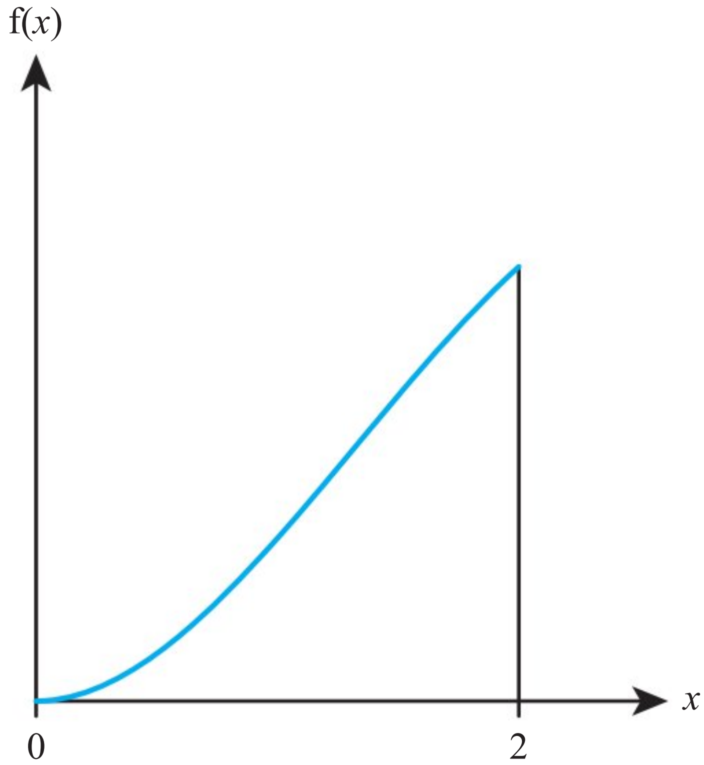
39 a $\frac{1}{100}$ **b** $\frac{1}{8}$ **c** 15.5 **d** 16.7

40 a



b $\frac{a+b}{2}$

41 a



b 2

c $\frac{36}{25}$

42 b $\frac{5}{\sqrt{3}}$

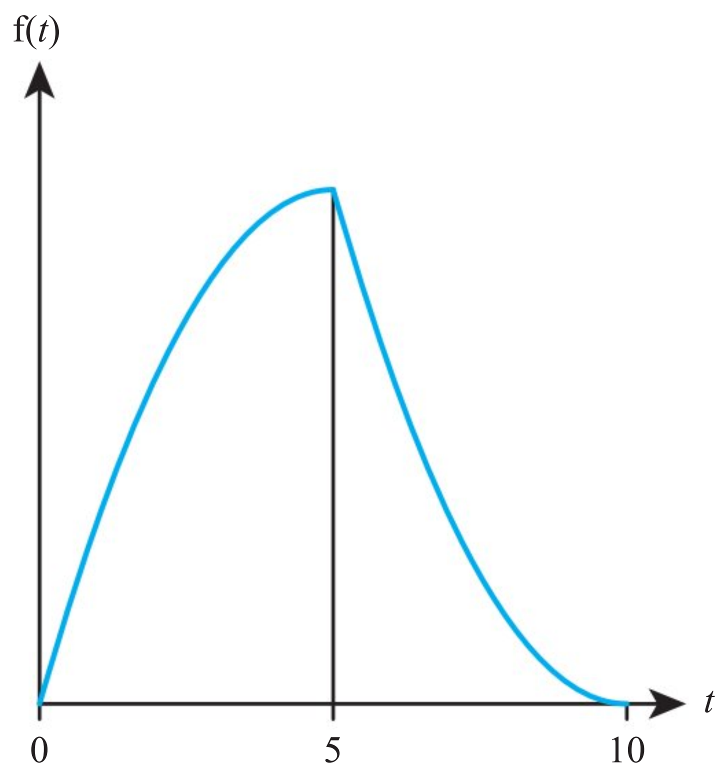
43 0.984

44 a 0.901

b 3.50

c $\frac{7}{2}$

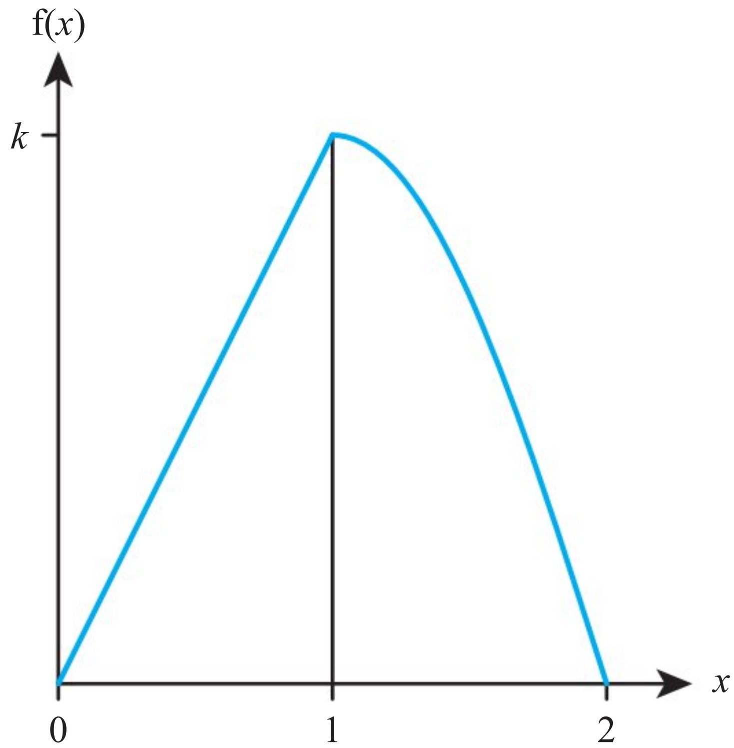
45 a



b $\frac{2}{3}$

46 a $\frac{2\pi}{\pi+4}$

b



c 1.07

Chapter 9 Mixed Practice

1 a $\mathrm{E}(X) = 3.2,\ \mathrm{Var}(X) = 1.56$

b $\mathrm{E}(Y) = -7.6,\ \mathrm{Var}(Y) = 14.04$

2 a 0.1

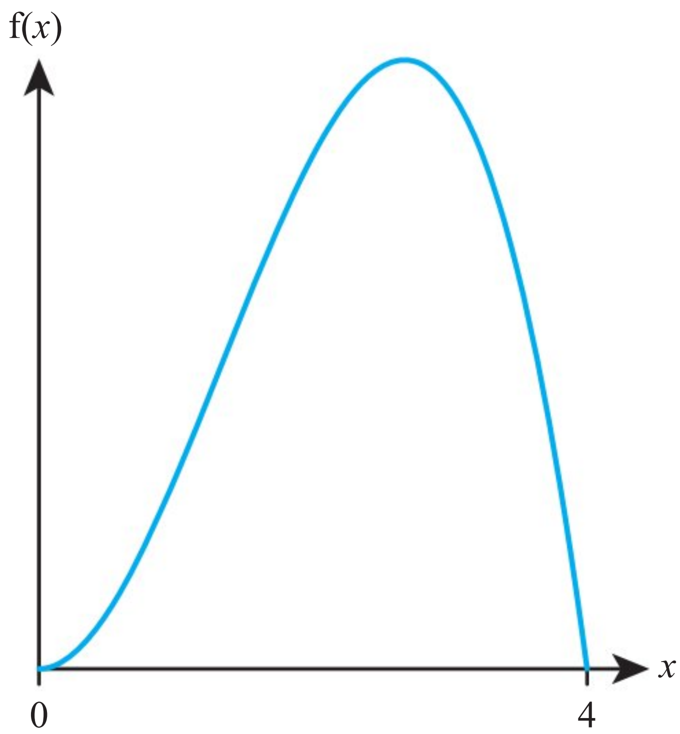
b $\mathrm{E}(V) = 4.2,\ \mathrm{SD}(V) = 2.14$

c $\mathrm{E}(W) = 5.8,\ \mathrm{SD}(W) = 2.14$

3 b $\frac{20}{27}$

c 3

4 a



b $\frac{8}{3}$ c 2.46

5 a 1.87, 0.379 b 6.47, 6.06

6 0.667

7 a

x	1	2	3	4
P($X = x$)	$\frac{1}{13}$	$\frac{5}{26}$	$\frac{4}{13}$	$\frac{11}{26}$

b $\frac{40}{13}$ d 23

8 a

h	0	1	2	3
P($H = h$)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b 1.5 c No d 6.75

9 1.61

10 a 4 b $a = 0.2$, $b = 0.3$

11 $a = 0.12$, $b = 0.26$

12 a 0.48 b $\frac{7}{8}$

13 a 0.6 b 0.538

14 $\frac{6}{23}$

15 a 0.0355 b 0.567

c 3 d $1.50

16 0.288

17 a 20, 20 b 0.156

c 0.132 d Student's

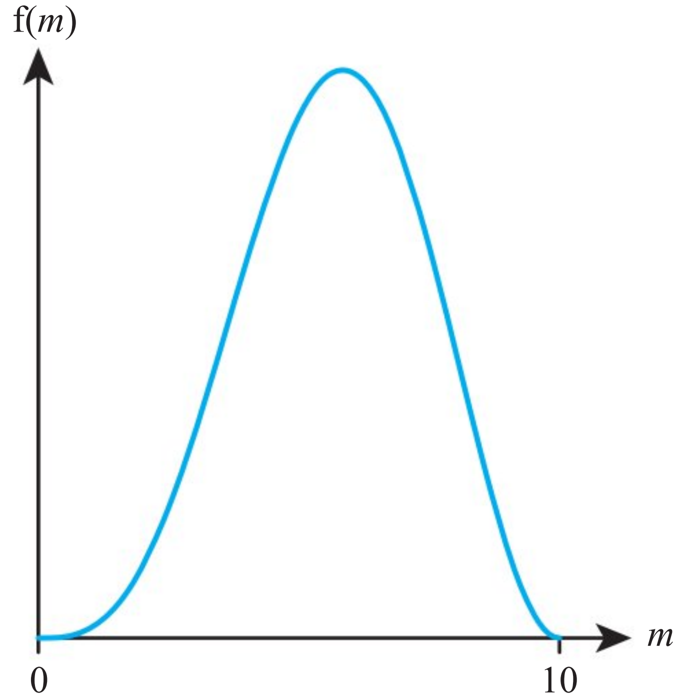
18 a 1.018 b 23 months c 0.269

d i 0.619 ii 0.919

19 $\frac{6}{19}$

20 a $\frac{1}{25}$

b

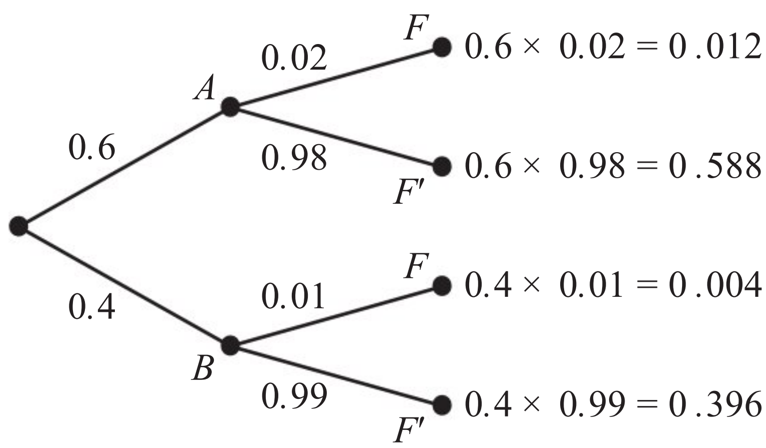


c 0.436 d 5.65, 1.69 e 4%

21 0.782

22 $\frac{11}{32}$

23 a i



ii 0.016 iii 0.75

b i $\frac{12}{35}$

ii

x	0	1	2	3
P($X = x$)	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

iii $\frac{9}{7}$

24 a $\frac{56}{45}$

b $\frac{2\sqrt{6}}{3}$

c $\sqrt{8 - 2\sqrt{10}}$

25 a i 0.407 ii 0.275

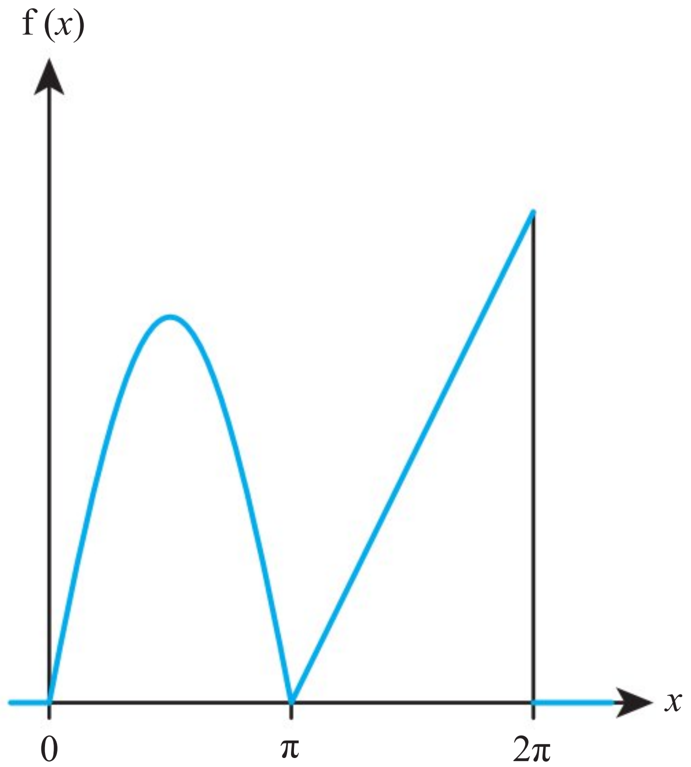
b 0.0676

c 0.0340

26 0.00258

27 b 2 c $\frac{1}{2}$

28 a



b $\frac{1}{2}$ d π e 3.40

f 3.87 g 0.375 h $\frac{1}{3}$

Chapter 10 Prior Knowledge

1 a $14(2x+1)^6$

b $x^2(3\ln x+1)$

c $\dfrac{\cos x-\sin x}{e^x}$

2 $(2+8x+4x^2)e^{2x}$

3 $(-2, 18)$

4 $\frac{3}{2}\ln x+\frac{10}{3}x^{\frac{3}{2}}+c$

5 $\frac{1}{5}$

6 $5+5\sec x-2\sec^2 x$

7 $\dfrac{1}{x-1}+\dfrac{1}{x+2}$

Exercise 10A

1 a -2 b 3

2 a 1.5 b 0.5

3 a 6 b -2

4 a 3 b 5

5 a $\frac{1}{2}\ln\frac{1}{2}$ b $\ln 4$

6 a e b $\frac{1}{2}e^4$

7 a $a=2, b=-1$ b $a=3, b=-2$

8 a $a=5, b=4$ b $a=11, b=6$

9 a $a=2, b=2$ b $a=3, b=3$

10 a $a=1, b=1$ b $a=e^2, b=-e^2$

11 a $a=\frac{1}{e}, b=0$ b $a=4e^{-2}, b=-e^{-2}$

12 a $a=2, b=1$ b $a=4, b=-3$

13 a 3 b 2

14 a 1 b 1.5

15 a 0 b 0

16 a 0.5 b 1

17 a 0.5 b -4

18 a 1 b -1

19 a Divergent b Divergent

20 a Convergent b Convergent

21 a Divergent b Divergent

22 a Convergent b Convergent

23 a 2 b 0

24 a $3x^2$ b $4x^3$

25 a $2x$ b 2

26 a $2x+2$ b $3x^2+1$

27 a $60x^2$ b $120x^3$

28 a $24x+12$ b $180x^2-48x$

29 a $\frac{2}{x^3}$ b $8e^{2x}$

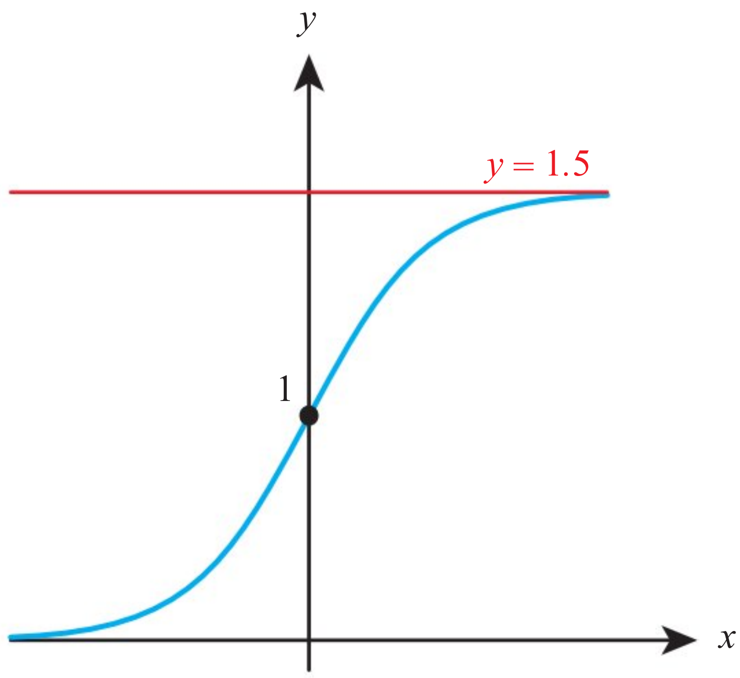
30 a $-\cos x$ b $\sin x$

32 a i 1.5

ii 0

iii 1

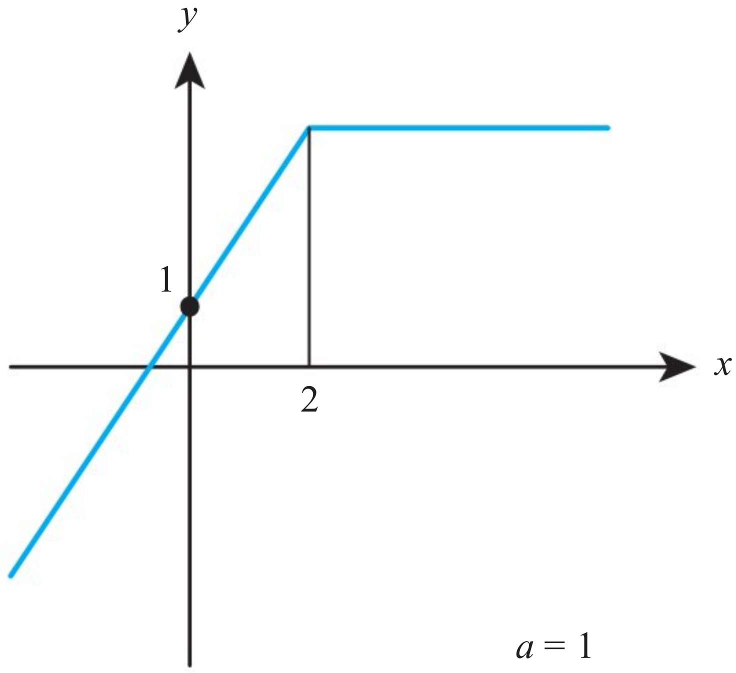
c

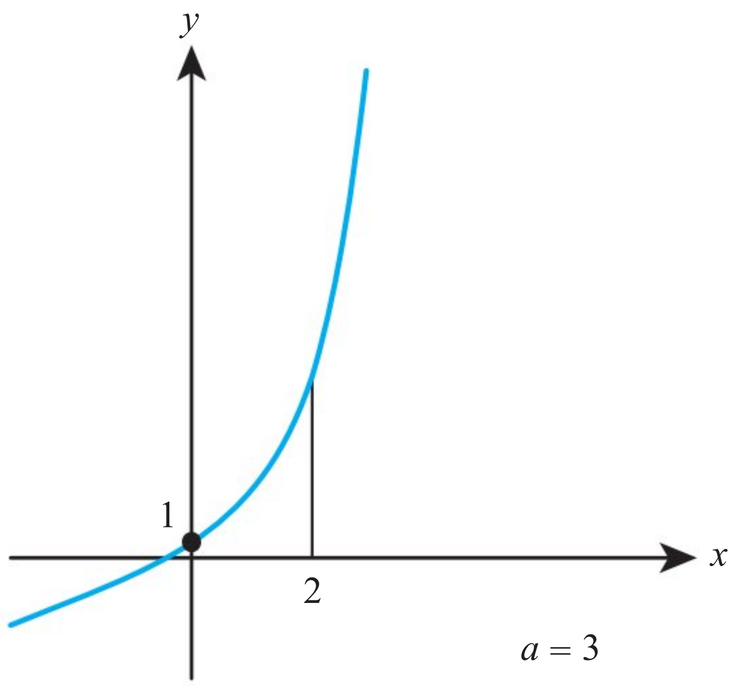


33 $\frac{\ln 2}{2}$

34 a Divergent b $a = 1$ or 3

c





35 $a = -0.5$, $b = -1$

Exercise 10B

1 a 4 b 2

2 a –2 b 2.5

3 a 1 b –2

4 a 1 b $\frac{1}{2}$

5 a 2 b 4

6 a 0 b 0

7 a ∞ b ∞

8 a ∞ b ∞

9 1

10 0

11 $\frac{1}{2}$

12 a –1 b 0

13 a ∞ b $\frac{1}{4}$

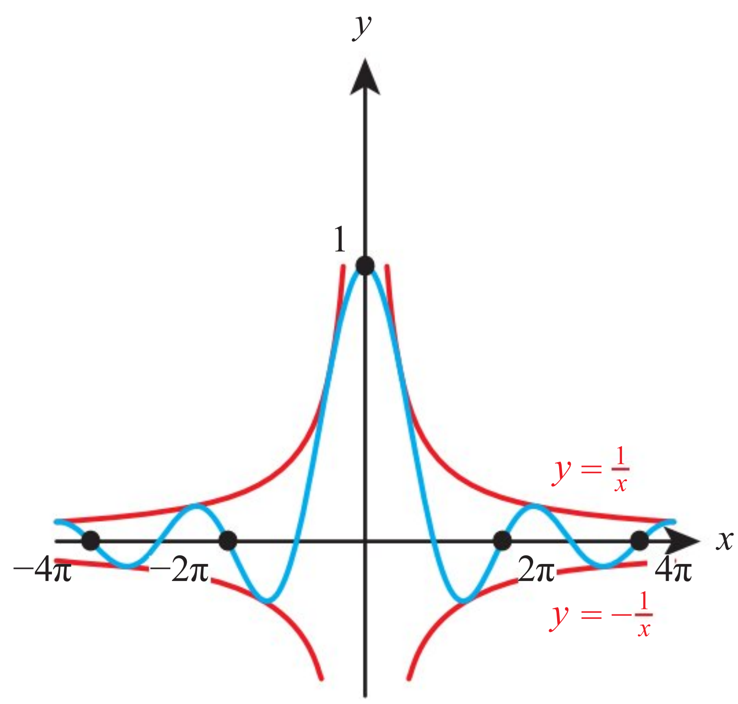
14 25

15 1

16 $\frac{1}{4}$

17 a 1 b 0

c



19 0

20 0

21 $\frac{1}{2}$

Exercise 10C

1 a $\frac{3}{4}$ b –6

2 a –1 b $-\frac{e}{2}$

3 a $-e^2$ b $-\frac{e}{2}$

4 a -3 b $-\frac{1}{4}$

5 a $-\frac{y(2x+y)}{x(x+2y)}$ b $-\frac{y}{x}$

6 a $\frac{x^2y^2-x^2y+y^3}{xy^2-x^3}$

b $\frac{y}{2x^2y+4xy^2+x+2y^3}$

7 a $-\frac{y^2}{x^2}$ b $\frac{2x^2+4xy-1}{x+2y+2}$

8 a $\frac{y^2(1-e^{(x+y)})}{y^2e^{(x+y)}+1}$ b $=-\frac{xe^{(x^2)}}{ye^{(y^2)}}$

9 a $\frac{1-y\cos(xy)}{x\cos(xy)-1}$ b $-\frac{\sin(x+y)}{\sin(x+y)+1}$

10 b $-\frac{1}{2}$ c $y=2x$

11 a $(0, 1)$ b $y=x+1$

12 a $-\ln 2$ b $-\frac{1}{2\sqrt{e}}$

13 $y=2x-5$

14 $y=4-x$

15 b $2-\sqrt{3}$

16 $y=1-1.5x$ and $y=-1-1.5x$

17 a $(0,0),(0,1),(0,-1)$

b -1, 0.5, 0.5 respectively.

18 a $(1,0)$, $(-1,0)$ b $x=1$, $x=-1$

19 a 0 or 5

b $y=\frac{2x}{5}-\frac{2}{5}$ or $y=\frac{23x}{5}+\frac{2}{5}$

20 $\frac{y\sin y}{ye^y-xy\cos y-1}$

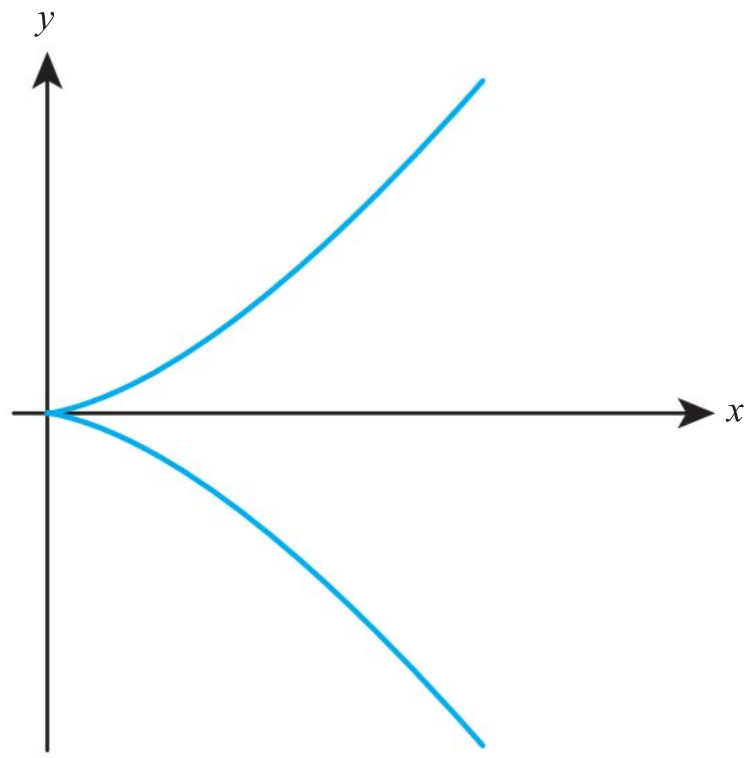
21 $\frac{\sin x+x\cos x}{\sin y+y\cos y}$

22 $-\frac{9}{y^3}$

23 $\left(-\sqrt{2},\frac{1}{\sqrt{2}}\right)$ and $\left(\sqrt{2},-\frac{1}{\sqrt{2}}\right)$

24 $\left(\sqrt[3]{\frac{8}{3}},\sqrt[3]{\frac{8}{3}}\right)$ and $(2,-2)$

25 a



c $(1,-1)$

Exercise 10D

1 a -3 b 4

2 a 12 b 7e

3 a $-\frac{3}{4}$ per second b -4 per second

4 a -2 per hour b $-\frac{3}{2}$ per hour

5 -2

6 -21

7 $\frac{15}{16}$

8 $20\,\text{cm}^2\,\text{s}^{-1}$

9 $22.6\,\text{mm}^2$ per day

10 $1.92\,\text{cm}\,\text{s}^{-1}$

11 a $10\,\text{cm}^2\,\text{s}^{-1}$

b $0.8\,\text{cm}\,\text{s}^{-1}$

12 $\frac{5}{9\pi}\,\text{cm}\,\text{s}^{-1}$

13 7.65 cm

14 $2.68\,\text{m}\,\text{s}^{-1}$

15 Increasing

16 $\frac{0.2}{\sqrt{5}}\approx 0.0894\,\text{m}\,\text{s}^{-1}$

Exercise 10E

1 a 0 b 8

2 a -6 b -7

3 a $-e^{-1}$ b $-2e^{-2}$

4 Max: 2 Min: -0.25

5 Max: $\frac{1}{e}$ Min: $\frac{2}{e^2}$

6 4.5

7 0.25

8 $2\sqrt{5}$

9 a $x(10-2x)^2$ $0 \leqslant x \leqslant 5$

b $\frac{5}{3}$ c $\frac{2000}{27}$ cm³ d 0

10 96 cm^2

11 a $0.654\text{ cm}^3\text{ s}^{-1}$ b 43.2 s

12 1.12

13 0.254

15 a $2\sqrt{2}$ m b $5\sqrt{5}$ m

Exercise 10F

1 a 7 b $\frac{10}{3}$

2 a $2-8\sqrt{3}$ b $-7\sqrt{2}$

3 a $324\ln 3$ b $-\frac{3}{64}\ln 2$

4 a $\frac{1}{2\ln 3}$ b $\frac{1}{\ln 5}$

5 a $\frac{3\sqrt{2}-4\sqrt{3}}{2}$ b $\frac{19}{5}$

6 a $2\tan x + 3\sec x + c$

b $-5\cot x - 2\operatorname{cosec} x + c$

7 a $2\tan x + 3\sec x + c$

b $-3\cot x + \operatorname{cosec} x + c$

8 a $\frac{2^x}{\ln 2} + c$ b $\frac{3^x}{\ln 3} + c$

9 a $3\arcsin x + c$ b $4\arctan x + c$

10 a $2\arcsin x + x + c$ b $2x - 3\arcsin x + c$

11 a $-4\cot(2x-1) +$ b $3\tan(3x+1) + c$

12 a $-15\cot\left(\frac{x}{3}\right) + c$ b $12\tan\left(\frac{x}{4}\right) + c$

13 a $\frac{3^{2x}}{2\ln 3} + c$ b $\frac{2^{5x}}{5\ln 2} + c$

14 a $\frac{1}{2}\arctan(2x) + c$ b $\frac{1}{4}\arctan(4x) + c$

15 a $\arcsin(3x) + c$ b $2\arcsin(5x) + c$

16 a $\arctan(x+2) + c$ b $\arctan(x-3) + c$

17 a $\frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + c$ b $\frac{1}{4}\arctan\left(\frac{x+2}{4}\right) + c$

18 a $\frac{1}{\sqrt{2}}\arctan\left(\frac{x+3}{\sqrt{2}}\right) + c$

b $\frac{1}{\sqrt{5}}\arctan\left(\frac{x-5}{\sqrt{5}}\right) + c$

19 a $\arcsin\left(\frac{x+2}{4}\right) + c$ b $\arcsin\left(\frac{x-1}{3}\right) + c$

20 a $\arcsin\left(\frac{x-2}{\sqrt{5}}\right) + c$ b $\arcsin\left(\frac{x+1}{\sqrt{3}}\right) + c$

21 a $\ln|x+1| + \ln|x+3| + c$

b $\ln|x+3| + \ln|x-2| + c$

22 a $\ln|x-3| - \ln|x+1| + c$

b $\ln|x+2| - \ln|x+3| + c$

23 a $\ln|2x-1| - \ln|x+1| + c$

b $\ln|3x+1| - \ln|x+1| + c$

24 $y - 4 = 8\left(x - \frac{\pi}{4}\right)$

25 $y - \sqrt{3} = -\frac{1}{8}\left(x - \frac{\pi}{6}\right)$

26 2

27 $\frac{6}{4+x^2}$

28 $\left(\frac{\pi}{4}, 2\right)$ and $\left(\frac{3\pi}{4}, -2\right)$

29 $(\log_3 4, 4)$

30 $\frac{\sqrt{3}}{2}$

31 $\frac{7}{\ln 2}$

32 $\pm\frac{1}{2}$

33 $y = 3\tan(\pi x) + 2$

35 a $\frac{1}{x-2} - \frac{1}{x+1}$ b $\ln\left|\frac{x-2}{x+1}\right| + c$

36 a $\frac{2}{x+2} - \frac{1}{x-2}$ b $\ln\left(\frac{9}{2}\right)$

43 352 ml

44 a $\frac{1}{2\sqrt{x-x^2}}$ b π

45 a $\tan x$ b $\frac{2}{3}\ln 2$

46 $\frac{\pi}{6}$

47 a $\frac{1}{4}\ln\left|\frac{x+1}{x+5}\right| + c$ b $\frac{1}{3}\arctan\left(\frac{x+3}{3}\right) + c$

c $\ln\left(x^2+6x+18\right) - 2\arctan\left(\frac{x+3}{2}\right) + c$

49 a $(2x-2)^2 + 25$

b $\frac{1}{10}\arctan\left(\frac{2x-2}{5}\right) + c$

50 a $\arcsin x + \frac{x}{\sqrt{1-x^2}}$

b $x\arcsin x + \sqrt{1-x^2} + c$

51 1

52 $\ln 2$

Exercise 10G

1 a $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

b $\frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + c$

2 a $\frac{1}{28}(2x+1)^7 - \frac{1}{24}(2x+1)^6 + c$

b $\frac{1}{81}(3x-2)^9 + \frac{1}{36}(3x-2)^8 + c$

3 a $\frac{2}{3}(2+x)^{\frac{3}{2}} - 4(2+x)^{\frac{1}{2}} + c$

b $\frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-2)^{\frac{1}{2}} + c$

4 a $e^x - \ln\left(e^x+1\right) + c$

b $e^x + \frac{1}{2}e^{2x} + \ln\left(e^x-1\right) + c$

5 a $2(1+\sin x) - \frac{1}{2}(1+\sin x)^2 + c$

b $\frac{1}{2}(1+\cos x)^2 - 2(1+\cos x) + c$

6 a $\frac{256}{15}$ b $\frac{10}{21}$

7 a $\ln\left(\frac{16}{9}\right)$ b $\frac{2}{3}\ln\left(\frac{5}{2}\right)$

8 a $\frac{\sqrt{2}}{2}$ b $\frac{\pi}{4}$

9 a $\frac{1}{4}(\pi-2)$ b π

10 $\frac{7}{10}$

11 $\frac{4}{105}(11\sqrt{2}-4)$

12 $6 + \frac{8\sqrt{2}}{3}$

13 $2e^{\sqrt{x}} + c$

14 $\frac{14}{3}$

15 a $\frac{1}{u} - \frac{1}{u+1}$ b $\ln\left(\frac{e^x}{e^x+1}\right) + c$

16 $2 + \ln 2$

17 $\frac{1}{3}(\ln x)^3 + c$

18 $\tan x + \frac{1}{3}\tan^3 x + c$

19 $\operatorname{arcsec}\left(e^x\right) + c$

20 $\arctan\left(\frac{1}{6}\right)$

21 $x - \ln\left|1+e^x\right| + c$

22 $\frac{1}{5}\arcsin\left(\frac{5x}{2\sqrt{2}}\right) + c$

23 $\frac{\pi}{4}$

Hint: If you use technology to sketch $y = \pm\sqrt{1-x^2}$ you might see a familiar shape which helps to explain this answer.

24 a $e^u = \frac{x+\sqrt{x^2+4}}{2}$ b $\ln\left(\frac{1+\sqrt{5}}{2}\right)$

Exercise 10H

1 a $\frac{1}{4}(2x\sin 2x + \cos 2x) + c$

b $\frac{1}{9}(3x\sin 3x + \cos 3x) + c$

2 a $2\left(-x\cos\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\right) + c$

b $3\left(-x\cos\left(\frac{x}{3}\right) + 3\sin\left(\frac{x}{3}\right)\right) + c$

3 a $-\frac{1}{4}e^{-2x}(2x+1) + c$ b $-\frac{1}{9}e^{-3x}(3x+1) + c$

4 a $\frac{1}{4}x^2(2\ln x - 1) + c$ b $\frac{1}{9}x^3(3\ln x - 1) + c$

5 a $-\frac{1}{x}(\ln x + 1) + c$ b $-\frac{1}{4x^2}(2\ln x + 1) + c$

6 a $\frac{2}{9}x\sqrt{x}(3\ln x - 2) + c$ b $2\sqrt{x}(\ln x - 2) + c$

7 a $\frac{1}{27}e^{3x}(9x^2 - 6x + 2) + c$

b $-\frac{1}{4}e^{-2x}(2x^2 + 2x + 1) + c$

8 a $\frac{1}{4}(-2x^2\cos 2x + 2x\sin 2x + \cos 2x) + c$

b $\frac{1}{27}(-9x^2\cos 3x + 6x\sin 3x + 2\cos 3x) + c$

9 a $3\left(x^2\sin\left(\frac{x}{3}\right) + 6x\cos\left(\frac{x}{3}\right) - 18\sin\left(\frac{x}{3}\right)\right) + c$

b $2\left(x^2\sin\left(\frac{x}{2}\right) + 4x\cos\left(\frac{x}{2}\right) - 8\sin\left(\frac{x}{2}\right)\right) + c$

10 $\frac{1}{4}(1+e^2)$

11 $\frac{\pi}{2} - 1$

12 $-\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C$

13 $\frac{5e^6+1}{36}$

15 $8\ln 2 - 4$

16 $-e^{-x}(x^2 + 2x + 2) + c$

17 $\pi^2 - 4$

18 $x(x+1)\ln x - \frac{1}{2}x^2 - x + c$

19 a $\frac{1}{2}\tan 2x + c$

b $\frac{1}{2}x\tan 2x - \frac{1}{4}\ln(\sec 2x) + c$

20 b $\frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + c$

21 a $\tan x$ b $1 - \frac{\sqrt{2}}{4}(2 + \ln 2)$

22 a $J = e^x\sin x - I$

b $\frac{1}{2}e^x(\sin x + \cos x) + c$

23 $x\ln x - x + c$

26 $\frac{1}{13}e^{3x}(3\sin 2x - 2\cos 2x) + c$

27 $\frac{1}{10}e^{-x}(3\sin 3x - \cos 3x) + c$

28 b $6 - 2e$

Exercise 10I

1 a $\frac{26}{3}$ b $\frac{45}{4}$

2 a 1.83   b 0.848

3 a 5.10   b 8.77

4 a 2.48   b 0.527

5 a 1370   b 230

6 a 91.7   b 512

7 a 11.8   b 33.0

8 a 3.14   b 7.07

9 a 101   b 134

10 a 12.6   b 45.7

11 a 3.59   b 0.771

12 a 93.2   b 48.7

13 a $\ln 5$ b $\frac{4}{5}\pi$

14 3

15 2

16 a -3 b 18π

17 a $\frac{32}{3}$ b 107 (3s.f.)

18 a 2.43 b 7.75

19 b 5.25

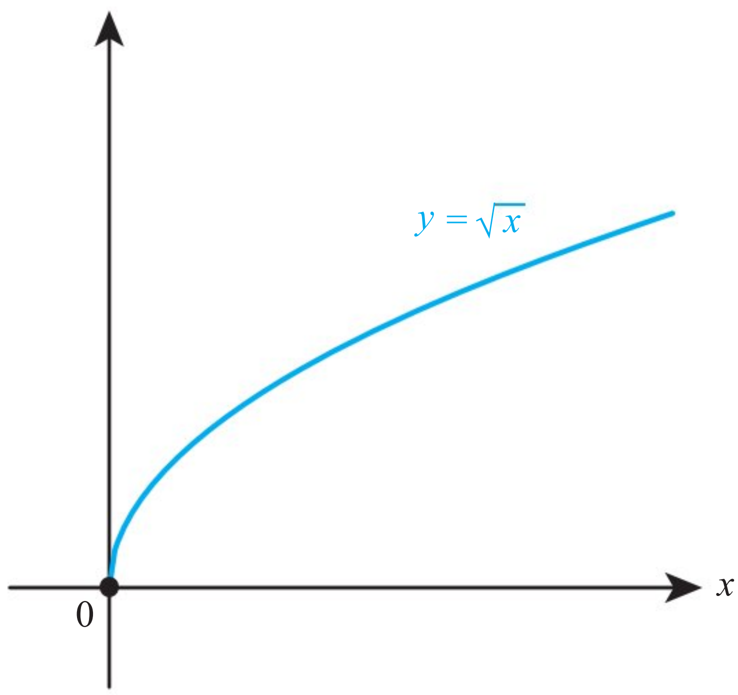
c i 44.1 ii 30.1

20 2π

21 3646

22 $\frac{\pi^2}{2}$

23 a



b 127 c 153

24 a $A(0, 2),\ B(2, 0)$

b 4.39

c 32.7

25 $\frac{20}{3}\pi$

26 $\frac{32}{5}\pi$

27 a $a-1$

28 a $hx+ry=rh$

29 a $x^2+y^2=r^2$

30 a (0, 0) and (1, 1) b $\frac{3\pi}{10}$

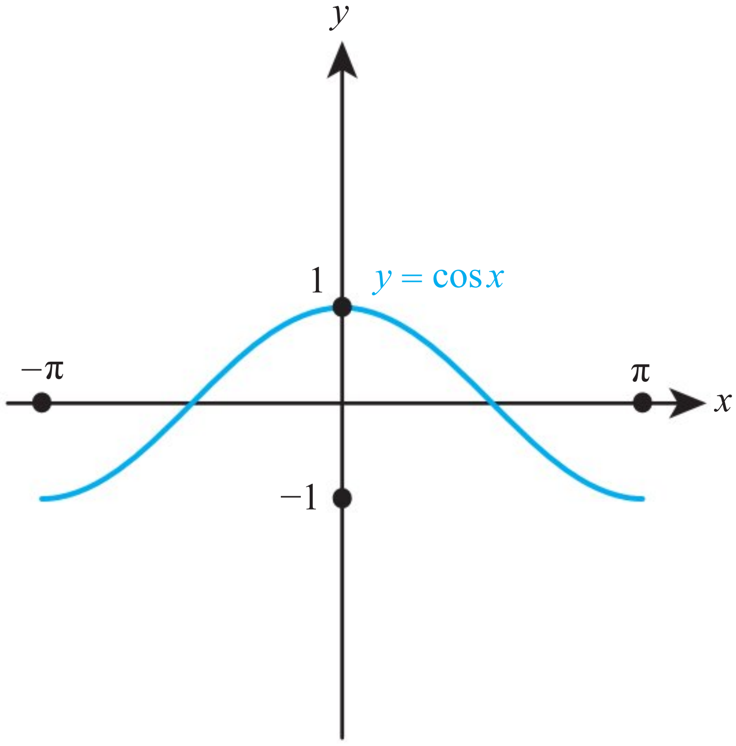
31 $\frac{\pi^2-2\pi}{4}$

32 a (0, 0) and (2, 4) b $\frac{8\pi}{3}$

33 1 : 3

34 a $y=\ln(x-2)$ b $\frac{\pi}{2}\left(e^2-8e-10\right)$

35 a



c $3\pi^2$

36 a $1-\sin\theta$ b $\sin\theta=\sqrt{1-a^2}$

Chapter 10 Mixed Practice

1 $2\sqrt{3}$

2 $\frac{\sqrt{3}}{3}$

3 a $\sec x\left(\sec^2 x+\tan^2 x\right)$

4 $10^x(\ln 10)^3$

5 19.0

6 $\frac{8}{5}$

7 $2\arctan(3x)+c$

8 $259.2\,\text{cm}^3\,\text{s}^{-1}$

9 -16.3

10 181

11 b 10 c $600\,\text{cm}^2$

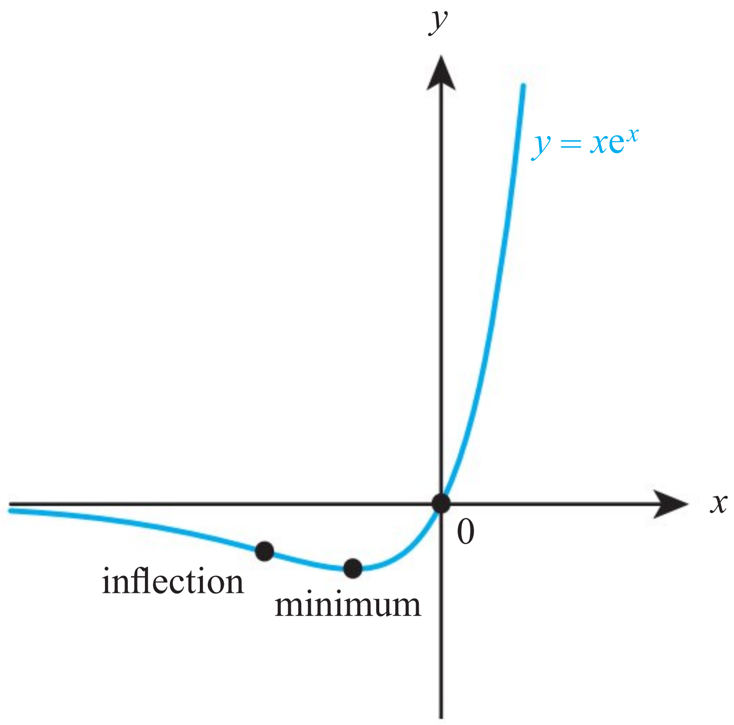
12 a $\frac{1}{x-2}-\frac{2}{x+1}$

13 $y=x$

14 a $e^{3x}+3x\,e^{3x}$ c $\left(-\frac{1}{3},\frac{1}{3e}\right)$

d $\left(-\frac{2}{3},-\frac{2}{3e^2}\right)$

e



19 a $\frac{4}{9}$ b No

20 $\ln 3$

21 $-\frac{1}{2}$

23 a $\ln 11$ b 1.52 c 5.33

24 a (1.73, 0) b 0

25 $\pi\left(e^4-1\right)$

26 $\frac{\pi^2}{4}$

27 π

29 $\frac{1}{16}\left(1+3e^4\right)$

30 $1+e^2$

31 a $-(t+1)e^{-t}+c$ b $1-\frac{2}{e}$

32 $\frac{1}{5}\left(x^2+1\right)^{\frac{5}{2}}-\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}+c$

33 b $4\left(e^2-1\right)$

35 $a=-2e^{-1}$, $b=4e^{-1}$

36 $\frac{1}{4}\pi\left(e^2-1\right)$

37 a $\frac{1}{\sqrt{1-x^2}}$

b $x\arcsin x+\sqrt{1-x^2}+c$

38 a $(x-4)^2+9$

b $\ln\left|x^2-8x+25\right|+5\arctan\left(\frac{x-4}{3}\right)+c$

40 a $\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{xa}{2}\sqrt{1-\frac{x^2}{a^2}}+c$

b $\frac{\sqrt{3}}{2}$ m, $\theta=\frac{\pi}{6}$

41 $\mathrm{In}\left(\frac{4+\sqrt{15}}{2+\sqrt{3}}\right)$

42 $0.243\,\mathrm{m\,s^{-1}}$

43 0.721

44 0

45 -1

47 $\left(3, e^6\right)$

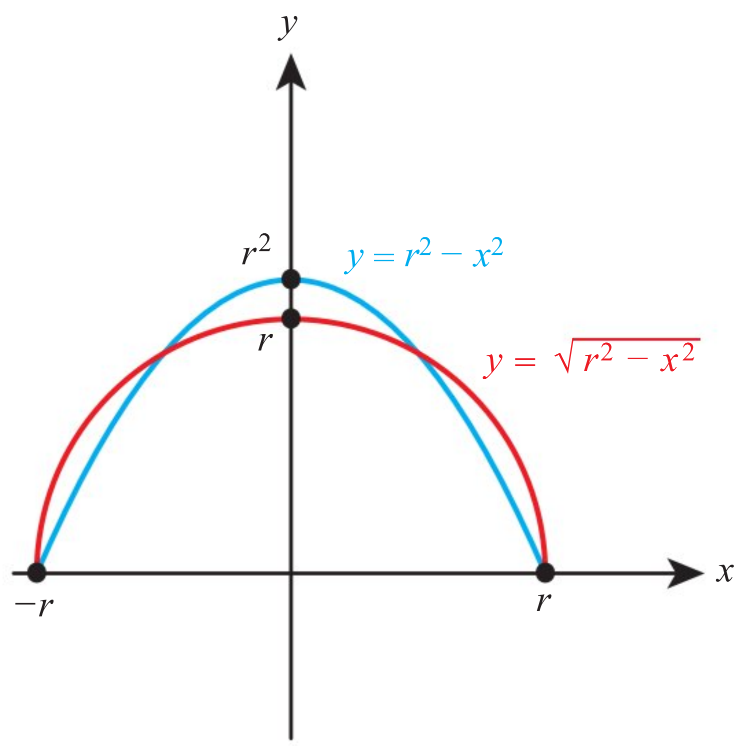
48 a $(-2,4)$, $(2,-4)$

c $(-2,4)$ is a max, $(2,-4)$ is a min

49 b $\frac{\sqrt{3}}{2}$m, $\theta=\frac{\pi}{6}$

50 a $y=-\frac{h}{a-b}(x-a)$

51 a



b $\frac{4}{3}$

52 a $y=0.000545x^3-0.0582x^2+1.69x+10$

b 74.4 l

53 a $\sqrt{1+k^2}$

c $k\arctan k-\frac{1}{2}\ln\left(1+k^2\right)$

54 a $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\frac{1}{9},\frac{1}{10}$

b 2 when n is even, -2 when n is odd

c 18

55 $(-3, 0)$, $(3, 0)$, $(-1, -4)$, $(-1, 4)$

56 a The total area under the curve must be 1.

b $X = \mu + \sigma Z$, so the mean is increased by μ and the variance is multiplied by σ^2.

Link: You learnt about linear transformations of random variables in Section B.

Chapter 11 Prior Knowledge

1 $\frac{\sin x^2}{2} + c$

2 $\frac{3e^x - y}{2y + x}$

3 -8

4 x^2

5 1035

6 90

Exercise 11A

1 a 1st order linear

b 1st order linear

2 a 2nd order linear

b 2nd order linear

3 a 2nd order linear

b 3rd order linear

4 a 1st order non-linear

b 1st order non-linear

5 a 1st order non-linear

b 2nd order non-linear

6 a 2nd order non-linear

b 1st order non-linear

7 a $\frac{dB}{dt} = kB$ b $\frac{dh}{dt} = \frac{k}{h}$

8 a $\frac{d^2s}{dt^2} = \frac{k}{s^2}$ b $\frac{d^2s}{dt^2} = k\sqrt{t}$

9 a $\frac{dI}{dt} = kI(N - I)$ b $\frac{dR}{dt} = \frac{kR(N - R)}{t}$

10 a $y = 1 + \frac{x^3}{3}$ b $y = 3 - \cos x$

11 a $y = +1 - \frac{e^{2x}}{2}$ b $y = 5 - \arctan x$

12 a $y = 1 + x - \ln x$ b $y = \frac{1}{x} + 3x$

13 a 2.49 b 1.4

14 a 2.98 b 0.655

15 a 1.36 b 1.46

16 a 4.56 b 3.82

17 a $y = A + Bx - 2e^{2x}$ b $y = 3 + 2x - 2e^{2x}$

18 a i 0.9 ii 3.8

b i 0.8 ii 3.6

c $y = x^2$, **b ii** is furthest away

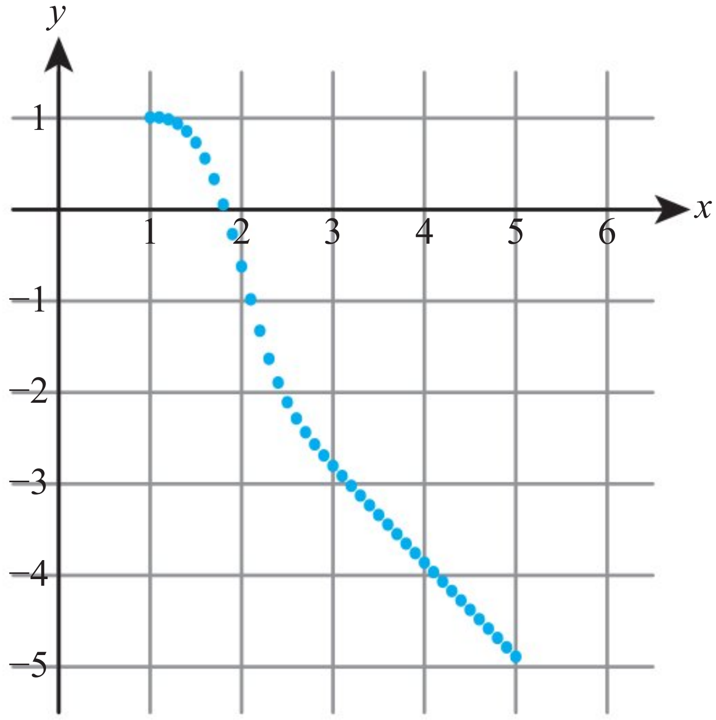
19 a 31.4

b Use a smaller step length

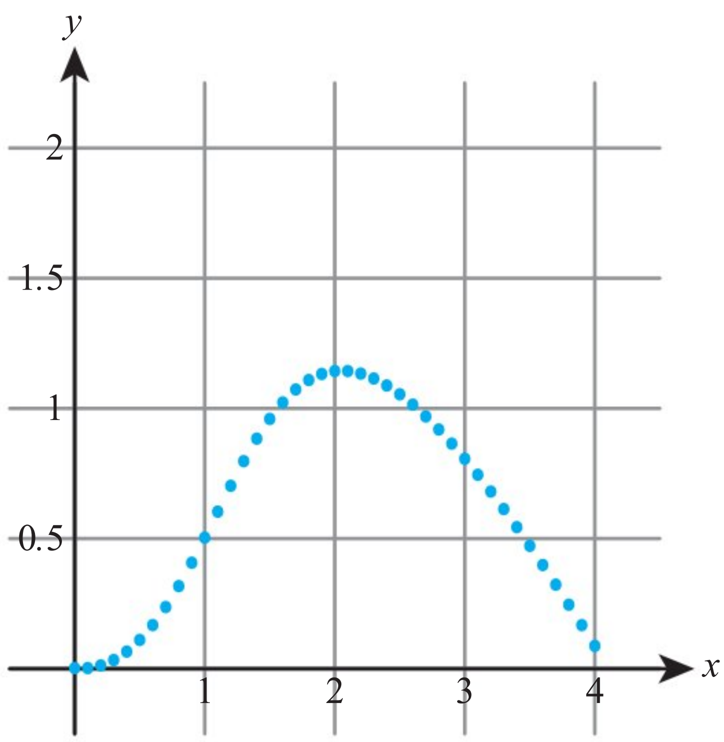
20 183 m

21 177 m

22



23



b 1.1

24 a 1.46 m b 3 seconds

25 a $\frac{dr}{dt} = -0.0328$ b 15 minutes

26 0.889

27 b 3.96

28 (9.46, 3.71)

Exercise 11B

1 a $y = Ae^{2x}$ b $y = Ae^{-x}$

2 a $y = Ae^{x} - 1$ b $y = Ae^{-x} + 1$

3 a $y = -\frac{3}{c + x^3}$ b $y = \sqrt{\frac{2}{c - x^4}}$

4 a $y = Ax$ b $y = \sqrt{x^2 + c}$

5 a $y = cx^2 - x$ b $y = \frac{c}{x} + \frac{x}{2}$

6 a $x\sqrt{c + 2\ln x}$ b $y = \sqrt{\frac{c}{x^2} + \frac{x^2}{2}}$

7 a $y = \frac{x(\ln Ax)^2}{4}$ b $y = x\sqrt{c + 2\ln x}$

8 $y = -\frac{-2}{x^3 + c}$

9 $y = \sqrt[3]{3\sin x + c}$

10 $y = \ln(x^2 + c)$

11 $y = 4e^{\tan x}$

12 $2y^2 = 3x^3 + 18$

13 $y = Ax$

14 $y = -\frac{1}{x^2 + c}$

15 $y = \sqrt[3]{\frac{9}{2}(x^2 + 2)}$

16 a $y = 1 + Ae^{(x+2)^2}$

b $y = 1 + e^{x^2 + 4x}$

18 a $k = \frac{1}{5}$

b $m = 25e^{-\frac{t}{5}}$

c 3.47 seconds

19 a $\frac{1}{4}$

b 24 000

20 b $V = \sqrt{6000t + 90000}$

21 a $v = 100 - 100e^{-0.1t}$ b 40.8 m

22 $y = -\sqrt{9 - 4e^{-2x}}$

23 $y = -\ln(c - e^{x})$

24 $y = \frac{1}{2}\ln(4e^{x} - 3)$

25 $A = 2,\ B = -2$

26 $y = \sqrt{102 - 2\cos x}$

27 a $\sin x + \cos y = 1$

b $\pm\frac{\pi}{2}$

28 $y = Ax^3 - x$

29 a $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} = f\left(\frac{y}{x}\right)$

b $y^2 = 2x^2\ln(Ax)$

30 a $\frac{(2x + y)^2}{4x^2} = 1 + \frac{y}{x} + \frac{1}{4}\left(\frac{y}{x}\right)^2 = f(x)$

b $y = \frac{x}{2}\tan\left(\frac{1}{4}\ln x\right)$

31 $y = x\ln\left(\frac{|Ax|}{(x - y)^2}\right)$

32 a $y = x + \frac{c}{x}$

b $y = x + \frac{4}{x}$

33 $y = x^3 e^{2 - \frac{4}{x}}$

34 $y = 3\tan(3\ln|Ax|)$

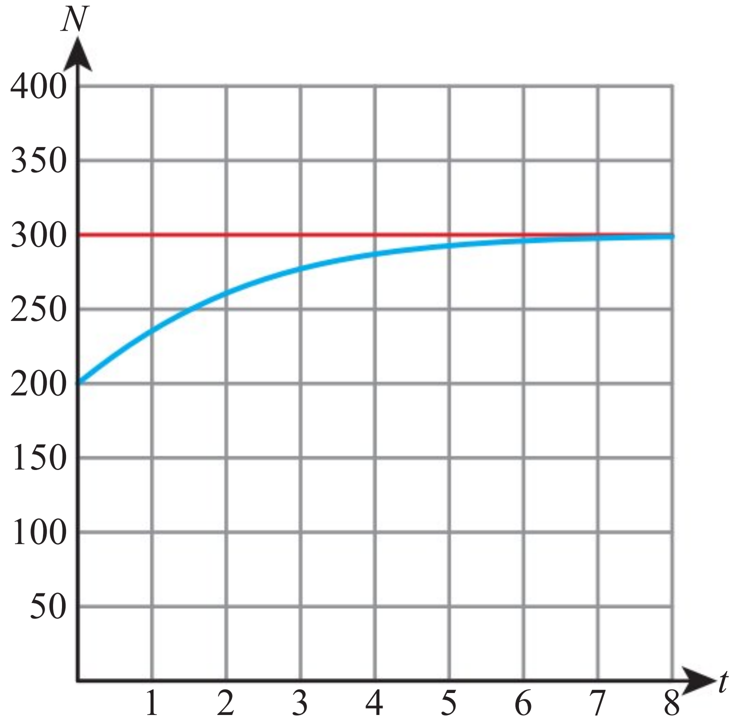
35 $y = \sin(\ln(1 + x^2))$

36 $y = 2(1 - x)$

37 $y = \arcsin(\tan x + c)$

38 a $N = \frac{600e^{\frac{3t}{5}}}{1 + 2e^{\frac{3t}{5}}}$

b Increases towards the limit of 300



40 b $y=\dfrac{1}{1+2e^{\frac{x^2}{2}}}$

41 b $y=Ax-\dfrac{1}{x}+1$

42 $(2x-3y+3)^2=6x-2x^2$

43 a $\dfrac{dv}{du}=\dfrac{4u-v}{2u+v}$

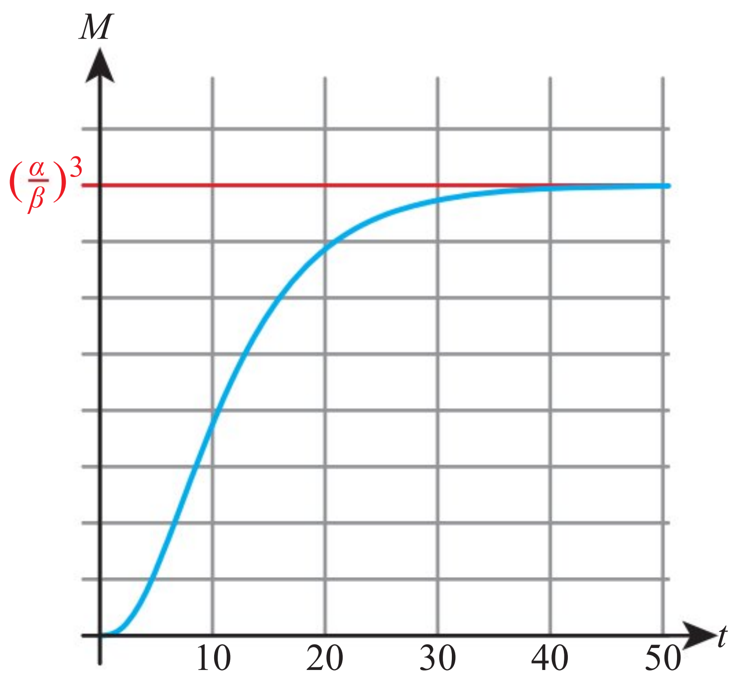
b $(y-x-4)^3(y+4x+1)^2=c$

44 $y=\arctan(x+c)-x$

45 $A=\dfrac{3}{4},B=-\dfrac{1}{2}$

46 b $M=\left(\dfrac{\alpha}{\beta}\right)^3\left(1-Ae^{-\frac{\beta}{3}}\right)^3$ c $\left(\dfrac{\alpha}{\beta}\right)^3$

d



Exercise 11C

1 a $y=ce^{-x}+\dfrac{e^x}{2}$ b $y=ce^x+xe^x$

2 a $y=\dfrac{c}{x}+\dfrac{x^2}{3}$ b $y=\dfrac{c}{x}+\dfrac{x^3}{4}$

3 a $y=cx+x^2$ b $y=\dfrac{c}{x^2}+\dfrac{x^2}{4}$

4 a $y=c\cos x-\cos x\ \ln(\cos^2 x)$

b $y=c\cos x+x\cos x$

5 a e^{3x} b $y=\dfrac{1}{4}e^x+ce^{-3x}$

6 $y=-e^x+ce^{2x}$

7 a e^{x^2} b $y=(x+c)e^{-x^2}+4$

8 $y=(5x+c)e^{-2x^2}+3$

9 b $y=2x^3\ln x+cx^3$

10 a x^2

b $y=3x^2+\dfrac{2}{x^2}$

11 $y=(2x+c)e^{\cos x}$

12 $y=(3x+4)e^{-5x^2}$

13 $y=\dfrac{c}{x}-\dfrac{1}{x^2}$

15 b $y=\left(x+4-\dfrac{\pi}{4}\right)\sec^2 x$

16 a $e^{\sin x}$

b $y=1+ce^{-\sin x}$

17 $y=\dfrac{\ln x+2}{x}$

18 $y=\dfrac{2x^3+3x^2-12x+c}{6(x+2)}$

19 $y=\dfrac{x^2(x^2+2)}{2(x^2+1)}$

20 $y=2+\dfrac{c}{x^2}$

21 $y=\dfrac{3x^2+c}{x-1}$

22 $y=\dfrac{1}{2}\sin x+\left(\dfrac{1}{2}x+c\right)\sec x$

23 $\dfrac{e(e-1)}{4}$

24 $y=(3\sin x+c)\sec^2 x$

25 $y=-\cot x+\sqrt{2}\ \operatorname{cosec} x$

26 $y=x^2\ln(x-3)+cx^2$

27 $y=(x+2)\cos x$

28 $y=\dfrac{x^2+c}{2(x^2-1)}$

29 **a** **i** $5+(v_0-5)\mathrm{e}^{-2t}$ **ii** 5

b **i** $5(t+1)+\dfrac{v_0-5}{t+1}$

30 **a** $2xy^2+2x^2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$

31 **b** $y=\sqrt[3]{\dfrac{1}{x}(\mathrm{e}^x+c)}$

32 **b** $y=-\sqrt{\dfrac{1}{2}x^2-x}$

33 $y=\sqrt{x^2+cx}$

34 **a** $\dfrac{\mathrm{d}z}{\mathrm{d}x}+z\tan x=2\cos^2 x$

35 **b** $z=x^3+cx^2$

c $y=\dfrac{1}{3}x^3+cx^2+d$

36 **a** **i** $P=\dfrac{\alpha}{\beta-\gamma}\left(\mathrm{e}^{-\gamma t}-\mathrm{e}^{-\beta t}\right)$

ii $\dfrac{1}{\beta-\gamma}\ln\left(\dfrac{\beta}{\gamma}\right)$

iii Diagram of growth that rises and falls, with t-value at maximum labelled

b $P=(\alpha t+c)\mathrm{e}^{-\beta t}$

37 **a** $[\mathrm{Bi}]=[\mathrm{Bi}]_0\,\mathrm{e}^{-k_1 t}$

b Rate of change of Po governed by generation from bismuth decay and loss due to Po decay

c $[\mathrm{Po}]=\dfrac{k_1[\mathrm{Bi}]_0}{k_2-k_1}\left(\mathrm{e}^{-k_1 t}-\mathrm{e}^{-k_2 t}\right)$

d $[\mathrm{Pb}]=\dfrac{k_1k_2[\mathrm{Bi}]_0}{k_2-k_1}\left(\dfrac{1}{k_2}\mathrm{e}^{-k_2 t}-\dfrac{1}{k_1}\mathrm{e}^{-k_1 t}\right)+[\mathrm{Bi}]_0$

e $[\mathrm{Bi}]_0$

Exercise 11D

1 **a** $x-\dfrac{x^3}{6}+\dfrac{x^5}{120}$ **b** $1-\dfrac{x^2}{2}+\dfrac{x^4}{24}$

2 **a** $1+x+x^2$ **b** $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}$

3 **a** $1+\dfrac{x}{2}-\dfrac{x^2}{8}$ **b** $x-\dfrac{x^2}{2}+\dfrac{3x^3}{8}$

4 **a** $x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}$ **b** $\dfrac{\pi}{2}-x-\dfrac{x^3}{6}$

5 **a** $1-\dfrac{x}{2}+\dfrac{x^2}{8}-\dfrac{x^3}{48}$ **b** $1-\dfrac{x}{3}+\dfrac{x^2}{18}-\dfrac{x^3}{162}$

b $y=\sqrt{\dfrac{c}{x^2}+1}$

6 **a** $1+3x^2+\dfrac{9x^4}{2}+\dfrac{9x^6}{2}$

b $1+2x^3+2x^6+\dfrac{4x^9}{3}$

7 **a** $1-\dfrac{9x^4}{2}+\dfrac{27x^8}{8}$ **b** $1-2x^6+\dfrac{2x^{12}}{3}$

8 **a** $4x-8x^2+\dfrac{64x^3}{3}$ **b** $-3x-\dfrac{9x^2}{2}-9x^3$

9 **a** $x+x^2+\dfrac{x^3}{3}$ **b** $1+x-\dfrac{x^3}{3}$

10 **a** $x+\dfrac{x^2}{2}-\dfrac{7x^3}{24}$ **b** $1+\dfrac{x}{3}-\dfrac{11x^2}{18}$

11 **a** $x-\dfrac{3x^2}{2}+\dfrac{11x^3}{6}$ **b** $x-x^2+\dfrac{23x^3}{24}$

12 **a** $\mathrm{f}'(x)=\sec^2 x,\ \mathrm{f}''(x)=2\sec^2 x\tan x,\ \mathrm{f}'''(x)$ $=4\sec^2 x\tan^2 x+2\sec^4 x$

b $x+\dfrac{x^3}{3}$

c 0.849%

13 **a** $\mathrm{f}'(x)=\sec x\tan x,\ \mathrm{f}''(x)=\sec x\tan^2 x+\sec^3 x,$

$\mathrm{f}'''(x)=\sec x\tan^3 x+5\sec^3 x\tan x$

b $1+\dfrac{x^2}{2}$

c 1.02, 0.0332%

14 $x-\dfrac{9}{2}x^3+\dfrac{27}{8}x^5$

15 **a** $-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}$ **b** $-\dfrac{79}{750}$

16 a $1+\frac{x}{e}-\frac{x^2}{2e^2}+\frac{x^3}{3e^3}$ b $\frac{6e^3+6e^2-3e+2}{6e^3}$

c 0.270%

17 $x+\frac{x^2}{2}-\frac{2x^3}{3}$

18 $2x+2x^2-\frac{4x^3}{3}$

19 a $1+2x+\frac{5x^2}{2}+\frac{5x^3}{3}$ b $1+x+\frac{3x^2}{2}+\frac{7x^3}{6}$

20 $1-\frac{x^2}{4}$

21 a $-3x-\frac{9x^2}{2}-9x^3$ b $-\frac{3}{2}$

22 2

23 a $x+\frac{x^3}{3}+\frac{2x^5}{15}$ b $x+x^2+\frac{5x^3}{6}+\frac{x^4}{2}$

24 a $x-\frac{2x^3}{3}$

b Use $\sin x\cos x=\frac{1}{2}\sin 2x$

25 a $x+\frac{x^3}{3}$ b $\frac{364}{375}$

26 a $2x-\frac{4x^3}{3}$ b $\frac{\sqrt{6}}{5}$

27 a $\frac{1}{\sqrt{1-x^2}}$, $x(1-x^2)^{-\frac{3}{2}}$, $(1-x^2)^{-\frac{3}{2}}+3x^2(1-x^2)^{-\frac{5}{2}}$

b $x+\frac{x^3}{6}$ c $2x-\frac{4x^3}{3}$ d $\sqrt{\frac{2}{3}}$

28 a $1+\frac{x}{e}-\frac{x^2}{2e^2}+\frac{x^3}{3e^3}$

29 a $1-x^2+\frac{x^4}{2}$ b $\frac{23}{30}$

c 2.66%

d $2.42\times10^{-6}\%$ – approximation is much better at small values of x

30 a $x-x^2+\frac{5x^3}{6}$

b $\frac{3}{8}$ c $\frac{1}{2}$

31 a $1+\frac{1}{2}x^2$ b $x+\frac{x^3}{6}$

32 b $\frac{2}{3}$

33 a $x-\frac{x^3}{2}+\frac{x^5}{24}$ b $\frac{1}{3}$

34 $-\frac{1}{2}$

35 a $x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}$ b 2

36 $1-\frac{3x}{2}+\frac{11x^2}{8}$

37 a $\ln 2+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{24}$

b $2\ln 2-x-\frac{7x^2}{4}+\frac{11x^3}{12}$

38 a $x^2-\frac{x^4}{6}+\frac{x^6}{120}$ b 0

39 a $x^2-\frac{x^3}{2}+\frac{x^4}{4}-\frac{x^5}{4}$ b $\frac{1}{2}$

40 a $e^{x\ln 2}$ b $1+(\ln 2)x+\frac{(\ln 2)^2}{2}x^2$

c $\frac{\ln 2}{\ln 3}$

41 a $f''(x)=\sec^2 x$, $f'''(x)=2\sec^2 x\tan x$, $f^{(4)}(x)$
$=2\sec^4 x+4\sec^2 x\tan^2 x$

b $x+\frac{x^2}{2}+\frac{x^4}{12}$ c $-\frac{1}{3}$

42 a $x-\frac{x^3}{6}+\frac{x^5}{120}$

43 a i $\ln(1+x)\approx x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$

ii $\arctan x\approx x-\frac{x^3}{3}+\frac{x^5}{5}$

b $x-\frac{x^2}{2}+\frac{x^4}{4}-\frac{11x^5}{60}$

44 a $\frac{(-1)^n}{(2n+1)!}$ b $\frac{(-1)^n}{(2n+1)!}$

45 b $\frac{2^n}{n!}-\frac{(-1)^n}{(2n)!}$

46 a $\frac{(-1)^{n-1}}{n}x^n$ (for $n\geqslant 1$)

47 $y=-\frac{1}{4}x+3$

48 (0, 2), local maximum

49 a $\frac{1}{2}$ b $\frac{1}{30}$

50 b $x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}$

Exercise 11E

1 a $(k+1)a_{k+1} = a_k$
b $(k+1)a_{k+1} = -2a_k$

2 a $(k+2)(k+1)a_{k+2} = -4a_k$
b $(k+2)(k+1)a_{k+2} = 3a_k$

3 a $(k+2)(k+1)a_{k+2} + (k+1)a_{k+1} = a_k$
b $(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} = -2a_k$

4 a $1 + 3x + \frac{3x^2}{2}$ b $1 + 2x + 3x^2$

5 a $2 + 4x + 8x^2$ b $3 - 9x + 27x^2$

6 a $1 + 2x - 3x^2$ b $-1 + 2x + \frac{x^2}{2}$

7 $1 + x + \frac{x^2}{2}$

8 a $1 + 2x + 2x^2$ b 5

9 a $1 + \frac{1}{2}x^3 - \frac{1}{8}x^6$ b $c + x + \frac{1}{8}x^4 - \frac{1}{56}x^7$
c 0.10001

10 $1 + 2x + 2x^2 - \frac{x^3}{6}$

11 a $1 - \frac{x^3}{6}$ b 0.979

12 $1 - x - \frac{x^2}{2} + \frac{x^3}{3}$

13 $y = -2 + 3x - \frac{13}{2}x^2 + \frac{91}{6}x^3$

14 a $\sum_{k=0} \frac{x^{k+1}}{k!}$ b $c + \sum_{k=0} \frac{x^{k+2}}{k!(k+2)}$
c 1.17

15 $1 + (1-e)x + \frac{e^2 - e}{2}x^2$

17 a $\sum_{n=0} \frac{x^{2n}}{2^n n!}$
b $e^{\frac{x^2}{2}}$

18 $x + \sum_{r=1} \frac{(-1)^r 2^2 5^2 \ldots (3n-1)^2}{(3r+1)!} x^{3n+1}$

19 $A\sum_{r=0} \frac{(-1)^r}{2^r r!} x^{2r} + B\sum_{r=0} \frac{(-2)^r r!}{(2r+1)!} x^{2r+1}$

Hint: You might not have found the answer in exactly this form. Can you see why?

$\frac{1}{2^r r!} = \frac{1}{2 \times 4 \times 6 \ldots \times 2r}$

20 b i x ii $\frac{1}{2}(3x^2 - 1)$

21 a $a_{r+2} = \frac{2(r-k)}{(r+2)(r+1)} a_r$
b k is an odd positive integer; $y = x$ and $y = x - \frac{2}{3}x^3$

22 b $2x^2 - 1$, $4x^3 - 3x$

23 b $y = A\sqrt{x} + \frac{B}{\sqrt{x}}$

Chapter 11 Mixed Practice

1 a $y = Ae^{\frac{3}{2}\sin 2x}$ b $y = 5e^{\frac{3}{2}\sin 2x}$

2 $y = Ae^{x^3 - 2x}$

3 a e^{-x^3} b $y = 3e^{x^3} - 2$

4 b $y = (x^2 + 1)(\arctan x + c)$

5 1.44

6 $x - \frac{7x^3}{6} + \frac{7x^5}{40}$

7 a $1 + x + \frac{x^2}{2}$

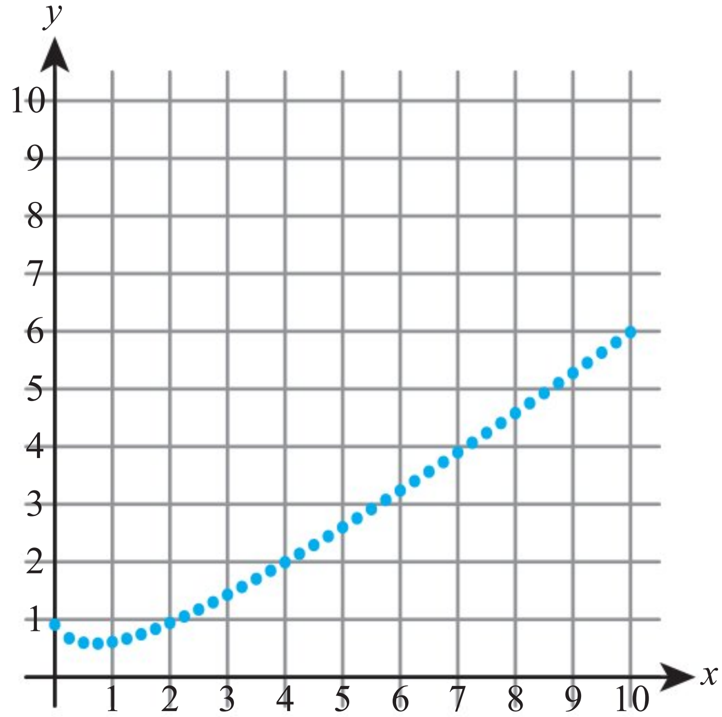
8 a $2x - \frac{8x^3}{3} + \frac{32x^5}{5}$ b $\frac{8}{3}$

9 a $1 + x^2$ b $A + Bx + \frac{x^2}{2} + \frac{x^4}{12}$

10 a 2.23
b (ii) $y = \cos x(\sin x + 2)$

11 a $R = R_0 e^{-kt}$ b $\frac{\ln 2}{k}$
c The time taken to go from $\frac{1}{2}$ to $\frac{1}{4}$ and from $\frac{1}{4}$ to $\frac{1}{8}$ etc will be the same.

12 a



b 0.6

13 a −0.599

b $-\ln\left(e^{-x}+e-1\right)$; 0.0181

c Decrease the step length

14 a −0.0392

b $e^{x^2}(x-1)$; 0.0392

15 $y=1+2e^{-\tan x}$

17 $y=(x+4)e^{2x^2}$

18 a 3.92 b $y=2e^{3-\frac{3}{x}}$ c 1.92%

19 a $y=\arctan\left(\ln|x-1|\right)$

b 2.39%

20 $N=2e^{0.2\left(t-\frac{12}{\pi}\cos\left(\frac{\pi t}{6}\right)\right)}$

21 a $\frac{dy}{dx}+(2x)y=2x\left(1+x^2\right)$

b $y=x^2+ce^{-x^2}$

22 a $\frac{dy}{dx}=3v+2v^2$ where $v=\frac{y}{x}$

b $y=\frac{x^3}{6-x^2}$

23 $\frac{2x^2}{9}$

24 $x-\frac{x^2}{2}-\frac{x^3}{3}$

25 $\frac{x^4}{6}$

26 a $\frac{1}{\sqrt{1-x^2}},\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}},\frac{1+2x^2}{\left(1-x^2\right)^{\frac{5}{2}}}$

b $x+\frac{x^3}{6}$ c $\frac{1}{3}$

27 $2+4x-\frac{x^3}{2}$

28 $1+2x+\frac{1-e}{2}x^2$

29 a $y=kx$ d 50

30 a ii $y=\pm\sqrt{\sin 2x+c}$

iii $c=1$

b i $y=\sqrt{\sin 2x+3},\ y\in\left[\sqrt{2},\,2\right]$

ii 2.99

iii 13.0

31 $y=\frac{1+x}{1-x}$

32 a $\frac{dy}{dx}=v-\frac{1}{\ln v}$, where $v=\frac{y}{x}$

b $y\left(\ln\left(\frac{y}{x}\right)-1\right)=x\ln\left(\frac{x}{e}\right)$

33 a $\frac{-\frac{1}{3}}{v+3}+\frac{-\frac{2}{3}}{v-3}$

b $\frac{dy}{dx}=v+\frac{9+v}{1+v}$, where $v=\frac{y}{x}$

34 $16e^{y-x}=(2x-3y-3)$

35 $y=\frac{(x-1)^2}{4x}$

37 a $\frac{\cos x-\sin x}{\sin x+\cos x},\frac{-2}{(\sin x+\cos x)^2},\frac{4(\cos x-\sin x)}{(\sin x+\cos x)^3}$

38 a $\cos x e^{\sin x}$ b $1+x+\frac{x^2}{2}-\frac{x^4}{8}$

c $\frac{1}{6}$

39 a $-\frac{5}{4}$ c 3 terms

40 $\sum_{n=0}\frac{1\times5\times9\ldots(4n+1)}{(2n+1)!}x^{2n+1}$

Analysis and approaches

HL: Practice Paper 1

1 $y - \frac{\pi}{4} = -\frac{2}{3}\left(x - \frac{1}{3}\right)$

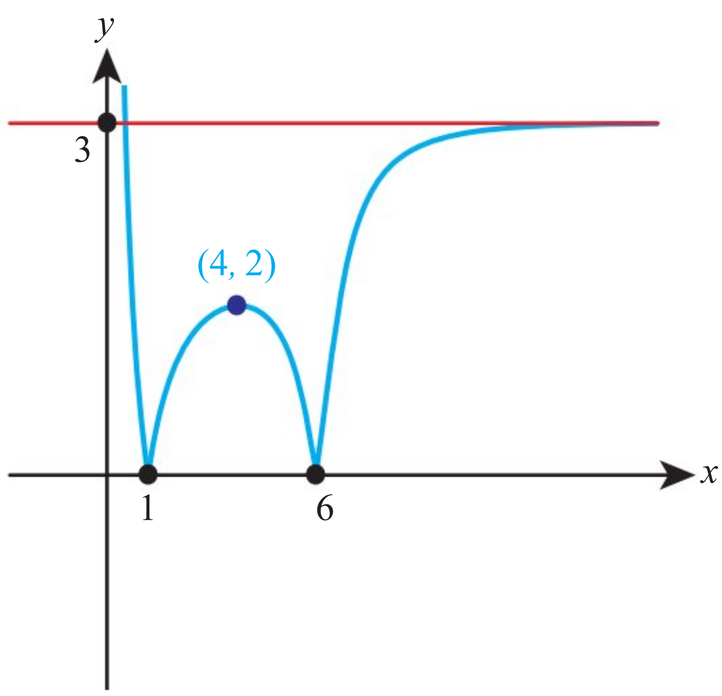
2 a $\frac{1}{3}$

b $\frac{5}{16}$

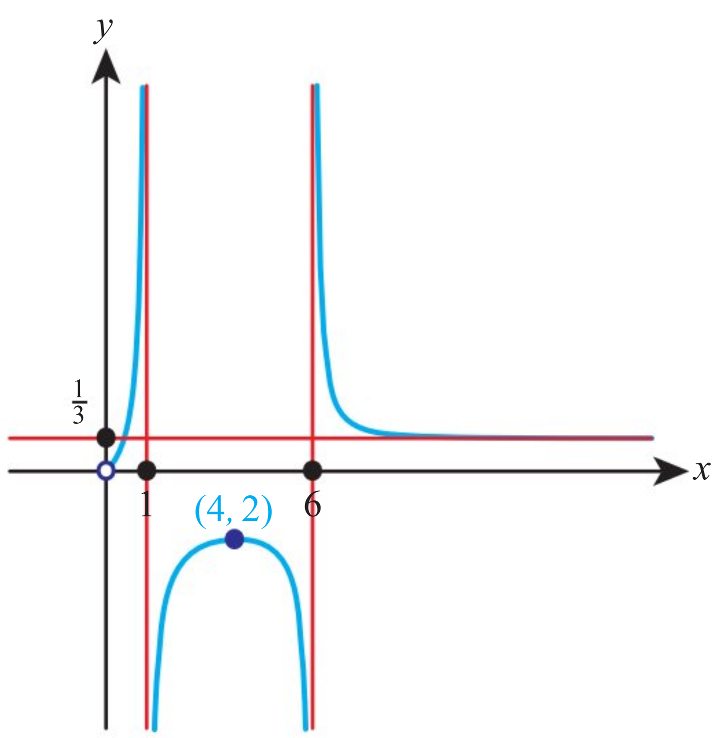
3 a $\frac{(4+3\mathrm{i})}{5}$

b $p = \frac{2}{5}, q = -\frac{1}{5}$

4 a



b



5 $\ln 9$

6 a $\frac{2}{x-1} - \frac{1}{x+1}$

b $\ln\left(\frac{27}{16}\right)$

7 a $\frac{2t}{1-t^2}$ **b** $\frac{\pi}{8}, \frac{5\pi}{8}$ **c** -2

9 $(0, 0), (-1, 1)$

10 a 3

b ii -3

iii $\mathbf{r} = \begin{pmatrix} -5 \\ 10 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$

c i $\sqrt{46}$ **ii** $\frac{3}{2\sqrt{23}}$

11 b e^{-t}

c i $F(t) = \begin{cases} 0 & \text{for } t \leqslant 0 \\ 1 - \mathrm{e}^{-t} & \text{for } t > 0 \end{cases}$

d i e^{-3} (ii) $\ln 2$

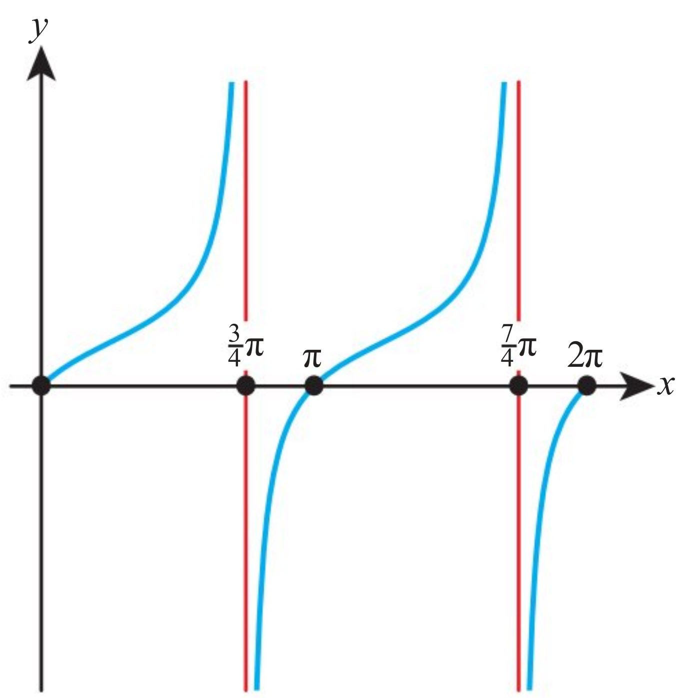
e i $C\frac{1}{1-\mathrm{e}^{-5}}$ **ii** $\frac{1-6\mathrm{e}^{-5}}{1-\mathrm{e}^{-5}}$

12 a i $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

ii $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

iii $f'(x) = \frac{1}{(\sin x + \cos x)^2} > 0$

iv



b i $S + C = \frac{\pi}{4}$ **ii** $\frac{\pi}{8} - \frac{1}{4}\ln 2$

Analysis and approaches HL: Practice Paper 2

1 **a** $u_{n+1} = 1.04u_n + 100$ **b** 32 **c** 33.5%

2 **a** $2 + x + 2y$ **b** $\frac{x}{y}$

3 **a** 21, −4 **b** 11

4 15.7

5 $\frac{1}{2}$

6 **a** 3 km per minute. **b** $\begin{pmatrix} 4 \\ 10 \\ 10 \end{pmatrix}$

c Goes through the point at $t = 3$ for A and $t = 6$ for B.

d No – closest distance is 6.44 km

7 **a** $x > 1.76$

b $x > \frac{1.76}{k}$

8 **a** 86 400

b **i** $\frac{1}{5}$

ii $\frac{1}{15}$

10 **a** $\sigma = 2.57$, $\mu = 11.3$

b Expectation: 2.4, Standard deviation: 2.11

c 0.64 **d** 0.684 **e** 0.6

11 **a** **ii** $y = \frac{1 + e^x}{1 + x}$

b **i** $e^x - 2\left(\frac{dy}{dx}\right)^2 - 2\frac{d^2y}{dx^2} - 2y\frac{d^2y}{dx^2}$

ii −3, 16, −65

ii $y \approx 2 - 3x + 8x^2 - \frac{113}{6}x^3$

c 1.47

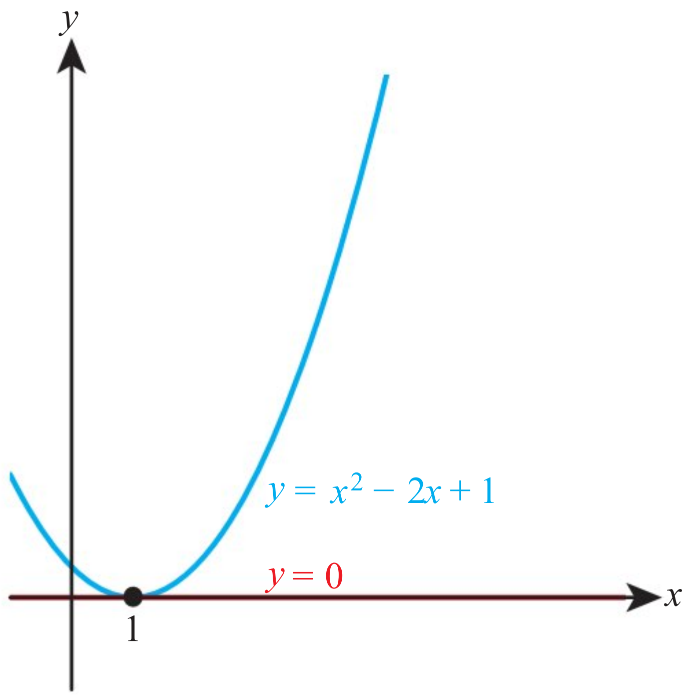
12 **a** **iii** $-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10$

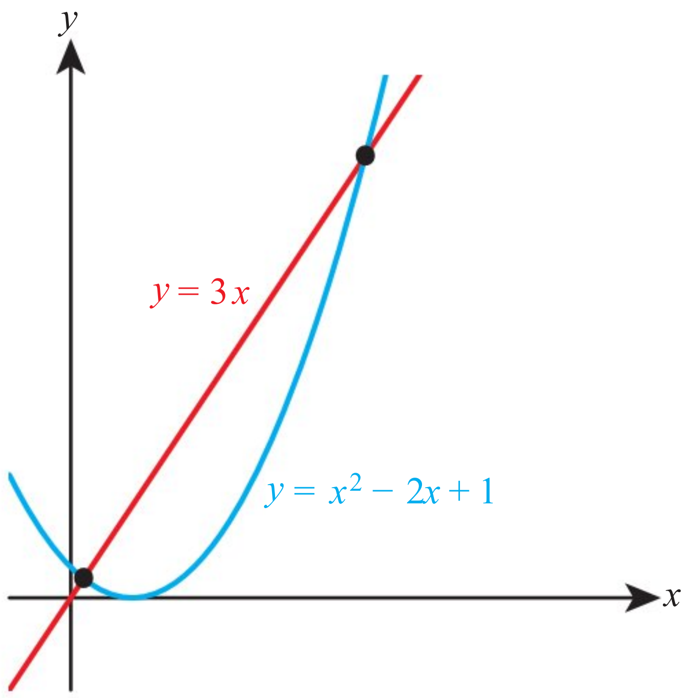
b **ii** $1 + 3x^2 + \frac{x^4}{2}$

c $\frac{5n\pi}{8}$

Analysis and approaches HL: Practice Paper 3

1 **a** **i and ii**

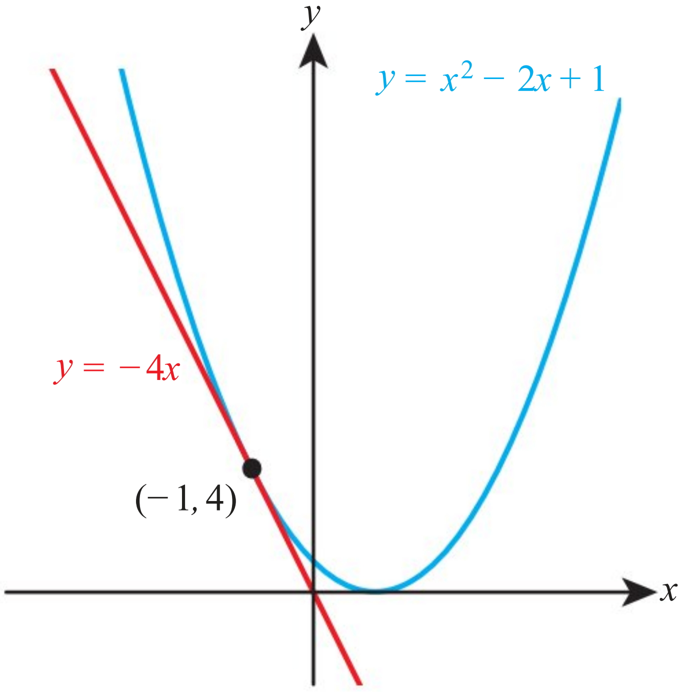




Two solutions

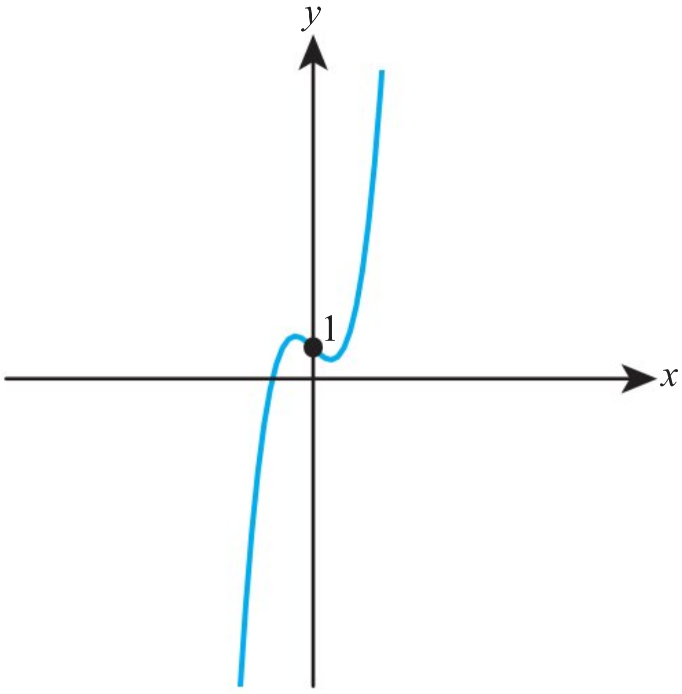
iii If $a = 0$ or $a = -4$ there is one solution.
If $-4 < a < 0$ there is no solution.
Otherwise there are two solutions.

iv

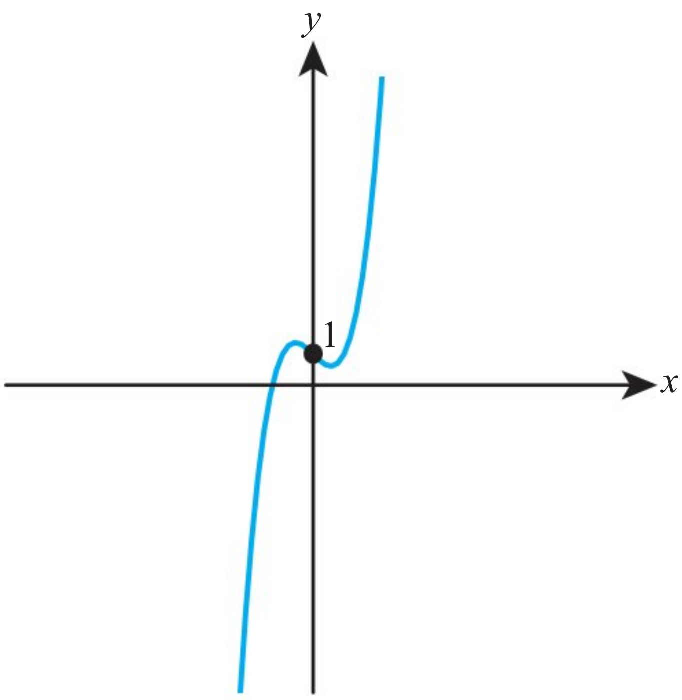


They are tangential.

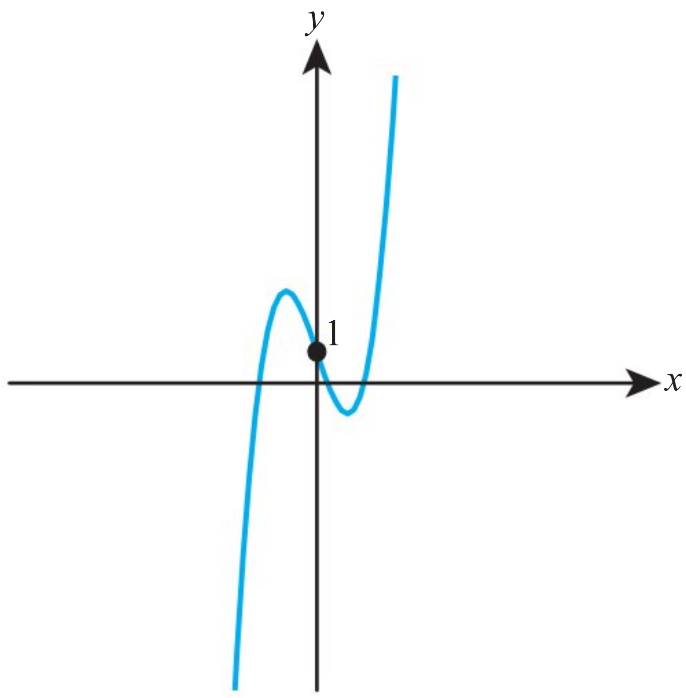
b i



ii



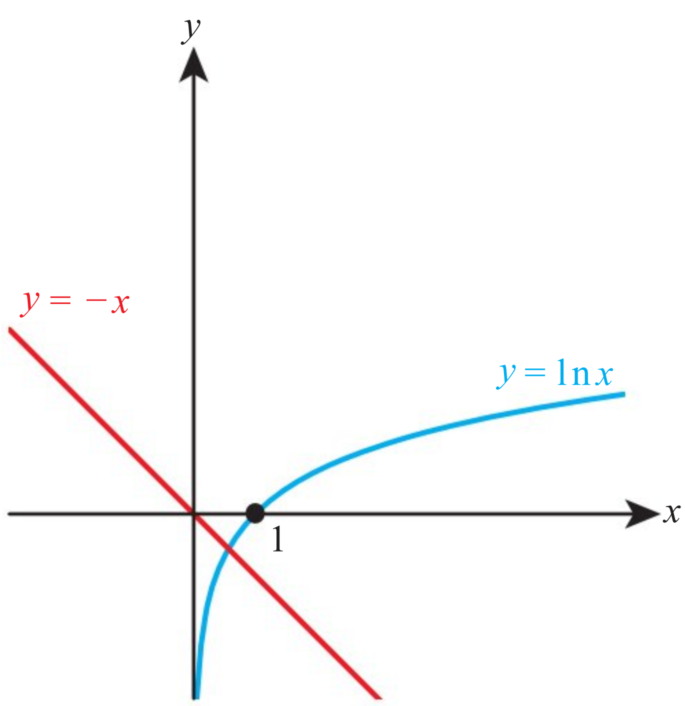
iii



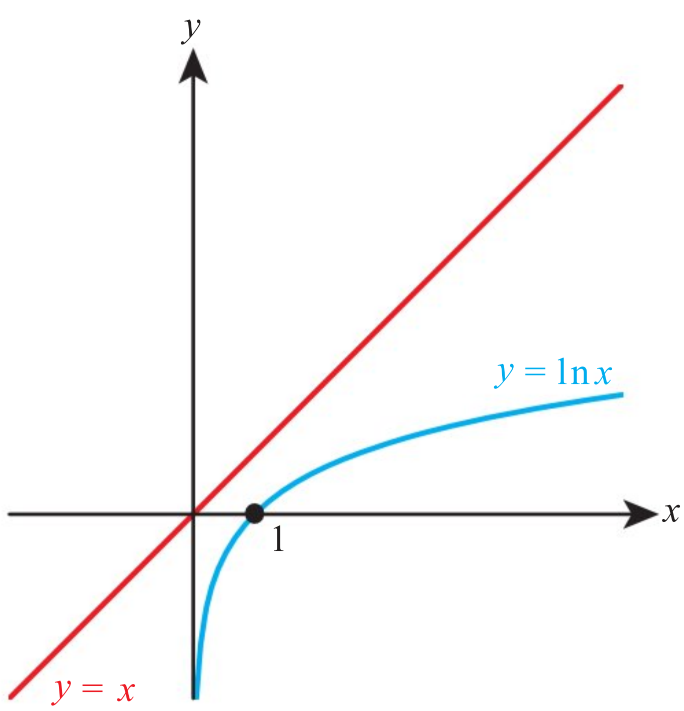
iv $\left(\sqrt{\frac{b}{3}}, 1 - \frac{2b}{3}\sqrt{\frac{b}{3}}\right)$

v $\frac{3}{\sqrt[3]{4}}$

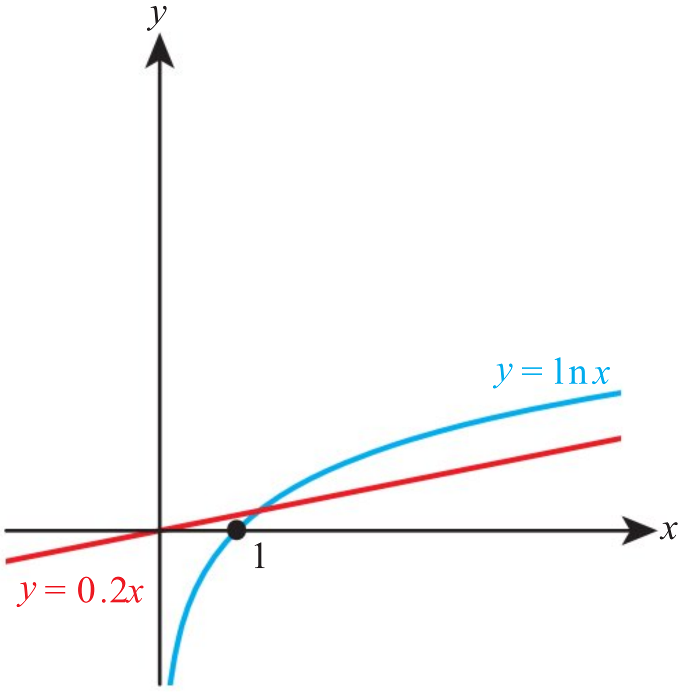
c i



ii



iii



iv $y - \ln p = \frac{1}{p}(x - p)$

v e

vi $k < 0$ or $k = \frac{1}{e}$

d 0.1286

2 a ii 5

b ii $B_n = (n+1)!A_n,\ B_3 = 120$

e $B_3 \times 4^3 = 7680$

Be the Examiner answers

2.1 Solution 2

3.1 Solution 2

3.2 Solution 2

4.1 Solution 1

4.2 Solution 3

5.1 Didn't prove true for $n = 1$
Should have said 'Assume true for $n = k$'
Can't use result for $n = k + 1$ in final step

6.1 Solution 3

6.2 Solution 2

7.1 Solution 3

7.2 Solution 2

8.1 Solution 3

8.2 Solution 2

8.3 Solution 3

8.4 Solution 2

9.1 Solution 3

9.2 Solution 2

10.1 Solution 2

10.2 Solution 3

11.1 Solution 1

Glossary

Argand diagram Another term for the complex plane

Argument (of a complex number) The angle a number in the complex plane makes with the real axis, measured anticlockwise

Base vectors The vectors **i**, **j** and **k**, which are of magnitude 1 and parallel to the x, y and z axes respectively

Boundary conditions Values of y or $\frac{dy}{dx}$ at other values of x (other than $x = 0$)

Cartesian equation (of plane) The form $n_1x + n_2y + n_3z = d$, where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is a normal to the plane and $d = \mathbf{a} \cdot \mathbf{n}$ for a point in the plane with position vector $\mathbf{a}$

Cartesian form (of complex number) A way of writing a complex number, z, in terms of its real and imaginary parts: $z = x + iy$, where $x, y \in \mathbb{R}$

Combination An arrangement of items where the order does not matter

Complex conjugate If $z = x + iy$, then the complex conjugate of z is $z^* = x - iy$

Complex number A number that can be written in the form $x + iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$

Complex plane A Cartesian plane where the x-axis represents the real part of a complex number and the y-axis the imaginary part

Components (of a vector) The number of units in the direction of the coordinate axes

Consistent (system of equation) A set of simultaneous equations that have solution(s)

Continuous function A function whose graph can be drawn without taking pen from paper

Continuous random variable A variable that can take any real value in a given interval (which may be finite or infinite)

Counterexample A particular case that disproves a statement

Cross product Another term for vector product

Degree (of a polynomial) The highest power of x in a polynomial

Direction vector (of line) A vector parallel to a given line

Displacement vector A vector from one point to another point

Dot product Another term for scalar product

Euler form A way of writing a complex number, z, in terms of its modulus, r, and argument, θ: $z = re^{i\theta}$

Euler's method An iterative method that approximates the solution of a differential equation

Even function A function such that $f(-x) = f(x)$ for all x in the domain of f

First-order differential equation An equation with a first derivative term but no higher derivatives

General solution (of differential equation) The solution containing an unknown constant

General solution (of system of equations) A form of solution in which the variables are expressed in terms of parameter(s)

Homogenous differential equation A differential equation that can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

Imaginary part If $z = x + iy$, then the imaginary part of z is the real number y

Inconsistent (system of equations) A set of simultaneous equations that do not have a solution

Inductive step A step in proof by induction that establishes the result for the next integer by building on the result for the previous integer

Initial conditions The value of y or $\frac{dy}{dx}$ at $x = 0$

Integrating factor A function that can be multiplied through a first order linear differential equation so that the LHS can be expressed as $\frac{d}{dx}(f(x, y))$

Integration by parts A method for integrating a product of two functions: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Linear differential equation A differential equation where neither y nor any of its derivatives are multiplied together, or have any non-linear function applied to them

L'Hôpital's rule A rule for finding limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Maclaurin series An infinite series in positive integer powers of x that represents a function

Modulus (of a complex number) The distance of a number from the origin in the complex plane

Modulus–argument (polar) form A way of writing a complex number, z, in terms of its modulus, r, and argument, θ: $z = r(\cos\theta + i\sin\theta)$

Oblique asymptote An asymptote that is neither horizontal nor vertical

Odd function A function such that $f(-x) = -f(x)$ for all x in the domain of f

Order (of a polynomial) Another term for degree

Parametric form (of equation of line) A form of the equation where x, y and z are expressed in terms of a parameter

Partial fractions Two or more rational functions that sum to give a more complicated rational function

Particular solution A solution where the values of any constants have been found

Permutation An arrangement of items where the order matters

Polynomial An expression that can be written as a sum of terms involving only non-negative integer powers of x

Position vector A vector from the origin to a point

Probability density function A function, f, that gives the distribution of a continuous random variable:

$$\mathrm{P}(a < X < b) = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$$

Proof by contradiction An indirect method of proof that starts by assuming the statement is false and shows that this leads to an impossible or contradictory conclusion

Real part If $z = x + \mathrm{i}y$, then the real part of z is the real number x

Recurrence relation A formula that defines the next term of a sequence from previous term(s)

Scalar product A scalar value given by $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

Scalar product form (of the equation of a plane) The form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where $\mathbf{a}$ is a point in the plane and $\mathbf{n}$ is a normal to the plane

Self-inverse function A function f such that $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$ for all x in the domain of f

Skew Straight lines that do not intersect and are not parallel

Solid of revolution A 3D shape formed by rotating part of a curve 360° around the x-axis (or y-axis)

Unit vector A vector of magnitude 1

Variance (of random variable) A measure of spread: $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Vector A quantity that has both magnitude and direction

Vector equation (of plane) The form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$, where $\mathbf{a}$ is a point in the plane and $\mathbf{d}_1$ and $\mathbf{d}_2$ are two vectors that lie in the plane

Vector product A vector perpendicular to the two given vectors with magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

Volume of revolution The volume of a solid of revolution

Index

3D shapes 57, 317

Abel, Niels 121
acceleration 303
algebra 16
  algebraic expressions 41
  binomial theorem 17–24
  fundamental theorem of 77
  partial fractions 22–4
  properties of a vector product 229
  properties of scalar product 202–3
  representation of a function 133
  systems of linear equations 25–7
  see also calculus
Al-Samawal 84
AND rule 4–8
angles
  finding with vectors 200–2, 213–14, 242–9
  see also triangles
approximation 262, 276, 307, 370
arbitrary constant 362
arccos 40–2
arcsin 39–40
arctan 41, 320
area
  bounded by a curve 332–3
  triangles 229–30
  vector products 229–31
Argand diagram 61–2, 65–8, 80–3, 474
argument (of a complex number) 65–6, 474
assumptions 102–3
  see also proof

base vectors 179–80, 190, 474
Bayes, Thomas 265
Bayes' theorem 264–7
binomial
  coefficients 6, 22
  expansion 6, 18–20, 22–4, 366
  theorem 17–24, 87
boundary conditions 352, 474
Brahmagupta 84

calculus 295
  applied 317–21
  area bounded by a curve 332–3
  with complex numbers 88
  continuous functions 297–8
  derivatives 302–3, 317, 319
  differentiation 300–1, 309–13, 320
  fundamentals of 297
  geometric interpretation of integrals 332
  higher derivatives 302–3
  implicit differentiation 309–13
  integration by parts 327–30
  integration by substitution 324–5
  L'Hôpital's rule 305–8, 371
  limits 299–300, 302, 305–8
  optimization 314–15
  partial fractions 321
  radians 302
  related rates of change 312–13
  solid of revolution 333, 475
  volumes of revolution 333–5, 475
  see also trigonomic functions
Cantor, Georg 103
Cardano, Girolamo 84
Cardano's formula 121
Cartesian
  equation of line 210–13
  equation of plane 236–9, 474
  form (of complex number) 58–61, 66, 69–71, 80, 474
Cauchy–Riemann conditions 88
choosing *r* items 6–7
chord, gradient of 300
circular reasoning 79
coefficients 6, 22, 74–7
  comparing 374
  constant 359
  polynomials 110, 118–20
  quadratic equations 122–5
  real 76
coincident lines 223
collinear points 193–4, 207
column vector 178
combination 474
combinations of *n* items 6–7
complex conjugates 59–90, 67, 74–7, 474
complex exponentials 88
complex numbers 474
  Argand diagram 61–2, 65–8, 80–3, 474
  calculations with 59–90, 68
  Cartesian form 58–61, 66, 69–71, 80, 474
  definition 474
  De Moivre's theorem 79–80, 87
  Euler form 69–71, 79, 474
  exponential form 69–70, 79
  factorizing polynomials 74–7
  indices 70
  modulus-argument (polar) form 65–8, 474
  powers 70
  quadratic equations 74–7
  representing 56, 63, 68
  roots 74–7, 79, 80–3
  two-dimensional coordinates of 61–2
  see also coefficients
complex plane 61–2, 474
components, of a vector 178–80, 200, 474
compound angle identities 44–6, 48–9, 302
conditional probability 263–4
conjugate pairs 77
consistent system of equation 474
constant coefficients 359
continuity 297–8
continuous functions 297–8, 474
continuous random variables 274–81, 474
contradiction, proof by 95, 102–3, 475
convergent functions 299–300
cosecant (cosec) 37
cosine (cos) 36–40, 44–5, 48–9, 87–8, 99, 163–5, 200, 302
cotangent (cot) 37–8
counterexample, disproof by 95, 104–5, 474
counting principles
  basic techniques 2–8
  problem solving 11–12
cross product 226, 237, 474
cubic inequalities 139–40
cubic polynomial graphs 110
curves 311
  gradient of 310

degree, of a polynomial 110, 474
De Moivre's theorem 79–80, 87, 99
dependent variables 352
derivative of the function 300
derivatives 302–3, 317, 319
  differential equations 352
determinant 28
differential equations 352–5, 357–9, 362–4, 374–6, 474
  see also calculus
differentiation 297–8
  from first principles 300–1
  implicit 309–13
diffusion 221
direction vectors 209, 474
Dirichlet function 298
discrete random variables 270–1
displacement 303
displacement vectors 190–1, 216–17, 474

disproof by counterexample 104–5, 474
  see also proof
divergent functions 299–300
divisibility
  and mathematical induction 98
  and polynomials 118–20, 136–7
domain restriction 165–6
dot product 199, 474
double angle identities 47

equations
  complex numbers 58–60, 74–7
  consistent 474
  determinant 28
  differential 352–5, 357–9, 362–4, 374–6
  inconsistent 28, 474
  of a line 207–17
  linear 25–6
  modulus function 146–7
  of a plane 234–9, 475
  quadratic 58, 74–6, 122–5
  simultaneous 25
  solving 25–7
  systems of 25–7
  see also trigonomic functions
equivalence 49
Euclid 102
Euler, Leonard 355
Euler form 69–71, 79, 474
Euler's method 353–5, 474
even functions 163–5, 474
excluded middle, **law of** 103
expanding brackets 17
expanding expressions 48–9
exponential form 69–70, 79, 88, 307
exponents, **laws of** 79, 363

factorization 19
factorizing polynomials 74–7, 113, 118–20
Fermat's little theorem 98
first-order differential equation 352, 474
first principles, differentiation from 300–1
formulas, as generalizations 6
fractals 57
fractions, partial 22–4, 321, 475
functions
  convergent 299–300
  cubic inequalities 139–40
  derivative of 300
  divergent 299–300
  domain restriction 165–6
  finding inverse function 165–6
  graphs of the functions *y* 143–55
  modulus 143–7
  odd and even 163–5, 474
  properties of 163–5
  rational functions of the form 134–7
  self-inverse 166–7, 475
  solving inequalities 139–41
  see also trigonomic functions
fundamental theorem of algebra 77

Gabriel's Horn, 335
Galois, Evariste 121
generalization 6, 94
  and validity 102
  see also binomial theorem
general solution
  of differential equation 352–3, 474
  of system of equations 27, 246, 474
geometric
  interpretation of integrals 332
  representations 57
  sequences 18
Gödel, Kurt 104
Gödel's incompleteness theorem 104
gradient
  of the chord 300
  of a curve 310
  of the tangent 300
graphs
  continuous functions 297–8, 474
  cubic inequalities 139–40
  horizontal asymptote 134–7, 140, 151–5
  modulus function 143–7
  oblique asymptote 135–6
  odd and even functions 163–5
  polynomials 110–13
  rational functions of the form 134–7
  symmetries 164–5
  symmetries of trigonometric graphs 48–9
  transformations 151–5
  vertical asymptote 134–7, 140, 151–5
  x-intercept 112–13, 134–7, 140, 151–5
  y-intercept 112–13, 134–7, 140, 151–5

heat equation 352
higher derivatives 302–3
higher order polynomials 125
Hilbert's Hotel 8
homogeneous differential equations 357–9, 474
horizontal asymptote 134–7, 140, 151–5
hyperbola 134

i, number 57–61
imaginary parts 59–60, 68, 474
implicit differentiation 309–13
inconsistent system of equations 28, 474
independent variables 352
induction, proof by 95–100
inductive reasoning 100
inductive step 96, 474
inequalities, solving 139–41, 146–7
infinite polynomials 17–18
infinite series 366–7
infinity 299, 335
initial conditions 352, 474
integrating factors 362–4, 474
integration
  by parts 327–30, 474
  by substitution 324–5
intersection of lines 221–4
inverse function 165–6
inverse trigonomic functions 39–42
irrational numbers 102–3
isoperimetric problem 317

kinematics 216–17, 303

Ladder problem 317
law of the excluded middle 103
L'Hôpital's rule 305–8, 371
linear differential equations 352, 474
linear equations 17, 25–6
lines
  coincident 223
  equations of 207–17
  intersection of 221–4
  parallel 223–4
  skew 221–3, 475
Liu Hui 84
logarithms 363
Lorentz force 227

Maclaurin series 366–71, 374–6, 474
magnitude, of a vector 185, 191–2, 226–9
mathematical induction 96–100
  see also proof
matrices 28
Maurolico, **Francesco** 96
mean 280–1
median 277–8
mode 277
modelling real life situations
  optimization 314–15
  using probability 263, 265
  using trigonomic functions 35
  using vectors 176–8
modular arithmetic 98

modulus (of a complex number) 65–6, 474
modulus-argument (polar) form 65–8, 474
modulus equations 146–7
modulus function 143–7
Monty Hall problem 267
Moving Sofa problem 317
multiple angle identities 48

***n*C** 6–8, 18
negative integers 18
  powers 79
negative numbers
  debts as 84
  history of 84
  square root of 58
neural networks 17
***n*P** 7–8, 12
number i 57–61
number line 57, 63
  complex numbers 61–2

oblique asymptote 135–6, 474
Occam's razor 112
odd functions 163–5, 474
optimization 314–15
order, of a polynomial 110, 474
OR rule 4–8

parallel lines 223–4
parallelogram 184, 192, 229–30
parallel vectors 183–5, 192, 203–4, 223–4, 229
parametric form 210, 475
partial differential equations 352
partial fractions 22–4, 321, 475
particular solution 352–3, 375, 475
patterns, as representations 56, 63
permutations of *n* items 4–7, 11–12, 475
perpendicular vectors 203–4, 214, 229, 237–8
physics, and vectors 227
polynomials 109–10, 475
  complex coefficients 75–7
  cubic 110
  degree of 110, 474
  division 118–20, 136–7
  expressions of higher powers 109–10
  factorizing 74–7, 113, 118–20
  factors 112
  function of a factor 112
  graphs 110–13
  higher order 125
  infinite 17–18
  number of roots of 77
  order of 110, 474
  quadratic 110, 122–5
  quartic 110
  quotient 118–20
  real coefficients 76
  remainder theorem 118–20
position vectors 190–5, 207–8, 475
positive integers 17–18
  mathematical proof 96
  powers 18, 79
powers
  complex numbers 70
  negative integers 79
  non-negative integers 110
  positive integers 17–18, 79
probability
  Bayes' theorem 264–7
  conditional 263–4
  continuous random variables 274–81
  density function 274–5, 475
  discrete random variables 270–1
  linear transformations 270–1
  random variables 263, 270–1
  tree diagram 265–6
  variance 270–1, 280–1, 475
problem solving
  counting techniques 11–12
  equivalence 49
  using vectors 176–8
  see also trigonomic functions
proof 94
  compound angle identities 44–6
  by contradiction 95, 102–3, 475
  deductive 95
  differentiation 302
  disproof by counterexample 95, 104–5
  and generalization 102
  by induction 95–100, 474
  validity of 102
Pythagoras' theorem 185

quadratic denominator 321
quadratic equations 58, 74–6, 122–5
quadratic polynomial graphs 110
quartic polynomial graphs 110

radians 302
Ramsey theory 8
random variables 263, 270–1
rational functions of the form 134–7
rational numbers 18
ratios
  calculus 306
  trigonomic functions 36
real coefficients 76
real parts 59, 61, 87, 475
recurrence relation 374–5, 475
remainder theorem 118–20
rhombus 192
right-angled triangle 36
right hand screw rule 227
roots
  complex numbers 74–7, 79, 80–3
  quadratic equations 122–5
  of unity 80–1
rotations 68
  see also volumes of revolution

scalar product 199–204, 214, 475
scalar product form 236–7, 475
scalar quantities 177
secant (sec) 36–9
self-inverse functions 166–7, 475
sequences 95
  see also series
series
  differential equations 352–5, 374–6
  Euler's method 353–5
  integrating factors 362–4
  Maclaurin series 366–71, 374–6, 474
  and mathematical induction 97
  recurrence relation 374–5
simultaneous equations 25
sine (sin) 36–42, 44–5, 48–9, 87–8, 99, 163–5, 229, 302
skew lines 221–3, 475
solid of revolution 333, 475
space 112, 176, 221
speed 177
square numbers 97
square root
  negative numbers 58
  see also roots
substitution 324–5

tangent (tan) 36–41, 46–7, 163–5, 300
Thomae function 298
three dimensions
  intersection of lines 221–4
  see also 3D shapes
Torricelli's trumpet 335
transformations 151–5
tree diagram 265–6
triangles
  areas of 229–30
  radians 302
  right-angled 36
  see also calculus; trigonomic functions
trigonometry 34, 57

trigonomic functions
- combining 35
- complex numbers 87–8
- compound angle identities 44–6, 48–9, 302
- differentiation 302
- double angle identities 47
- expanding expressions 48–9
- identities for powers 87–8
- inverse trigonomic 39–42
- multiple angle identities 48, 87–8
- proving identities 44–8
- range and domain of 37–42
- reciprocal 36–8
- symmetries of trigonometric graphs 48–9
- uses of 35
- *see also* calculus

two dimensions, intersection of lines 221

unit vectors 185, 190, 475

validity, and generalization 102

variables 474
- continuous random 280–1, 474
- dependent 352
- discrete random 270–1
- independent 352
- separation of 357–9

variance 270–1, 475
- continuous random variables 280–1

vectors 68, 475
- addition 180–1
- algebraic properties of a vector product 229
- angles 200–2, 213–14, 242–9
- areas 229–31
- Cartesian form 210–13, 236–9
- collinear points 193–4, 207
- components 178–80, 200, 474
- cross product 226, 237, 474
- defining 176–8
- direction 209, 474
- displacement 190–1, 216–17, 474
- distances 191–2
- equation of a line 207–17
- equation of a plane 234–9, 475
- and geometry 190–5, 227
- intersection of lines 221–4
- kinematics 216–17
- Lorentz force 227
- magnitude of 185, 191–2, 226–9
- parallel 183–5, 192, 203–4, 223–4, 229
- parametric form 210
- perpendicular 203–4, 214, 229, 237–8
- and physics 227
- position 190–5, 207–8, 475
- representing 178–82
- right hand screw rule 227
- scalar multiplication 183–5
- scalar product 199–204, 214, 475
- scalar product form 236–7, 475
- subtraction 181
- unit vectors 185, 190, 475
- vector product 226–31, 475
- zero vector 181

velocity 177, 216–17, 303

vertical asymptote 134–7, 140, 151–5

volumes of revolution 333–5, 475

wave equation 352

***x*-intercept** 112–13, 134–7, 140, 151–5

***y*-intercept** 112–13, 134–7, 140, 151–5

zero
- denominator as 135, 301
- divided by zero 301, 305–8
- factorizing polynomials 112
- gradient as 315
- numerator as 301
- remainder theorem 119
- vector 181

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The authors are all University of Cambridge graduates and have a wide range of expertise in pure mathematics and in applications of mathematics, including economics, epidemiology, linguistics, philosophy and natural sciences.

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